A generalization of the Bernstein-Walsh-Siciak theorem on uniform approximation of functions by polynomials on compact sets.

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Contents:

1. Background

We let $\mathcal{P}_m(\mathbb{C}^n)$ denote the space of polynomials of degree $\leq m$ in n complex variables and let

$$d_{K,m}(f) = \inf\{\|f - p\|_K \,;\, p \in \mathcal{P}_m(\mathbb{C}^n)\}$$

denote the smallest error in an approximation of f by polynomials of degree $\leq m$, i.e., the distance from f to $\mathcal{P}_m(\mathbb{C}^n)$ in the supremum norm $\|\cdot\|_K$ on $\mathcal{C}(K)$.

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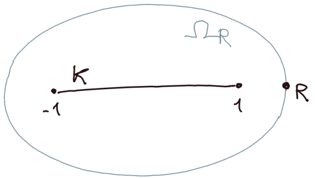
Theorem (Bernstein 1912) Let $f : [-1,1] \to \mathbb{C}$ and $R > 1$. Then

$$\overline{\lim_{m\to\infty}} \left(d_{[-1,1],m}(f)\right)^{1/m} \leq \frac{1}{R}$$

if and only if f has a holomorphic extension to the domain bounded by the ellipse with focii -1 and 1 and semi-major axis R.



Theorem (Bernstein-Walsh ∼1925) Let $K \subset \mathbb{C}$ be compact, f a holomorphic function in some neighborhood of K, and assume that $\mathbb{C} \setminus K$ is connected and a regular domain for the Dirichlet problem for harmonic functions with logarithmic growth at ∞,

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Then for a given real number $R \geq 1$ the inequality

$$\overline{\lim_{m\to\infty}} \left(d_{K,m}(f)\right)^{1/m} \leq \frac{1}{R}$$

holds if and only if f has a holomorphic extension to

$$\Omega_R = \{z \in \mathbb{C}\,;\, g_K(z,\infty) < \log R\},$$

where $g_K(\cdot,\infty)$ is the Green function of K with logaritmic pole at infinity, which is the unique function on $\mathbb{C}$, which is 0 on K, harmonic on $\mathbb{C} \setminus K$, and has logarithmic growth at ∞.

For the closed unit disc $\overline{\mathbb{D}}$ we have

$$g_{\overline{\mathbb{D}}}(z) = \log^+(|z|), \qquad z \in \mathbb{C},$$

and R is the radius of convergence of the power series of the given function at the origin.

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We have an explicit formula for the Green function for $K = [-1, 1]$,

$$g_K(z, \infty) = \log |z + (z^2 - 1)^{1/2}|, \qquad z \in \mathbb{C} \setminus [-1, 1],$$

where the branch of the square root is chosen such that for $t > 1$ the value $t + (t^2 - 1)^{1/2} > 0$.

A review of some results from approximation theory

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- **Mergelyan (1951):** The Runge theorem holds for $\mathbb{C} \setminus K$ connected even if it is only assume that the given function is continuous on K and holomorphic in the interior of K.

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$$\widehat{K} = \{z \in \mathbb{C}^n \,;\, |p(z)| \leq \sup_K |p|, \forall p \in \mathcal{P}(\mathbb{C}^n)\},$$

  and that for $n = 1$ the compact set K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected.
  The Runge-Oka-Weil theorem states that every holomorphic function on some neighborhood of a polynomially convex set in $\mathbb{C}^n$ can be approximated uniformly on K by polynomials.

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  It is not clear what would be the correct statement, because we have to take into account the complex structure in ∂K.

  Only fragmentary results are known for special classes of compacts K, e.g., products of compact sets, convex domains, closure of a strictly pseudoconvex domain.

