Homework 2

Due Monday, April 15.

(1) Prove that if $\phi_t : X \to Y$ is a homotopy and h is the path $\phi_t(x_0)$ formed by the images of the basepoint $x_0 \in X$, then the two maps

$$\pi_1(X, x_0) \stackrel{(\phi_1)_*}{\to} \pi_1(Y, \phi_1(x_0)) \stackrel{\beta_h}{\to} \pi_1(Y, \phi_0(x_0))$$

and

$$\pi_1(X, x_0) \stackrel{(\phi_0)_*}{\to} \pi_1(Y, \phi_0(x_0))$$

agree.

(2) Prove that if $\phi : G \to H$ is a group homomorphism which has both a left inverse and a right inverse (they are a priori not necessarily the same), then ϕ is an isomorphism.

(3) Let X and Y be topological spaces. The *product topology* on $X \times Y$ is given as follows: $W \subset X \times Y$ is open if and only if it is a union of $U_\alpha \times V_\beta$ where $U_\alpha \subset X$ and $V_\beta \subset Y$ are open. (In other words, $\{U_\alpha \times V_\beta\}$ is a *basis* for the product topology.) Prove the product topology on $X \times Y$ is indeed a topology, and it is the smallest topology that makes the projection maps $X \times Y \stackrel{p_1}{\to} X$, $(x, y) \mapsto x$, and $X \times Y \stackrel{p_2}{\to} Y$, $(x, y) \mapsto y$, continuous.

(4) Show that if $Z \stackrel{f}{\to} X$ and $Z \stackrel{g}{\to} Y$ are continuous maps, then $Z \stackrel{(f,g)}{\to} X \times Y$ is continuous. (Unless stated otherwise, we will always use the product topology when we write $X \times Y$.)

(5) Using (3) and (4), prove that $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

(6) Modify the proof of the Path Lifting Property to prove the Homotopy Lifting Property.

(7) Verify that an open set of $(0, 1)$ is a *disjoint* union of open intervals (a_i, b_i).

(8) Hatcher, Section 1.1, Exercises 5, 10, 11, 16(a,b,c). (Refer to the text for the definition of a *retraction*, not to be confused with a *deformation retraction*, which is a special type of retraction.)