Homework 9

(1) Show that two smooth maps $f : L \to N$ and $g : M \to N$ with L, M compact, L, M, N oriented, and $\dim L + \dim M = \dim N$ are transverse if and only if $f \times g : L \times M \to N \times N$ is transverse to $\Delta \subset N \times N$. Also verify that $I(f, g) = (-1)^{\dim M} I(f \times g, \Delta)$.

(2) Let x be a Lefschetz fixed point of $f : M \to M$. Then show that the local contribution of x to the global Lefschetz number $L(f)$ is $\mathrm{sgn}(\det(df(x) - I))$.

(3) Let X be a vector field on M with isolated zeros. If $X(x) = 0$, then show that the definition of $\mathrm{ind}(X, x)$ from class does not depend on the choice of oriented coordinates.

(4) In the proof of the Poincaré-Hopf theorem, it was stated that

$$\deg \left. \frac{f_\epsilon(z) - z}{|f_\epsilon(z) - z|} \right|_{\partial U} = \deg \left. \frac{X(z)}{|X(z)|} \right|_{\partial U},$$

where X is a vector field on M with isolated zeros, $X(x) = 0$, $U = \{z \in \mathbb{R}^n \mid |z| < \delta\}$ is a small open ball centered at $x = 0$, and $f_\epsilon : M \to M$ is the time-ϵ flow of X. Give a careful proof of this.

(5) Show that, if $m < n$, then every smooth map $f : M^m \to S^n$ is smoothly homotopic to a constant.

(6) Show that, if two smooth maps $f, g : M \to S^n$ satisfy $|f(x) - g(x)| < 2$ for all x, then f is smoothly homotopic to g. Here we are taking $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

(7) (20 pts) If M is compact, then:
  - (a) Show that every continuous map $f : M \to S^n$ admits a C^0-close smooth approximation $g : M \to S^n$ (i.e., for any $\epsilon > 0$ there exists a smooth g such that $|f(x) - g(x)| \leq \epsilon$ for all x). Moreover there is a continuous homotopy from f to g. (Hint: use the fact that $S^n \subset \mathbb{R}^{n+1}$.)
  - (b) Show that if two smooth maps $M \to S^n$ are continuously homotopic, then they are smoothly homotopic.

  This implies that the space of continuous maps $f : M \to S^n$ modulo continuous homotopy is in bijection with the space of smooth maps $f : M \to S^n$ modulo smooth homotopy.

(8) (20 pts) Recall that a submanifold $L \subset \mathbb{R}^n$ admits a normal bundle

$$N(L) := \{(x, v) \in L \times \mathbb{R}^n \mid v \perp T_x L\},$$

where $\perp$ is with respect to the standard inner product on $\mathbb{R}^n$, and that

$$N(L) \to \mathbb{R}^n, \quad (x, v) \mapsto x + v,$$

maps an ϵ-neighborhood of the zero section $L \times \{0\}$ diffeomorphically onto an ϵ-neighborhood N_ϵ of L in $\mathbb{R}^n$. Consider the retraction

$$r : N_\epsilon \to L, \quad x + v \mapsto x.$$

  - (a) Show that $r(x + v)$ is closer to $x + v$ than any other point of L.
  - (b) Using the retraction r, prove (7) with S^n replaced by L.

(9) (10 pts each, extra credit) Guillemin and Pollack, Chapter 3, Section 6, Exercises 1–13. This gives an alternate proof of the Hopf degree theorem.