Siciak's extremal functions

$$\Phi_{K,m} = \sup\{|p|^{1/m}\,;\ p \in \mathcal{P}_m(\mathbb{C}^n), \|p\|_K \leq 1\} \qquad \Phi_K = \overline{\lim_{m\to\infty}}\, \Phi_{K,m},$$

The Fekete lemma implies

$$\Phi_K = \lim_{m\to\infty} \Phi_{K,m} = \sup_{m\in\mathbb{N}} \Phi_{K,m},$$

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Theorem (Bernstein-Walsh-Siciak 1961)

Let K be polynomially convex, $f \in \mathcal{O}(K)$, $R \geq 1$, and assume that Φ_K is continuous. Then

$$\varlimsup_{m\to\infty} \big(d_{K,m}(f)\big)^{1/m} = \frac{1}{R}.$$

if and only if f has a holomorphic extension to the open set

$$\Omega_R = \{z \in \mathbb{C}^n\,;\, \log \Phi_K(z) < \log R\}.$$

Siciak-Zakharyuta theorem

We have

$$\log|p(z)|^{1/m} \leq c_p + \log^+ \|z\|, \qquad z \in \mathbb{C}^n, \quad p \in \mathcal{P}_m(\mathbb{C}^n).$$

We let $\mathcal{L}(\mathbb{C}^n)$ denote the class of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying

$$u(z) \leq c_u + \log^+ \|z\|_\infty, \qquad z \in \mathbb{C}^n,$$

and set

$$V_K = \sup\{u\,;\ u \in \mathcal{L}(\mathbb{C}^n), u|_K \leq 0\}$$

then we have $\log \Phi_K \leq V_K$

Theorem (Siciak-Zakharyuta) For every compact K we have

$$\log \Phi_K = V_K$$

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S-polynomial spaces:

For every $m \in \mathbb{N}$ we associate to S the space $\mathcal{P}_m^S(\mathbb{C}^n)$ of all polynomials in n complex variables of the form

$$p(z) = \sum_{\alpha \in (mS) \cap \mathbb{N}^n} a_\alpha z^\alpha, \qquad z \in \mathbb{C}^n,$$

with the standard multi-index notation and let $\mathcal{P}^S(\mathbb{C}^n) = \cup_{m=0}^{\infty} \mathcal{P}_m^S(\mathbb{C}^n)$.

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$\mathcal{P}^S(\mathbb{C}^n)$ is a graded ring:

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The standard simplex:

Observe that for $\Sigma = \{x \in \mathbb{R}_+^n ; \sum_{j=1}^n x_j \leq 1\}$ we have

$$\mathcal{P}_m^\Sigma(\mathbb{C}^n) = \mathcal{P}_m(\mathbb{C}^n).$$

The supporting function of S

For every compact subset S of $\mathbb{R}^n$ we define the *supporting function*

$$\varphi_S(\xi) = \sup_{x \in S} \langle x, \xi \rangle = \max_{x \in \operatorname{ext} S} \langle x, \xi \rangle, \qquad \xi \in \mathbb{R}^n.$$

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There is a bijective correspondence between compact convex subsets of $\mathbb{R}^n$ and *1-positively homogeneous convex functions* φ on $\mathbb{R}^n$

$$\varphi = \varphi_S \qquad \Leftrightarrow \qquad S = \{x \in \mathbb{R}^n \,;\, \langle x, \xi \rangle \leq \varphi(\xi),\ \forall \xi \in \mathbb{R}^n\}.$$

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For every $S \subset \mathbb{R}^n_+$ and cone $\Gamma \subseteq \mathbb{R}^n$ we define the Γ-*hull* of S by

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The standard simplex is $\Sigma = \operatorname{ch}\{0, e_1, \dots, e_n\}$, so

$$\varphi_\Sigma(\xi) = \max\{\xi_1^+, \dots, \xi_n^+\} = \|\xi^+\|_\infty,$$

where $\xi_j^+ = \max\{\xi_j, 0\}$ and $\xi^+ = (\xi_1^+, \dots, \xi_n^+)$.

The logarithmic supporting function of S

From now on we take $S \subseteq \mathbb{R}^n_+$, $0 \in S$, and $S \neq \{0\}$.

