Homework 3

(1) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective. Also show that there is no submersion from a compact manifold to $\mathbb{R}^n$.

(2) Let N be an n-dimensional submanifold of a manifold M of dimension m and let $x \in N$. Prove that there exists an open set U of M containing x and a local coordinate system $\{x_1, \ldots, x_m\}$ on U such that $U \cap N = \{x_{n+1} = 0, \ldots, x_m = 0\}$.

(3) Let $F(x_1, \ldots, x_n)$ be a *homogeneous function* of degree d in n real variables, i.e.,

$$F(tx_1, \ldots, tx_n) = t^d \cdot F(x_1, \ldots, x_n).$$

(a) Prove Euler's identity:

$$\sum_{i=1}^{n} x_i \cdot \frac{\partial F}{\partial x_i} = d \cdot F.$$

(b) Prove that the set $F^{-1}(a)$, $a \neq 0$, is a submanifold of $\mathbb{R}^n$.

(4) Let $f : M \to N$ be a proper submersion between connected manifolds of the same dimension. Show that f is a *covering map*, i.e., for each $p \in N$ there exists an open neighborhood U of p, such that $f^{-1}(U)$ is a union of disjoint open sets in M, each of which is mapped diffeomorphically onto U.

(5) Let $f : \mathbb{C} \to \mathbb{C}$ the the map given by $z \mapsto z^m + a_1 z^{m-1} + \cdots + a_m$, where $a_1, \ldots, a_m \in \mathbb{C}$ are fixed and $z \in \mathbb{C}$. Prove that f is a submersion except at finitely many points.

(6) (20 points) Show that the space of all $m \times n$ matrices with real entries of rank k forms a smooth submanifold of $\mathbb{R}^{mn}$ of dimension $k(m + n - k)$.

(7) (30 points) Give a detailed proof of the equivalence of the three definitions of T_pM given in class. Pay special attention to good exposition.