Symplectic fillings and Lefschetz fibrations
Austin Christian
February 22, 2019

1. Symplectic fillings

- Recall that, under certain conditions, a symplectic manifold with boundary endows its boundary with a contact structure. In such a case, we say that the symplectic manifold *fills* the contact manifold, and today we want to discuss a classification result for symplectic fillings.

- Fix a co-oriented contact manifold (M, ξ) and suppose (W, ω) is a symplectic manifold with $\partial W = M$ as oriented manifolds. We say that (W, ω) is
  - a *weak symplectic filling* of (M, ξ) if $\omega|_\xi > 0$;
  - a *strong symplectic filling* of (M, ξ) if there is a 1-form λ on W such that $\omega = d\lambda$ on some neighborhood of ∂W and $\xi = \ker(\lambda|_{\partial W})$;
  - an *exact filling* of (M, ξ) if there is a 1-form λ on W such that $\omega = d\lambda$ on all of W and $\xi = \ker(\lambda|_{\partial W})$.

  We say that (M, ξ) is *weakly symplectically fillable*, *strongly symplectically fillable*, or *exactly fillable* if it admits a weak symplectic, strong symplectic, or exact fillling, respectively.

- We showed earlier in the quarter that fillable contact manifolds must be tight ([Eli90],[Gro85]). Certainly every exact filling is a strong filling and every strong filling is a weak filling, so we have inclusions of contact manifolds
$$\{\text{exactly fillable}\} \subseteq \{\text{strongly fillable}\} \subseteq \{\text{weakly fillable}\} \subseteq \{\text{tight}\}.$$
  Less obvious is the fact that each of these inclusions is strict. The first example of a weakly-but-not-strongly fillable contact manifold was produced by Eliashberg in [Eli96], and Ghiggini constructed an infinite family of strongly-but-not-exactly fillable contact manifolds in [Ghi05]. In [EH02], Etnyre and Honda gave examples of tight contact manifolds admitting no symplectic fillings.

- The best classification results would identify the symplectic fillings of a contact manifold up to symplectomorphism, but this is a lot to ask. We lower the bar a bit with the following notions of equivalence:
  1. We will say that symplectic fillings (W, ω) and (W', ω') are *symplectically deformation equivalent* if there exists a diffeomorphism $\varphi\colon W \to W'$ and a smooth family ω_t, $t \in [0, 1]$, of symplectic forms on W so that
     (a) $\omega_0 = \omega$ and $\omega_1 = \varphi^*\omega'$;
     (b) for each $t \in [0, 1]$, (W, ω_t) is a symplectic filling.
  2. Given a weak filling (W, ω) of a contact manifold, we can obtain another filling by *blowing up* (W, ω) at a point. As with other notions of blowup, symplectic blowup consists of replacing a point with a projective space. We'll avoid the details of this construction today and just point out that a blowup of W is diffeomorphic to $W \# \overline{\mathbb{CP}}^2$. Today we'll classify some symplectic fillings *up to blowup*.

  We say that a symplectic 4-manifold (W, ω) is *minimal* if it contains no smoothly embedded 2-spheres with homological self-intersection number -1. More concisely, (W, ω) cannot be obtained by blowing up any other symplectic manifold.

- The first big classification result is then:

  Theorem 1.1 (Gromov, [Gro85]; Eliashberg [Eli90])**.** *Up to symplectic deformation equivalence and blowup, (B^4, ω_{std}) is the unique weak symplectic filling of (S^3, ξ_{std}).*

- Various other classification results exist in dimension three, many of them obtained by means similar to those used in the proof of Theorem 1.1. For instance, the fillings of $S^1 \times S^2$ with its standard structure were classified by Eliashberg ([Eli90]) and McDuff ([McD90]), and more general lens spaces had their fillings classified by McDuff ([McD90]) and Lisca ([Lis08]).

- Our goal today is to discuss a more recent result which repackages the main ideas in the above classifications into a single, ready-to-use tool. In particular, we'll derive Theorem 1.1 as a corollary of this newer technology. The following result is due to Wendl:

  Theorem 3.1 (Wendl, [Wen10])**.** Suppose (W, ω) is a symplectic filling of a contact 3-manifold (M, ξ), and that ξ is supported by a planar open book $\pi\colon M\backslash B \to S^1$. Then (W, ω) admits a symplectic Lefschetz fibration over $\mathbb{D}$ such that the induced open book at the boundary is isotopic to $\pi\colon M\backslash B \to S^1$. Moreover, the Lefschetz fibration is allowable if and only if (W, ω) is minimal.