We let $\operatorname{Log}\colon \mathbb{C}^{*n} \to \mathbb{C}$ by $\operatorname{Log} z = (\log|z_1|, \dots, \log|z_n|)$, define H_S by

$$H_S(z) = (\varphi \circ \operatorname{Log})(z) = \varphi_S(\log|z_1|, \dots, \log|z_n|), \qquad z \in \mathbb{C}^{*n},$$

and extend the definition to the coordinate hyperplanes by

$$H_S(z) = \overline{\lim_{\mathbb{C}^{*n} \ni w \to z}} H_S(w), \qquad z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}.$$

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For the standard simplex we have

$$H_\Sigma(z) = \log^+ \|z\|_\infty, \qquad z \in \mathbb{C}^n.$$

Proposition: $H_S \in \mathcal{PSH}(\mathbb{C}^n) \cap C(\mathbb{C}^n)$ for every S.

The class $\mathcal{L}^S(\mathbb{C}^n)$

consists of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying a growth estimate

$$u(z) \leq c_u + H_S(z), \qquad z \in \mathbb{C}^n.$$

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Proof:
For $\alpha \in \mathbb{N}^n \setminus mS$ we find $\langle \alpha, \xi \rangle > m\varphi_S(\xi)$

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{C_t} \frac{p(\zeta)}{\zeta^\alpha} \, \frac{d\zeta_1 \cdots d\zeta_n}{\zeta_1 \cdots \zeta_n}.$$

where C_t is the polycircle with center 0 and polyradius $(e^{t\xi_1}, \dots, e^{t\xi_n})$.

For $\zeta = (e^{t\xi_1 + i\theta_1}, \dots, e^{t\xi_n + i\theta_n}) \in C_t$ we have

$$|p(\zeta)|/|\zeta^\alpha| \leq Ce^{-t(\langle \alpha, \xi \rangle - m\varphi_S(\xi))} \to 0, \qquad t \to +\infty.$$

A Liouville type theorem

The Liouville theorem tells us that an entire function $p \in \mathcal{O}(\mathbb{C}^n)$, which for some $m \in \mathbb{N}$ and $a \in [0, 1[$ satisfies a growth estimate

$$|p(z)| \leq C(1 + |z|)^{a+m}, \qquad z \in \mathbb{C}^n$$

is a polynomial of degree $\leq m$, i.e., $p \in \mathcal{P}_m(\mathbb{C}^n)$.

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The following is a Liouville type theorem for the classes $\mathcal{P}^S_m(\mathbb{C}^n)$:

Proposition: Let $p \in \mathcal{O}(\mathbb{C}^n)$ and assume that for some $C > 0$ and $a \geq 0$ less than the euclidean distance between mS and $\mathbb{N}^n \setminus mS$ we have

$$|p(z)| \leq C(1 + |z|)^a e^{mH_S(z)}, \qquad z \in \mathbb{C}^n.$$

Then $p \in \mathcal{P}^S_m(\mathbb{C}^n)$.

Proof: Choose ξ such that $s_\alpha \in mS$ satisfies

$$|\alpha - s_\alpha| = d(\alpha, mS) = \langle \alpha - s_\alpha, \xi \rangle = \langle \alpha, \xi \rangle - m\varphi_S(\xi) > a.$$

Then

$$\begin{aligned} |p(\zeta)|/|\zeta^\alpha| &\leq C\big(1 + (e^{2t\xi_1} + \cdots + e^{2t\xi_n})^{1/2}\big)^a e^{-t(\langle \alpha, \xi \rangle - m\varphi_S(\xi))} \\ &\leq C(1+\sqrt{n})^a e^{-t(\langle \alpha, \xi \rangle - m\varphi_S(\xi) - a)} \to 0, \qquad t \to +\infty, \end{aligned}$$

The convexity of φ_S implies

$$H_S(z_1 w_1, \dots, z_n w_n) \leq H_S(z) + H_S(w),$$

and as a special case when all w_j are equal we get

$$H_S(\lambda z_1, \dots, \lambda z_n) \leq H_S(z) + H_S(\lambda 1) = H_S(z) + \varphi_S(1) \log^+ |\lambda|.$$

and

$$H_S(z) \leq \varphi_S(1) \log^+ \|z\|_\infty, \qquad z \in \mathbb{C}^n.$$

Hence $\mathbb{B}_\infty = \overline{\mathbb{D}}^n \subseteq \mathcal{N}_S = \{z \in \mathbb{C}^n \, ; \, H_S(z) = 0\}$.

