Homework 4

(1) (20 points) Recall that $\mathcal{F}_p$ is the set of germs of functions on a manifold M which vanish at $p \in M$. Let $\mathcal{F}_p^k$ be the ideal of $C^\infty(p)$ generated by $f_1 \cdots f_k$, where $f_i \in \mathcal{F}_p$. (This means that every element of $\mathcal{F}_p^k$ is a sum $\sum_i g_i f_{i1} \cdots f_{ik}$, $g_i \in C^\infty(p)$, $f_{ij} \in \mathcal{F}_p$.)

  (a) Prove that, in every coordinate system $(x_1, \dots, x_n)$, an element $f \in \mathcal{F}_p^k$ has a Taylor expansion which vanishes up to order k.

  (b) Compute the dimension of $\mathcal{F}_p^k / \mathcal{F}_p^{k+1}$.

  (c) Construct a smooth manifold $E \stackrel{\pi}{\to} M$ whose fiber at $p \in M$ is $\mathcal{F}_p^1 / \mathcal{F}_p^3$. (This involves writing down coordinate charts and computing transition functions.)

(2) Prove that the tangent bundle TS^2 defined as $\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| = 1, x \cdot y = 0\}$ is diffeomorphic to our "official definition" of the tangent bundle.

(3) Consider the cotangent bundle $\pi : T^*M \to M$ of a manifold M. In class we gave an atlas for T^*M in terms of $\pi^{-1}(U_\alpha)$, where $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M. Compute the Jacobian for the transition functions on the overlaps $\pi^{-1}(U_\alpha \cap U_\beta)$.

(4) If $g \in \Omega^0(M)$, then show that dg is indeed a 1-form over M, i.e., a smooth map $s : M \to T^*M$ such that $\pi \circ s = \mathrm{id}$.

(5) In class we defined the *derivative map* as follows: Let $f : M \to N$ be a smooth map between manifolds. Then the derivative $f_* : T_pM \to T_{f(p)}N$ is given by $X \mapsto X \circ f^*$, where $X : C^\infty(p) \to \mathbb{R}$ is a derivation and f^* is the pullback $C^\infty(f(p)) \to C^\infty(p)$. Give an equivalent definition for f_* in terms of Definition 1 of a tangent space and show the equivalence with the above definition.

(6) Given a smooth map $f : M \to N$ between manifolds, we defined the maps $f_* : T_pM \to T_{f(p)}N$. Show that they can be combined into a smooth map $f_* : TM \to TN$ of tangent bundles. (First rephrase f_* in terms of local coordinates.)

(7) Prove that $SO(2)$ consists of the 2×2 matrices $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\theta \in \mathbb{R}$. Show that $SO(2)$ is diffeomorphic to S^1.

(8) Explain why $SO(n)$ is a Lie group.