- This theorem allows us to transform the problem of classifying minimal symplectic fillings into the problem of classifying allowable Lefschetz fibrations inducing a particular monodromy at the boundary. The latter is in some cases a more tractable problem.

- Before unpacking this newer classification theorem, we should contrast the situation in dimension three with that in higher dimensions, where far less is known. Here the classic result is the Eliashberg-Floer-McDuff theorem:

  Theorem 1.2 (Eliashberg-Floer-McDuff, [McD91])**.** *If (W, ω) is a strong symplectic filling of (S^{2n-1}, ξ_{std}) and $[\omega]$ vanishes on $\pi_2(W) \subset H_2(W; \mathbb{R})$, then W is diffeomorphic to B^2n.*

- The original proof of this theorem is very similar in spirit to the classification of fillings of (S^3, ξ_{std}), but notice that the classification in higher dimensions is up to *diffeomorphism*. In particular, when our fillings have dimension greater than four, we lose the benefit of positivity of intersections, and geometric classifications are harder to come by. A discussion of these two results and their proofs can be found in [Wen18, Chapter 9].

- We now turn to some background material on open book decompositions and Lefschetz fibrations.

2. Open book decompositions, contact structures, and Lefschetz fibrations

Let M be a closed, oriented 3-manifold.

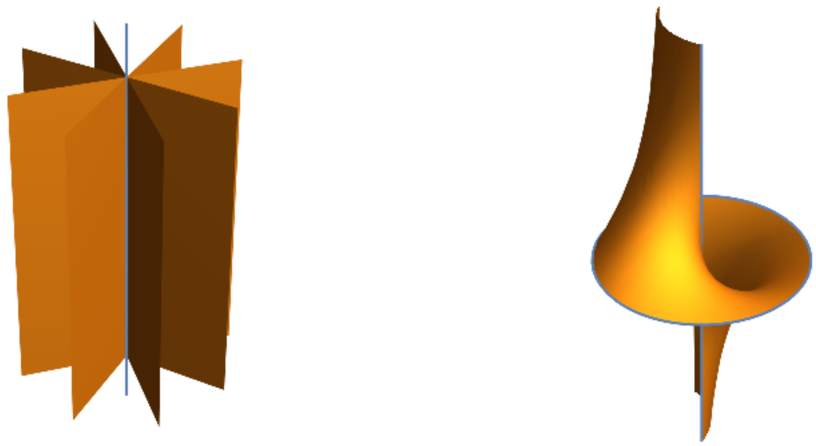
2.1. Open book decompositions.

- An *open book decomposition* of M is a fibration $\pi\colon M\backslash B \to S^1$, where
  (1) $B \subset M$ is an oriented link, which we call the *binding* of the open book decomposition;
  (2) for each $t \in S^1$, the fiber $\pi^{-1}(t)$ is the interior of a compact surface $\Sigma_t \subset M$ with $\partial\Sigma_t = B$. Each fiber is a *page* of the open book decomposition.

- **Example.** Consider $S^3 = \mathbb{R}^3 \cup \{\infty\}$, with cylindrical coordinates (r, θ, z). We may let $B = \{r = 0\} \cup \{\infty\}$, and define $\pi\colon S^3\backslash B \to S^1$ by $(r, \theta, z) \mapsto \theta$. This example of an open book decomposition is indicative of the name: each fiber of π indeed looks like a page, with the z-axis as our binding. See Figure 1a.

  Another open book decomposition of S^3 has binding given by the Hopf link. To see this decomposition, write
  $$S^3 = \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2} | r_1^2 + r_2^2\} \subset \mathbb{C}^2.$$
  We then let $B = \{r_1 = 0\} \cup \{r_2 = 0\}$ be the Hopf link and define
  $$\pi(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1 + \theta_2.$$
  Each page $\pi^{-1}(t)$ is then an annulus. Indeed, we have
  $$\pi^{-1}(t) = \{(r_1 e^{i\theta_1}, r_2 e^{i(t-\theta_1)}) | r_1^2 + r_2^2 = 1\} \simeq \{(r_1, \theta_1) | r_1 \in (0,1), \theta_1 \in S^1\}.$$
  Figure 1b attempts to depict a page of this open book decomposition in $\mathbb{R}^3$. Notice that all of S^3 is swept out as the page rotates around the binding.