Admissible weight function and external fields

Definition: Let $E \subseteq \mathbb{C}^n$ and $w\colon E \to \mathbb{R}_+$ be a function and set

$$q = -\log w\colon E \to \mathbb{R} \cup \{+\infty\}.$$

The function w is said to be an *admissible weight* and q is said to be an *admissible external field with respect to S on E* if

(i) w is upper semi-continuous ($\Leftrightarrow$ q is lower semi-continuous),

(ii) the set

$$\{z \in E\,;\, w(z) > 0\} = \{z \in E\,;\, q(z) < +\infty\}$$

is non-pluripolar, and

(iii) if E is unbounded, then

$$\lim_{\substack{|z|\to+\infty \\ z\in E}} e^{H_S(z)} w(z) = 0 \quad \Leftrightarrow \quad \lim_{\substack{|z|\to+\infty \\ z\in E}} \big(q(z) - H_S(z)\big) = +\infty.$$

Some authors call q admissible weight function rather than $w = e^{-q}$.

3. Weighted extremal functions

Siciak functions:

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\}.$$

for $z \in \mathbb{C}^n$, $m = 1, 2, 3, \ldots,$ and

$$\Phi^S_{E,q}(z) = \overline{\lim_{m\to\infty}}\, \Phi^S_{E,q,m}(z)$$

Siciak-Zakharyuta functions:

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\ u|_E \leq q\}.$$

We obviously have $\log \Phi^S_{E,q} \leq V^S_{E,q}$

Density of rational points

Recall that $S \subset \mathbb{R}^n_+$, $0 \in S$, and $S \neq \{0\}$
The smallest such S it a line segment. If its endpoints have both rational and irrational coordinates, then mS does not have any integer points except 0 and $\mathcal{P}^S_m$ only consists of constants.

Proposition: Let $K \subset \mathbb{C}^n$ be a compact with $\partial\mathbb{B}_\infty \subseteq K$. Then

$$V^S_K = H_S.$$

Observe that if $S' = \overline{S \cap \mathbb{Q}^n}$, then $\mathcal{P}^{S'}(\mathbb{C}^n) = \mathcal{P}^S(\mathbb{C}^n)$ and $\Phi^{S'}_{K,q} = \Phi^S_{K,q}$, so if $S' \neq S$, the proposition tells us that at some point $z \in \mathbb{C}^n$

$$\Phi^{S'}_{\mathbb{B}_\infty}(z) = \Phi^S_{\mathbb{B}_\infty}(z) \leq V^{S'}_{\mathbb{B}_\infty}(z) < V^S_{\mathbb{B}_\infty}(z).$$

It is neccessary to assume that $S \cap \mathbb{Q}^n$ is dense in S in order to have a Siciak-Zakharyuta type theorem.

4. A Siciak-Zakharyuta type theorem

Theorem (BSM, ÁES, RS and BS, 2023):

Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$, let q be an admissible weight on a compact subset K of $\mathbb{C}^n$ and assume that $V^S_{K,q}$ is continuous. Then

$$V^S_{K,q} = \log \Phi^S_{K,q}$$

if and only if $S \cap \mathbb{Q}^n$ is dense in S.

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if and only if $S \cap \mathbb{Q}^n$ is dense in S.

A convex compact $S \subseteq \mathbb{R}^n_+$ is said to be a *lower set* if for every $s \in S$

$$C_s = [0, s_1] \times \cdots \times [0, s_n] \subseteq S.$$

Bayraktar, Hussung, Levenberg, and Perera, 2020 proved the theorem for convex bodies that are lower sets.

