Homework 1

Due Monday, 4/8.

(1) Let X, Y be topological spaces and $f : X \to Y$ be a map. Suppose $X = X_1 \cup X_2$ is a union of two closed subsets X_1 and X_2, and the restrictions $f|_{X_1}$ and $f|_{X_2}$ are continuous. Then prove that f is continuous. (This result will be useful in the next two problems.)

(2) In class, we wrote down three homotopies during the proof that $\simeq$ is an equivalence relation. Verify that these homotopies are indeed continuous.

(3) Give a careful proof of the following: Let X be a topological space and $\simeq$ denote homotopy of paths in X relative to the endpoints. If $f_0 \simeq f_1$, $g_0 \simeq g_1$, and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 g_0 \simeq f_1 g_1$.

(4) Write a careful proof of the homotopy associativity (namely $(fg)h \simeq f(gh)$ whenever it makes sense to compose paths f, g, and h), using the Reparametrization Trick.

(5) Let $\sim$ be an equivalence relation on a topological space $(X, \mathcal{T})$, and $\pi : X \to X/\sim$ be the standard projection taking x to its equivalence class $[x]$. Then define the *quotient topology* on $X/\sim$ by stipulating that $V \subset X/\sim$ be open iff $\pi^{-1}(V)$ is open. Prove that the quotient topology is indeed a topology on $X/\sim$ and that it is the largest topology which makes π continuous.

(6) Show that the following are all homotopy equivalent to S^1.
  (a) The annulus $A = \{\frac{1}{2} \leq |x| \leq \frac{3}{2}\} \subset \mathbf{R}^2$.
  (b) $B = \mathbf{R}^2 - \{0\}$.
  (c) $S^1 \times I$.
  (d) $S^1 \times \mathbf{R}^n$.

(7) Prove that if A is a deformation retract of X, then A is homotopy equivalent to X.

(8) Hatcher, Section 1.1, Exercises 1, 2, 3, 6. (When you are doing the Hatcher problems, you may assume all the results and theorems presented in class – there is no need to reprove any results.)