- In practice, we're often more interested in the *abstract open book* associated to an open book decomposition. An abstract open book consists of a *page* Σ and a *monodromy* φ. Here Σ is an oriented



(A) Disk page.

(B) Annulus page.

FIGURE 1. Open book decompositions of S^3.

compact surface with boundary and $\varphi\colon \Sigma \to \Sigma$ is a diffeomorphism which restricts to the identity in a neighborhood of the boundary $\partial\Sigma$. We may construct a 3-manifold from this data according to

$$M_{(\Sigma,\varphi)} = \Sigma \times [0,1]/\sim,$$

where
(1) $(x,t) \sim (x,0)$ for all $x \in \partial\Sigma$ and $t \in [0,1]$;
(2) $(\varphi(x),1) \sim (x,0)$ for all $x \in \Sigma$.
A binding B_ϕ is given for this manifold by the image of $\partial\Sigma$, with the fibration $\pi\colon M_\phi \backslash B_\phi \to S^1 = \mathbb{R}/\mathbb{Z}$ given by projection onto the $[0,1]$-coordinate.

- Conversely, an open book decomposition can be used to determine an abstract open book. Let Σ be the closure of the fiber $\pi^{-1}(0)$. The monodromy $\varphi\colon \Sigma \to \Sigma$ is given as follows: given a point $x \in \pi^{-1}(0)$, let $\tilde{\gamma}_x\colon [0,1] \to M\backslash B$ be a path lifting the identity map $\gamma\colon S^1 \to S^1$. That is, $\pi \circ \tilde{\gamma}_x = \gamma$. Then we define $\varphi(x) = \tilde{\gamma}_x(1) \in \pi^{-1}(0)$.

- Today we'll use abstract open books and open book decompositions more or less interchangeably (calling both *open books*), but there are slight differences. See, for instance, [Etn06, Remark 2.6].

- **Example.** The first open book decomposition for S^3 given above has page $\Sigma = D^2$ and monodromy the identity map. We observed above that the second open book decomposition has annulus page. One can check that the monodromy for this open book is a positive Dehn twist about a boundary-parallel arc.

- Today we're particularly interested in *planar* open book decompositions. These are open book decompositions whose page Σ has interior diffeomorphic to a sphere with finitely many punctures. Thankfully, we have an existence result, even for this restricted class of decompositions:

  Theorem 2.1 (Alexander[1]). *Every closed, oriented 3-manifold admits an open book decomposition with planar pages.*

- What do open book decompositions have to do with contact structures? An open book can *support* a contact structure in the following sense: We say that a contact structure ξ on M is *supported* by an open book decomposition $\pi\colon M\backslash B \to S^1$ if there is a contact 1-form α on M such that
  (1) ξ is isotopic to $\ker\alpha$;

[1] I haven't actually verified the history here. Alexander showed in [Ale20] that all 3-manifolds admit open book decompositions, and the proof of this fact in [Rol03, Chapter 10] produces an OBD with planar pages, but I don't know who first showed that the pages could always be made planar.

(2) $d\alpha|_{\Sigma_t} > 0$ for each page Σ_t of the open book;
(3) $\alpha|_B > 0$ on the binding B.

How should we think of this relationship? Perhaps the most helpful intuition comes from Reeb vector fields. The latter two conditions in the above definition are equivalent to the statement that the Reeb vector field R_α is tangent to B and transverse to the pages Σ_t (with appropriate orientation). Recall that R_α is also transverse to $\ker \alpha$.

- **Example.** Both of the open book decompositions defined above for S^3 support the standard contact structure. Notice that in the second open book we could have instead defined
$$\pi(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1 - \theta_2.$$
Again we will have annulus pages, but the orientations will be reversed, and the resulting open book decomposition does not support ξ_{std}.

- Here are some important results we won't have time to prove today:

Theorem 2.2 (Thurston-Winkelnkemper, [TW75]). *Every open book decomposition of a closed, oriented 3-manifold supports a contact structure.*

Theorem 2.3 (Giroux, [Gir91]). *Every contact structure on a closed, oriented 3-manifold is supported by an open book.*

- In fact, the work of Giroux reaches even further: he showed that there is a bijective correspondence between the open book decompositions of a 3-manifold (up to *stabilization*) and the contact structures on that manifold. In particular, if contact structures ξ and ξ' are supported by a common open book decomposition, then ξ and ξ' are isotopic.

