Homework 6

(1) Prove that if $i : M \to N$ is an immersion, then the induced map $i_* : T_x M \to T_{i(x)} N$ is injective for each $x \in M$.

(2) Let V, W be $\mathbb{R}$-vector spaces. Show that the map $\tau : V^* \otimes W \to \mathrm{Hom}(V, W)$ defined in class is linear and prove that it is an isomorphism if W is finite-dimensional. It follows that $V \otimes W$ has dimension $\dim V \cdot \dim W$ if V and W are finite-dimensional. Is τ an isomorphism when W is not finite-dimensional?

(3) Let V, W, and U be $\mathbb{R}$-vector spaces. Prove that $(V \otimes W) \otimes U$ is naturally isomorphic to $V \otimes (W \otimes U)$. Here a *natural* map is a map which does not depend on a choice of basis.

(4) Let V be a 2-dimensional $\mathbb{R}$-vector space with basis $\{v_1, v_2\}$ and $A : V \to V$ be a linear map given by $v_1 \mapsto 5v_1 + 6v_2$, $v_2 \mapsto 3v_1 + 2v_2$. Then write a matrix for $A \otimes A : V \otimes V \to V \otimes V$ in terms of the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$.

(5) Let V be an $\mathbb{R}$-vector space of dimension n.
   (a) Find an alternating multilinear form $\phi : V \times \cdots \times V \to \mathbb{R}$, where we are taking n copies of V. In particular you need to show that your choice of ϕ is alternating.
   (b) Explain how to use the universal property to show that $\bigwedge^n V \simeq \mathbb{R}$.

(6) Let V be an $\mathbb{R}$-vector space. Show there exists a (well-defined) linear map $\bigwedge^k V \otimes \bigwedge^l V \to \bigwedge^{k+l} V$ which sends $(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_l$.

(7) If V is an $\mathbb{R}$-vector space of dimension n, then show there exists an isomorphism $\bigwedge^{n-k} V \simeq (\bigwedge^k V)^*$ which only depends on a choice of nontrivial $\omega \in (\bigwedge^n V)^*$.

(8) Let $\phi : M \to N$ be a smooth map between manifolds and $\omega \in \Omega^k(N)$. With respect to local coordinates $x_1, \ldots, x_m$ for M and $y_1, \ldots, y_n$ for N, if

$$\omega = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k}(y) dy_{i_1} \ldots dy_{i_k},$$

then we defined

$$\phi^* \omega = \sum_{i_1, \ldots, i_k} f_{i_1, \ldots, i_k}(y(x)) dy_{i_1}(x) \ldots dy_{i_k}(x).$$

Show that $\phi^* \omega$ is well-defined.

(9) Show that d_k, as defined in class, is independent of the choice of local coordinates.

(10) Let M be a manifold. Prove that d satisfies the formula $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$.