5. L^2 estimates and polynomial spaces

Theorem (BSM, ÁES, RS, and BS, 2023):

Let S be a compact convex subset of $\mathbb{R}^n_+$, $0 \in S$, $m \in \mathbb{N}^*$, and $d_m = d(mS, \mathbb{N}^n \setminus mS)$ denote the the euclidean distance between the sets mS and $\mathbb{N}^n \setminus mS$. Let $f \in \mathcal{O}(\mathbb{C}^n)$, assume that

$$\int_{\mathbb{C}^n} |f|^2 (1 + |\zeta|^2)^{-\gamma} e^{-2mH_S} \, d\lambda < +\infty$$

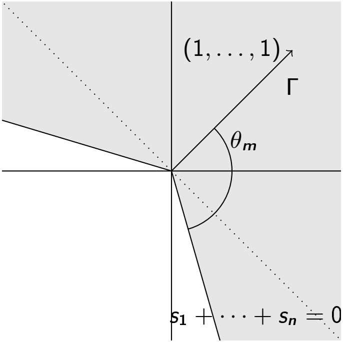
for some $0 \leq \gamma < d_m$, and let γ_0 denote the infinum of such γ. Let

$$\Gamma = \{\xi \in \mathbb{R}^n \, ; \, \langle 1, \xi \rangle \geq -(d_m - \gamma_0)|\xi|\},$$

be the cone consisting of all ξ such that the angle between the vectors $1 = (1, \dots, 1)$ and ξ is $\leq \arccos(-(d_m - \gamma_0)/\sqrt{n})$ and let

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+ \, ; \, \langle x, \xi \rangle \leq \varphi_S(\xi), \forall \xi \in \Gamma\}$$

be the hull of S with respect to the cone Γ. Then $f \in \mathcal{P}_m^{\widehat{S}_\Gamma}(\mathbb{C}^n)$.



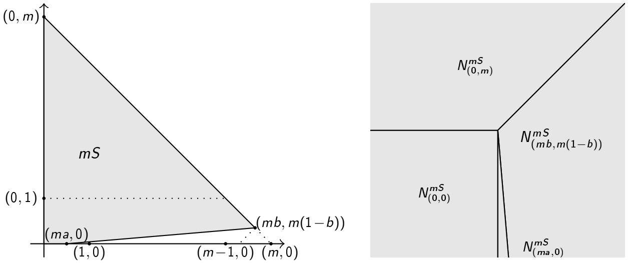
$$\theta_m = \arccos(-(d_m - \gamma_0)/\sqrt{n})$$

Example showing that the hull is optimal

Fix m. Let $0 < a < b < 1$ and define $S \subseteq \mathbb{R}^2_+$ as the quadrangle

$$S = \mathrm{ch}\{(0,0),(a,0),(b,1-b),(0,1)\}.$$

We show that for $f(z) = z^\alpha$, $\alpha = (k,0)$, $k = 1,\dots,m-3$, and $\gamma = 0$, the L^2 estimate in the theorem holds, but $f \notin \mathcal{P}^S_m(\mathbb{C}^n)$.



6. A weighted Bernstein-Walsh-Siciak theorem

Weighted distances to the polynomial spaces

For every bounded function $f : E \to \mathbb{C}$ we define the *distance of f from $\mathcal{P}_m^S(\mathbb{C}^n)$ with respect to the weight q* by

$$d_{E,q,m}^S(f) = \inf\{\|(f-p)e^{-mq}\|_E \,;\, p \in \mathcal{P}_m^S(\mathbb{C}^n)\}, \qquad m = 0, 1, 2, \dots.$$

and we say that f can be *approximated by S-polynomials with respect to q on E* if

$$\lim_{m\to\infty} d_{E,q,m}^S(f) = 0.$$

Recall that the Runge-Oka-Weil theorem says that if $K = \widehat{K}$ and f is holomorphic in some neighborhood of K, then

$$\lim_{m\to\infty} d_{K,m}(f) = 0,$$

and that the Bernstein-Walsh-Siciak theorem says that f extends as a holomorphic function to $\{z \in \mathbb{C}^n\,;\,,V_K < \log R\}$ if and only if

$$\overline{\lim_{m\to\infty}}\, d_{K,m}(f)^{1/m} \leq 1/R.$$

Pointwise convergence

Although f can be approximated by S-polynomials with respect to q we can not claim that f is a uniform limit of S-polynomials on K.