- There's an important caveat to Giroux's result: while every contact structure is supported by *some* open book decomposition, not all contact structures are supported by *planar* open books (see [Etn04]). We call (M, ξ) a *planar contact manifold* if ξ is supported by a planar open book[2].

- **Note.** Thurston-Winkelnkemper's result gives another (very short) proof that every closed, oriented 3-manifold admits a contact structure.

2.2. **Lefschetz fibrations.**

- Our interest today is in characterizing the symplectic 4-manifolds (W, ω) which have our contact manifold (M, ξ) for their boundary. If a 4-manifold with boundary W admits the structure of a *bordered Lefschetz fibration* (as defined in [Wen17]), this structure will naturally induce an open book decomposition on ∂W.

- **Definition.** Let W be a compact, oriented 4-manifold with boundary and corners, and write
$$\partial W = \partial_h W \cup \partial_v W,$$
decomposing the boundary into its *horizontal* and *vertical* parts. Each of these parts is a smooth manifold with boundary. A *bordered Lefschetz fibration* of W over the unit disk $\mathbb{D} \subset \mathbb{C}$ is a smooth map
$$\pi \colon W \to \mathbb{D}$$
such that
(1) π has finitely many critical points $W^{\text{crit}} := \text{Crit}(\pi) \subset \text{int}(W)$ and critical values $\mathbb{D}^{\text{crit}} := \pi(W^{\text{crit}}) \subset \text{int}(\mathbb{D})$;
(2) near each critical point $p \in W^{\text{crit}}$ there are complex coordinates (z_1, z_2) compatible with the orientation of W such that
$$\pi(z_1, z_2) = z_1^2 + z_2^2$$
on a neighborhood of p;
(3) each fiber $\pi^{-1}(z)$ has non-empty boundary;

[2]Etnyre and Ozbagci [EO08] have defined the *support genus* of a contact manifold to be the minimal possible genus for a page of an open book supporting ξ. Planar contact manifolds are then those with support genus zero.

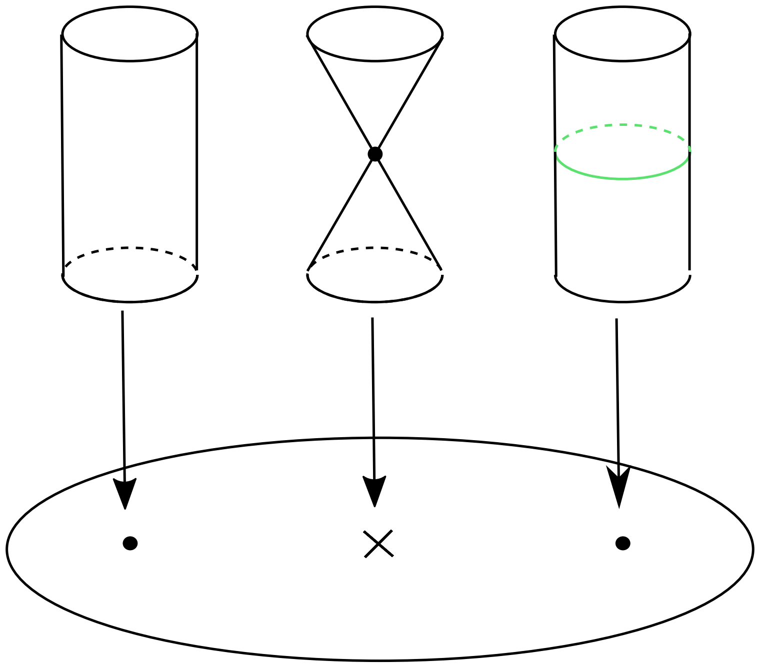


FIGURE 2. A bordered Lefschetz fibration over $\mathbb{D}$ with one singular fiber. The boundary ∂W inherits an open book decomposition whose page is an annulus.

(4) $\partial_v W = \pi^{-1}(\partial\mathbb{D})$, and $\pi|_{\partial_v W} : \partial_v W \to \partial\mathbb{D}$ is a smooth fibration;
(5) $\partial_h W = \bigcup_{z\in\mathbb{D}} \partial(\pi^{-1}(z))$, and $\pi|_{\partial_h W} : \partial_h W \to \mathbb{D}$ is a smooth fibration.

