Homework 9

(1) The vector field $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$, defined almost everywhere on $\mathbb{R}^2$, has curl zero, but it cannot be written as the gradient of any function. Explain what this means in terms of de Rham cohomology.

(2) For each integer n, exhibit a smooth map $S^1 \to S^1$ of degree n. (You must prove that your map has degree n.)

(3) For each integer n, exhibit a *smooth* map $S^2 \to S^2$ of degree n. (You must prove that your map has degree n.)

(4) Let V be an $\mathbb{R}$-vector space. Prove that the interior product

$$i : V \otimes \wedge^k V^* \to \wedge^{k-1} V^*,$$
$$v \otimes (f_1 \wedge \cdots \wedge f_k) \mapsto \sum_l (-1)^{l+1} f_1 \wedge \ldots f_l(v) \cdots \wedge f_k,$$

is well-defined.

(5) Let M be a manifold and X a global vector field on M. Show that

$$\mathcal{L}_X = d \circ i_X + i_X \circ d : \Omega^k(M) \to \Omega^k(M)$$

is a derivation, i.e., it satisfies $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$. Here i_X denotes the interior product with X.

(6) Let M be a compact orientable manifold of dimension $2n$ without boundary and let ω be a *symplectic form* on M, i.e., a closed 2-form such that $\omega^n = \omega \wedge \cdots \wedge \omega$ is nowhere zero. Prove that $H^2(M) \neq 0$.

(7) Verify that Lie brackets satisfy anticommutativity and the Jacobi identity.

(8) Prove that if $X, X_1, \ldots, X_k$ are vector fields on M and $\omega \in \Omega^k(M)$, then

$$\mathcal{L}_X(\omega(X_1, \ldots, X_k)) = (\mathcal{L}_X \omega)(X_1, \ldots, X_k) + \sum_i \omega(X_1, \ldots, [X, X_i], \ldots, X_k).$$

(9) Prove that if $X_1, \ldots, X_{k+1}$ are vector fields on M and $\omega \in \Omega^k(M)$, then

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}). \end{aligned}$$

Here $\widehat{X_i}$ means omit the term X_i.

(10) Consider the vector field $X = x^2 \frac{d}{dx}$ on $\mathbb{R}$. Compute its integral curves. Explain why X does not admit a global flow $\Phi : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}$.

(11) Find a piecewise smooth curve connecting $(0, 0, 0)$ to any point (x, y, z), where each smooth piece an integral curve of the 2-plane field distribution $\xi = \ker(xdy - ydx + dz)$. (Such curves are far from unique.)