Assume first that $f : K \to \mathbb{C}$ is any bounded, K is not necessarily polynomially convex, and $\lim_{m\to\infty} d^S_{K,q,m}(f) = 0$. If q is bounded above we can find $p_m \in \mathcal{P}_m(\mathbb{C}^n)$ with $\|(f - p_m)e^{-mq}\|_K = d^S_{K,q}(f)$, which is equivalent to

$$|f(z) - p_m(z)| \leq d^S_{E,q,m}(f)e^{mq(z)}, \qquad z \in K,$$

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$$L = \{z \in K\,;\, \lim_{m\to\infty} d^S_{E,q,m}(f)e^{mq(z)} = 0\}.$$

If $\sup_L q$ is attained at some point in L, then $p_m \to f$ uniformly on L, and if $q \leq 0$ on K, then $p_m \to f$ uniformly on K.

Assume now that $V^S_{K,q}$ is continuous and define for $r >\in \mathbb{R}$

$$\Omega_r = \{z \in \mathbb{C}^n\,;\; V^S_{K,q}(z) < \log r\}.$$

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Let $R > 0$, $\sup_K q < \log R$, and assume that

$$\overline{\lim_{m\to\infty}} \left(d^S_{K,q,m}(f)\right)^{1/m} \leq \frac{1}{R}.$$

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Then for every $0 < \gamma < R$ there exists a constant $A_\gamma > 0$ such that

$$d^S_{K,q,m}(f) \leq \|(f - p_m)e^{-mq}\|_K \leq \frac{A_\gamma}{(R-\gamma)^m}, \qquad m \in \mathbb{N}.$$

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For every $j = 1, 2, 3, \dots$ and every $z \in K$ we have

$$\begin{aligned} |p_j(z) - p_{j-1}(z)| &\leq |f(z) - p_j(z)| + |f(z) - p_{j-1}(z)| \\ &\leq \frac{A_\gamma e^{jq(z)}}{(R-\gamma)^j}\left(1 + \frac{R-\gamma}{e^{q(z)}}\right). \end{aligned}$$

Since $q \in \mathcal{LSC}(K)$ takes its minimum a at some point in K, we have

$$\frac{1}{j} \log \big((R-\gamma)^j |p_j(z) - p_{j-1}(z)| / B_\gamma \big) \leq q(z), \qquad z \in K,$$

where $B_\gamma = A_\gamma(1 + (R-\gamma)/e^a)$, and by the definition of $V^S_{K,q}$

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$$\big|p_j(z)-p_{j-1}(z)\big| \le \frac{B_\gamma e^{jV^S_{K,q}(z)}}{(R-\gamma)^j}, \qquad z \in \mathbb{C}^n.$$

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If $0 < \varrho < 1$, then

$$|p_j(z) - p_{j-1}(z)| \leq B_\gamma \varrho^j, \qquad z \in \Omega_{\varrho(R-\gamma)},$$

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and this estimate implies that $p_m = \sum_{j=1}^m (p_j - p_{j-1})$ converges locally uniformly on $\Omega_{R-\gamma}$ to a holomorphic function F_γ. If $L \neq \varnothing$, then $F_\gamma = f$ on L.

We sum up our observations so far in:

Theorem (BSM, ÁES, RS, and BS, 2023):
Assume that $V^S_{K,q}$ is continuous, $f\colon K \to \mathbb{C}$ is bounded, and

$$\varlimsup_{m\to\infty} \left(d^S_{K,q,m}(f)\right)^{1/m} \leq \frac{1}{R}.$$

holds with $R > 0$ such that $\sup_K q < \log R$, and that

$$L = \{z \in K\,;\, \lim_{m\to\infty} d^S_{E,q,m}(f) e^{mq(z)} = 0\} \neq \varnothing.$$

Then for every $0 < \gamma < R$ the function $f|_L$ extends to a holomorphic function $F_\gamma \in \mathcal{O}(\Omega_{R-\gamma})$. If X is an open component of Ω_R, $L_X = L \cap X$ is non-pluripolar, and f is holomorphic in some neighborhood of L_X, then $f|_{L_X}$ extends to a unique holomorphic function on X.