Notice that π may be restricted to a smooth fiber bundle over $\mathbb{D}\backslash\mathbb{D}^{\text{crit}}$. The fibers of this bundle are called *regular fibers*, while for each $z \in \mathbb{D}^{\text{crit}}$ we have a *singular fiber* $\pi^{-1}(z) \subset W$.

- Consider a loop $\gamma\colon S^1 \to \mathbb{D}$ which avoids the critical values $\mathbb{D}^{\text{crit}}$. Then $F := \pi^{-1}(\gamma(0))$ is a regular fiber, and we may define a monodromy map $\varphi_\gamma\colon F \to F$ which is trivial near ∂F. This is obtained as before: for each point $p \in F$, let $\tilde{\gamma}_p\colon [0,1] \to W$ be a path which lifts γ and has $\tilde{\gamma}_p(0) = p$. Then $\varphi_\gamma(p) = \tilde{\gamma}_p(1) \in F$. In particular, since all critical points of π are interior, we have a monodromy map along $\partial\mathbb{D} \subset \mathbb{D}$.

- Some observations are now in order. First, each fiber $\pi^{-1}(z)$ is a smooth surface with boundary, so $\partial_h W$ is a disjoint union of k S^1-bundles over $\mathbb{D}$, for some $k \geqslant 1$. Since bundles over $\mathbb{D}$ are trivial, we may write
$$\partial_h W = \bigsqcup_{i=1}^{k} (S^1 \times \mathbb{D})$$
and let $B = \partial(\pi^{-1}(0)) \subset \partial_h W$. Then B is a k-component link, and $\partial_h W$ is a tubular neighborhood of B in ∂W. After rounding the corners of ∂W we obtain a smooth manifold M, and a fibration
$$\pi\colon M\backslash B \to S^1$$
is determined by $\pi|_{\partial_v W}\colon \partial_v W \to \partial\mathbb{D}$. That is, the Lefschetz fibration on W determines an open book decomposition of M. The page of this open book is given by a regular fiber of the Lefschetz fibration, and its monodromy is the monodromy along $\partial\mathbb{D}$.

- **Example.** Consider the unit 4-ball
$$B^4 = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 \leqslant 1\} \subset \mathbb{C}^2$$
and the map $\pi\colon B^4 \to \mathbb{D}$, given by
$$\pi(z_1, z_2) = z_1 z_2.$$

This map has a single critical point at $(0,0) \in \mathbb{C}^2$, so we investigate the fiber $\pi^{-1}(0)$. We see that

$$\pi^{-1}(0) = \{(0, z_2) | |z_2|^2 \leqslant 1\} \cup \{(z_1, 0) | |z_1|^2 \leqslant 1\},$$

so the singular fiber is a cone in B^4 with vertex at $(0,0)$, and the boundary of this cone is a Hopf link in $\partial B^4 = S^3$. Away from this singular fiber we see that

$$\pi^{-1}(re^{i\theta}) = \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) | r = r_1 r_2, \theta = \theta_1 + \theta_2\},$$

for any $0 < r < 1$ and any θ. So the regular fibers of π are annuli whose boundaries are also Hopf links in S^3, and in particular the regular fibers over $\partial\mathbb{D}$ are the pages of the open book we constructed above for S^3.

- One reason it is helpful to relate open book decompositions to Lefschetz fibrations is that monodromies of Lefschetz fibrations are relatively well understood. Specifically, fix a point $z_0 \in \partial\mathbb{D}$, and let F be the fiber at this point. We can describe the monodromy $\varphi \colon F \to F$ as a composition of Dehn twists along particular circles in F. First choose a smooth path

$$\gamma_z \colon [0,1] \to \mathbb{D}$$

for each critical value $z \in \mathbb{D}^{\text{crit}}$. These paths should be chosen so that $\gamma_z(0) = z_0$, $\gamma_z(1) = z$, and so that distinct paths intersect only at z_0. It can be shown that for each critical point $p \in W^{\text{crit}}$, there is a smoothly embedded circle $C_p \subset F$, unique up to isotopy, which collapses to p under parallel transport along $\gamma_{\pi(p)}$. This circle is called the *vanishing cycle* of p, and we will call our bordered Lefschetz fibration *allowable* if none of the vanishing cycles is homologically trivial. The collection of vanishing cycles generate the monodromy:

