Homework 5

(1) Show that the map $\phi^* : \Omega^1(N) \to \Omega^1(M)$, $\theta \mapsto \phi^*\theta$, is well-defined, i.e., indeed takes a smooth section of T^*N to a smooth section of T^*M.

(2) Prove that $d(fg) = fdg + gdf$, where f, g are smooth functions on a manifold M.

(3) Let $\phi : M \to N$ be a smooth map between manifolds. Prove that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d\downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array}$$

(4) Let $\phi : L \to M$ and $\psi : M \to N$ be smooth maps between smooth manifolds and let ω be a 1-form on N. Prove that $(\psi \circ \phi)^*\omega = \phi^* \circ (\psi^*\omega)$.

(5) Consider the map $i : S^1 = [0, 2\pi]/(0 \sim 2\pi) \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$. Carefully compute $i^*((x^2 + y)dx + (3 + xy^2)dy)$.

(6) Prove that every element $A \in SO(3)$ that is not the identity has a unique line through the origin in $\mathbb{R}^3$ which is fixed by A and A is given by a rotation about this axis.

(7) Prove that $U(n)$, the set of $n \times n$ matrices A with entries in $\mathbb{C}$ satisfying $AA^* = id$, is a Lie group. Here A^* is the conjugate transpose of A.

(8) Let $\mathcal{C}$ be an open cover of a manifold M. Prove that a collection $\{\Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ of maps (ranging over all $U_\alpha, U_\beta \in \mathcal{C}$) satisfying Conditions 1 and 2 in the notes gives a well-defined rank n vector bundle over M.

(9) Explain why $GL(n, \mathbb{C})$ is a Lie group. Show that $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$.