Homework 1

(1) (20 points) Recall that $\mathcal{F}_p$ is the set of germs of functions on a manifold M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of $\mathcal{F}_p^k$ is a sum $\sum_i g_i f_{i1} \cdots f_{ik}$, $g_i \in C^\infty(p)$, $f_{ij} \in \mathcal{F}_p$.)

(a) Prove that, in every coordinate system $(x_1, \dots, x_n)$, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes up to order k.

(b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.

(c) Construct a smooth manifold $E \xrightarrow{\pi} M$ whose fiber at $p \in M$ is $\mathcal{F}_p^1 / \mathcal{F}_p^3$. (This involves writing down coordinate charts and computing transition functions.)

(2) Consider the cotangent bundle $\pi : T^*M \to M$ of a manifold M. In class we gave an atlas for T^*M in terms of $\pi^{-1}(U_\alpha)$, where $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M. Compute the Jacobian for the transition functions on the overlaps $\pi^{-1}(U_\alpha \cap U_\beta)$.

(3) Prove that $d(fg) = fdg + gdf$, where f, g are smooth functions on a manifold M.

(4) Let $\phi : M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d\downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array}$$

(5) Let $\phi : L \to M$ and $\psi : M \to N$ be smooth maps between smooth manifolds and let ω be a 1-form on N. Prove that $(\psi \circ \phi)^*\omega = \phi^* \circ (\psi^*\omega)$.

(6) Let $\phi : M \to N$ be a smooth map between manifolds and $\omega \in \Omega^k(N)$. With respect to local coordinates $x_1, \dots, x_m$ for M and $y_1, \dots, y_n$ for N, if

$$\omega = \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(y) dy_{i_1} \dots dy_{i_k},$$

then we defined

$$\phi^*\omega = \sum_{i_1,\dots,i_k} f_{i_1,\dots,i_k}(y(x)) dy_{i_1}(x) \dots dy_{i_k}(x).$$

Show that $\phi^*\omega$ is well-defined.

(7) Show that d_k, as defined in class, is independent of the choice of local coordinates.

(8) Let M be a manifold. Prove that d satisfies the formula $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.

(9) Suppose the manifold M is the disjoint union of manifolds M_1 and M_2. Then prove that $H^k_{dR}(M) = H^k_{dR}(M_1) \oplus H^k_{dR}(M_2)$.