The converse

Theorem (BSM, ÁES, RS, and BS, 2023):

Assume that $V^S_{K,q}$ is continuous, $R > 0$, $\sup q < \log R$ and

$$a = \varliminf_{m\to\infty} \big(d(mS, \mathbb{N}^n \setminus mS)\big)^{1/m} > 0,$$

If $f \in \mathcal{O}(\Omega_R)$ can be approximated by S-polynomials on K with respect to q, then

$$\varlimsup_{m\to\infty} \big(d^{\widehat{S}_{\Gamma_m}}_{K,q,m}(f)\big)^{1/m} \leq \frac{1}{a^{1/2}R},$$

where $\widehat{S}_{\Gamma_m}$ and Γ is the same as in the previous L^2-theorem.

Proposition:

If S is a polytope with rational vertices, then $a = 1$

The converse

Theorem (BSM, ÁES, RS, and BS, 2023):

Assume that $V^S_{K,q}$ *is continuous,* $R > 0$, $\sup q < \log R$ *and*

$$a = \varliminf_{m\to\infty} \big(d(mS, \mathbb{N}^n \setminus mS)\big)^{1/m} > 0,$$

If $f \in \mathcal{O}(\Omega_R)$ *can be approximated by* S*-polynomials on* K *with respect to* q*, then*

$$\varlimsup_{m\to\infty} \big(d^{\widehat{S}_{\Gamma_m}}_{K,q,m}(f)\big)^{1/m} \leq \frac{1}{a^{1/2}R},$$

where $\widehat{S}_{\Gamma_m}$ *and* Γ *is the same as in the previous* L^2*-theorem.*

Proposition:

If S is a polytope with rational vertices, then $a = 1$

The proof is based on construction of entire functions with the aid of Hörmander's existence theorem for the Cauchy-Riemann system with weighted L^2-estimates.

Hörmander's L^2-estimates

Theorem(Hörmander) Let X be a pseudoconvex domain of $\mathbb{C}^n$, $\varphi \in \mathcal{PSH}(X)$, and define for $a \in \mathbb{R}$,

$$\varphi_a(z) = \varphi(z) + a \log(1 + |z|^2), \qquad z \in X.$$

Then for every $a > 0$ and $f \in L^2_{(0,1)}(X, \varphi_{a-2})$ satisfying $\bar\partial f = 0$ there exists a solution $u \in L^2(X, \varphi_a)$ of the inhomogeneous Cauchy-Riemann equation $\bar\partial u = f$ satisfying the estimate

$$\begin{aligned} \|u\|^2_{\varphi_a} &= \int_X |u|^2(1 + |z|^2)^{-a} e^{-\varphi}\, d\lambda \\ &\le \frac{1}{a} \int_X |f|^2(1 + |z|^2)^{-a+2} e^{-\varphi}\, d\lambda = \frac{1}{a} \|f\|^2_{\varphi_{\gamma-2}}. \end{aligned}$$

If $f_j \in C^\infty(X)$ for $j = 1, \ldots, n$, then $u \in C^\infty(X)$.

From L^2 estimates to uniform estimates

Let $\bar{B}(z,\delta) \subseteq X$ with $\bar{B}(z,\delta) \cap \operatorname{supp} f = \varnothing$, then u is holomorphic in $B(z,\delta)$ and the mean value theorem gives

$$u(z) = \mathcal{M}_\delta u(z) = \frac{1}{c_n \delta^{2n}} \int_{B(z,\delta)} u \, d\lambda.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} |u(z)| &\leq \frac{1}{c_n \delta^{2n}} \int_{B(z,\delta)} |u| e^{-\varphi_a/2} \cdot e^{\varphi_a/2} \, d\lambda \\ &\leq \frac{1}{c_n \delta^{2n}} \|u\|_{\varphi_a} \left(\int_{B(z,\delta)} e^{\varphi_a} \, d\lambda \right)^{1/2} \\ &= c_n^{-1/2} a^{-1/2} \|f\|_{\varphi_{a-2}} \cdot \delta^{-n} \big(\mathcal{M}_\delta(e^{\varphi_a})(z)\big)^{1/2}. \end{aligned}$$

For every $v \in \mathcal{PSH}(X)$ we have $\mathcal{M}_\delta v(z) \searrow v(z)$ as $\delta \searrow 0$, so the art of applying this estimate is the choice of δ as a function of z.

