Homework 1

(1) Show that the induced topology and the quotient topology satisfy the axioms of a topological space.

(2) Prove that $S^1 = \{x^2 + y^2 = 1\}$ with the induced topology (from $\mathbb{R}^2$) is homeomorphic to $S^1 = [0,1]/\sim$ with the quotient topology.

(3) Let $V \subset W$ be a vector subspace. Prove that the quotient W/V is a vector space.

(4) Let V, W be k-vector spaces. Show that $\mathrm{Hom}_k(V, W) = \{k\text{-linear maps } \phi : V \to W\}$ is a k-vector space. Compute its dimension if V, W are finite-dimensional.

(5) Read pp. 15–34 of Spivak, *Calculus on Manifolds*.

(6) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, then there is a unique L which satisfies

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - L(h)|}{|h|} = 0.$$

(7) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x,y) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}}, & (x,y) \neq 0, \\ 0, & (x,y) = 0. \end{cases}$$

Show that f is not differentiable at $(0,0)$.

(8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq 0, \\ 0, & (x,y) = 0. \end{cases}$$

Show that $\frac{\partial f}{\partial y}(x,0) = x$ for all x and $\frac{\partial f}{\partial x}(0,y) = -y$ for all y. Then show that $\partial_x \partial_y f(0,0) \neq \partial_y \partial_x f(0,0)$.

(9) Prove that S^1 (with either topology in Problem (2)) is a topological manifold.

(10) Give an example of a Hausdorff, second countable topological space which is not a topological manifold. *You must justify your example.*