Theorem 2.4. *If $\pi \colon W \to \mathbb{D}$ is a bordered Lefschetz fibration, then the monodromy $\varphi \colon F \to F$ of the induced open book at the boundary is a composition of positive Dehn twists along the vanishing cycles $C_p \subset F$.*

- For example, in the Lefschetz fibration $B^4 \to \mathbb{D}$ considered above, the vanishing cycle C_0 is a boundary-parallel circle in the annulus fiber. This can be seen in Figure 2, where a boundary-parallel curve in a regular fiber is pinched down to the critical point in the singular fiber. The monodromy of our open book is a single positive Dehn twist along this circle.

- As with open books and contact structures, we have a notion of compatibility between bordered Lefschetz fibrations and symplectic structures. A Lefschetz fibration $\pi \colon W \to \mathbb{D}$ *supports* a symplectic structure ω on W if
  (1) each fiber of $\pi|_{W \backslash W^{\text{crit}}} \colon W \backslash W^{\text{crit}} \mathbb{D}$ is a symplectic submanifold[3];
  (2) there is an almost complex structure J on W which preserves the fiber directions and is tamed by ω near W^{crit};
  (3) there is a 1-form λ so that $d\lambda = \omega$ near ∂W, $\lambda|_{\partial_h W}$ and $\lambda|_{\partial_v W}$ are both contact forms, and the Reeb field on $\partial_h W$ is positively tangent to the fibers of π.

- Say $\pi \colon W \to \mathbb{D}$ is a bordered Lefschetz fibration supporting the symplectic structure ω on W. Then we have two ways of obtaining a contact structure on (a smoothed version of) ∂W: either via the induced open book decomposition, followed by the Thurston-Winkelnkemper construction, or by letting $\xi = \ker \lambda$, where λ is a 1-form witnessing the fact that π supports ω. The following result says that these are the same.

Theorem 2.5 ([Wen17, Theorem 5.5], [LVHMW18, Theorem 1.24])**.** *Let $\pi \colon W \to \mathbb{D}$ be a bordered Lefschetz fibration supporting the symplectic form ω on W. Then ∂W can be smoothed so that (W, ω) becomes a symplectic filling of the contact structure supported by the induced open book at the boundary, and this smoothing is canonical up to symplectic deformation.*

- This allows us to give the last definition we'll need before stating Wendl's result. Let (W, ω) be a symplectic filling of a contact 3-manifold (M, ξ). A *symplectic Lefschetz fibration over $\mathbb{D}$ for (W, ω)*

[3] Notice that this includes the singular fibers of π, with the critical points removed.

consists of a bordered Lefschetz fibration

$$\pi \colon \overline{W} \to \mathbb{D}$$

and a symplectic form $\overline{\omega}$ such that

(1) the form $\overline{\omega}$ is supported by π;

(2) the filling $(\overline{W}, \overline{\omega})$ obtained by smoothing the corners of $\partial\overline{W}$ is symplectomorphic to (W, ω).

Wendl showed in [Wen10] that the isotopy class of $(\overline{W}, \overline{\omega})$ is determined by the deformation class of the symplectic structure ω. Namely, if (W, ω) and (W', ω') are deformation equivalent and admit symplectic Lefschetz fibrations over $\mathbb{D}$, then these Lefschetz fibrations are isotopic.

3. Wendl's result

- At last we repeat

Theorem 3.1 (Wendl, [Wen10])**.** *Suppose (W, ω) is a symplectic filling of a contact 3-manifold (M, ξ), and that ξ is supported by a planar open book $\pi \colon M\backslash B \to S^1$. Then (W, ω) admits a symplectic Lefschetz fibration over $\mathbb{D}$ such that the induced open book at the boundary is isotopic to $\pi \colon M\backslash B \to S^1$. Moreover, the Lefschetz fibration is allowable if and only if (W, ω) is minimal.*