Proof:

Let $\varepsilon > 0$ and $0 < \gamma < R$ such that $\sup q < \log(R - \gamma)$ and let $0 < \delta < 1$ be so small that $K_\delta = \{z \in \mathbb{C}^n\,;\, d(z, K) \leq \delta\}$ is contained in $\Omega_{R-\gamma}$ and

$$|V^S_{K,q}(w) - V^S_{K,q}(z)| < \varepsilon, \qquad |w - z| < \delta,\ z \in K.$$

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Take $\chi \in C^\infty(\Omega_R)$, with $0 \le \chi \le 1$, and $\chi = 1$ in some nbh of $\overline{\Omega}_{R-\gamma}$.

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Take $\chi \in C^\infty(\Omega_R)$, with $0 \leq \chi \leq 1$, and $\chi = 1$ in some nbh of $\overline{\Omega}_{R-\gamma}$. Set $\varphi_m = 2mV^S_{K,q}$, $a_m = \frac{1}{2}d(mS, \mathbb{N}^n \setminus mS)$, and define for $z \in \mathbb{C}^n$

$$\psi_m(z) = \varphi_m(z) + a_m \log(1+|z|^2) \quad \text{and} \quad \eta_n(z) = \psi_m(z) - 2\log(1+|z|^2).$$

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We have $\|f\bar\partial\chi\|_{\eta_m} < +\infty$. By Hörmander there exists a solution $u_m \in C^\infty(\mathbb{C}^n)$ of $\bar\partial u_m = f\bar\partial\chi$ satisfying

$$\|u_m\|^2_{\psi_m} = \int_{\mathbb{C}^n} |u_m|^2(1+|z|^2)^{-a_m} e^{-2mV^S_{K,q}}\, d\lambda \leq \frac{1}{a_m}\|f\bar\partial\chi\|^2_{\eta_m}.$$

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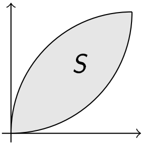
We define $p_m = f\chi - u_m$. Then the L^2-estimate and the previous theorm imply that $p_m \in \mathcal{P}^{\widehat{S}_\Gamma}_{K,q}(\mathbb{C}^n)$ and that there exists a constant $C_\gamma > 0$ such that

$$\|(f - p_m)e^{-mq}\|_K \leq \frac{C_\gamma e^{m\varepsilon}}{a_m^{1/2}(R-\gamma)^m}, \qquad m = 1, 2, 3, \ldots.$$

Example: of S for which

$$\lim_{m\to\infty}\left(d(mS,\mathbb{N}^n\setminus mS)\right)^{1/m}=1.$$



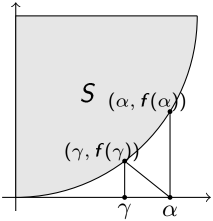
$$S=D_1(0,1)\cap D_1(1,0)$$

$$d(mS,\mathbb{N}^n\setminus mS)=\sqrt{1+m^2}-m=\frac{1}{\sqrt{1+m^2}+m}.$$

Example: of S for which

$$\lim_{m\to\infty} \left(d(mS, \mathbb{N}^n \setminus mS)\right)^{1/m} = 0.$$

Take $f \in C^2[0,1]$, with $f(0) = 0$, $f'(x) > 0$, and $f''(x) > 0$, $x \in]0,1]$.



$$d(mS, \mathbb{N}^n \setminus mS) = md(S, (1/m)\mathbb{N}^n \setminus S) \leq mf(1/m)$$

Take $f(x) = e^{-c/x^2+c}$, $x \in]0,1]$, and $f(0) = 0$, where $c > \frac{3}{2}$.

Thank you for your attention!

Congratulations Anders!