Homework 2

(1) Consider the map $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$. Carefully compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.

(2) Let $\phi : M \to N$ be a smooth map between manifolds. If $\omega \in \Omega^k(N)$, then prove that $d(\phi^*\omega) = \phi^*(d\omega)$.

(3) Compute the de Rham cohomology of S^2. Then inductively compute the de Rham cohomology of S^n for all $n \geq 2$.

(4) Compute the de Rham cohomology of a compact 2-dimensional manifold (i.e., surface) of genus $g \geq 1$ without boundary. [You can do this informally, by drawing pictures of the surface of genus g and its decomposition into pieces.]

(5) Compute the de Rham cohomology of $\mathbb{RP}^n$ for all $n \geq 2$.

(6) (30 points) Complete the proof that the short exact sequence of cochain maps

$$0 \to \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

gives rise to a long exact sequence on cohomology. (This problem should be done carefully and completely — it's worth 30 points.)

(7) Let M be an oriented manifold of dimension n and $\omega \in \Omega^n(M)$. Let $\{\phi_\alpha : U_\alpha \to \mathbb{R}^n\}$ be an oriented atlas and $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. We defined

$$\int_M \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(f_\alpha \omega).$$

Prove that this definition does not depend on the choice of oriented atlas as well as on the choice partition of unity.