- Let's see how Theorem 3.1 implies Theorem 1.1. Since Theorem 1.1 classifies fillings up to blowup, we may take (W, ω) to be a minimal filling of (S^3, ξ_{std}). We let $\pi \colon M\backslash B \to S^1$ be the open book decomposition of S^3 induced by the Lefschetz fibration on B^4 as above. Namely, this open book as annular pages, Hopf link binding, and the monodromy is given by a single positive Dehn twist along a boundary-parallel circle. According to Theorem 3.1, (W, ω) admits an allowable symplectic Lefschetz fibration over $\mathbb{D}$ inducing this open book on $\partial W = S^3$. In particular, this symplectic Lefschetz fibration has an annulus for its regular fiber, and exactly one singular fiber (corresponding to the single Dehn twist). This is the same symplectic Lefschetz fibration carried by (B^4, ω_{std}), and thus Theorem 2.5 tells us that (W, ω) is deformation equivalent to (B^4, ξ_{std}).

- This same proof will apply to any contact manifold supported by an open book with annular pages and monodromy equal to some non-negative power of a positive Dehn twist. The number of singular fibers will match the number of positive Dehn twists. Namely, Theorem 3.1 implies that $(S^1 \times S^2, \xi_{std})$ and $(L(k, k-1), \xi_{std})$, $k \geqslant 2$, have unique fillings up to deformation equivalence and blowup.

- A proof sketch for Theorem 3.1 can be found in [Wen17, Section 5.3]

References

[Ale20] James W Alexander. Note on Riemann spaces. *Bulletin of the American Mathematical Society*, 26(8):370–372, 1920.

[EH02] John B Etnyre and Ko Honda. Tight contact structures with no symplectic fillings. *Invent. Math.*, 148(3):609–626, 2002.

[Eli90] Yakov Eliashberg. Filling by holomorphic discs and its applications. *London Math. Soc. Lecture Note Ser., Geometry of Low-Dimensional Manifolds*, 2(151):45–67, 1990.

[Eli96] Y. Eliashberg. Unique holomorphically fillable contact structure on the 3-torus. *Int. Math. Res. Not. IMRN*, 1996(2):77–82, 1996.

[EO08] John Etnyre and Burak Ozbagci. Invariants of contact structures from open books. *Trans. Amer. Math. Soc.*, 360(6):3133–3151, 2008.

[Etn04] John B Etnyre. Planar open book decompositions and contact structures. *Int. Math. Res. Not. IMRN*, 2004(79):4255–4267, 2004.

[Etn06] John B Etnyre. Lectures on open book decompositions and contact structures. In *Floer Homology, Gauge Theory, and Low-dimensional Topology: Proceedings of the Clay Mathematics Institute 2004 Summer School, Alfréd Rényi Institute of Mathematics, Budapest, Hungary, June 5-26, 2004*, volume 5, page 103. American Mathematical Soc., 2006.

[Ghi05] Paolo Ghiggini. Strongly fillable contact 3–manifolds without Stein fillings. *Geom. Topol.*, 9(3):1677–1687, 2005.

[Gir91] Emmanuel Giroux. Convexité en topologie de contact. *Comment. Math. Helv.*, 66(1):637–677, 1991.

[Gro85] Mikhael Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Lis08] Paolo Lisca. On symplectic fillings of lens spaces. *Trans. Amer. Math. Soc.*, 360(2):765–799, 2008.

[LVHMW18] Samuel Lisi, Jeremy Van Horn-Morris, and Chris Wendl. On symplectic fillings of spinal open book decompositions i: Geometric constructions. *arXiv preprint arXiv:1810.12017*, 2018.

[McD90] Dusa McDuff. The structure of rational and ruled symplectic 4-manifolds. *J. Amer. Math. Soc.*, 3(3):679–712, 1990.

[McD91] Dusa McDuff. Symplectic manifolds with contact type boundaries. *Invent. Math.*, 103(1):651–671, 1991.

[Rol03] Dale Rolfsen. *Knots and Links*, volume 346. American Mathematical Soc., 2003.

[TW75] William P Thurston and Horst E Winkelnkemper. On the existence of contact forms. *Proc. Amer. Math. Soc.*, pages 345–347, 1975.

[Wen10] Chris Wendl. Strongly fillable contact manifolds and J-holomorphic foliations. *Duke Math. J.*, 151(3):337–384, 2010.

[Wen17] Chris Wendl. Contact 3-manifolds, holomorphic curves and intersection theory. *arXiv preprint arXiv:1706.05540*, 2017.

[Wen18] Chris Wendl. *Holomorphic Curves in Low Dimensions*. Springer, 2018.