Metric graphs, stable polynomials and Fourier quasicrystals

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Joint work with Peter Sarnak

1. Crystalline measures

Crystalline measure: the measure μ itself and its Fourier transform $\hat{\mu}$ are sums of delta functions with **discrete** supports:

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda \qquad \hat{\mu} = \sum_{s \in S} b_s \delta_s$$

Example: **Poisson summation formula**

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m) \Leftrightarrow \sum_{n \in \mathbb{Z}} \delta(x-n) = \sum_{m \in \mathbb{Z}} e^{i2\pi m x}$$

Fourier transform of the Dirac comb is a Dirac comb.

The sets Λ (the support) and S (the spectrum) need to be very well structured.

Fourier quasicrystal: both $|\mu|$ and $|\hat{\mu}|$ are in addition tempered distributions.

1. Crystalline measures

Any linear combination of Poisson formulas gives crystalline measures.

- Kahane and Mandelbrojt (1958) - an advanced collection of properties, no explicit examples.

Problem: Construct examples of crystalline measures that are **not** finite combinations of Poisson summation formulas.

"Negative" results

- Lev-Olevskii (2015)

  Theorem. *If the support of both a crystallline measure and its Fourier transform are* **uniformly discrete***, then the corresponding summation formula is a finite sum of Poisson formulas with the same period.*
- Yves Meyer: working hypothesis: no non-trivial crystalline measures with uniformly discrete support exist

"Positive" examples

- Lev-Olevskii (2016): there exist non-trivial crystallline measures, **not explicit**.
- Kolountzakis (2016): simplifies construction as above, **not explicit**.
- Y. Meyer (2016): an explicit example of crystalline measure, **assuming Riemann hypothesis**

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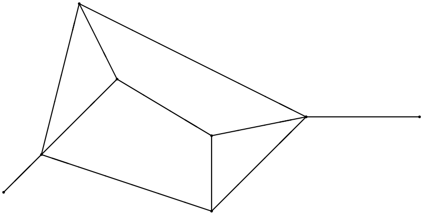
2. Trace formula for metric graphs

- **Metric graph**: collection of compact intervals (edges) e_n with certain end points identified (equivalence classes are vertices V_m)



- **Laplacian** on the edges:

$$\tau = -\frac{d^2}{dx^2}$$

- **Standard vertex conditions**: (free, natural, Neumann)

$$\begin{cases} u(x_i) = u(x_j), \quad x_i, x_j \in V_m & \text{continuity condition;} \\ \sum_{x_j \in V_m} \partial u(x_j) = 0, & \text{Kirchhoff condition.} \end{cases}$$

2. Trace formula for metric graphs

1. Solution of the differential equation $-\psi''(x) = \lambda\psi(x), \quad \lambda = k^2$ on the edges is a combination of exponentials for $x \in [v_{2n-1}, x_{2n}]$:

$$\begin{array}{rcccl} \psi(x)|_{e_n} & = & a_{2n-1}e^{ik|x-x_{2n-1}|} & + & a_{2n}e^{ik|x-x_{2n}|} & \text{incoming waves} \\ & \equiv & b_{2n}e^{-ik|x-x_{2n}|} & + & b_{2n-1}e^{-ik|x-x_{2n-1}|} & \text{outgoing waves} \end{array}$$

a_j - amplitudes of the waves coming into the edges
b_j – amplitudes of the waves leaving the edges

2. Equation above implies the first relation between a and b amplitudes:

$$\mathbf{S}_{\rm e}(k)\vec{a} = \vec{b},$$

where $\mathbf{S}_{\rm e}(k) = \mathrm{diag}\left\{\begin{pmatrix} 0 & e^{ik\ell_n} \\ e^{ik\ell_n} & 0 \end{pmatrix}\right\}_{n=1,2,\dots,N}$.

$\mathbf{S}_{\rm e}(k)$ is unitary for $k \in \mathbb{R}$ and contracting for $\Im k > 0$.

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3. Vertex conditions imply another relation between the amplitudes

$$\mathbf{S}_{\rm v} \vec{b} = \vec{a},$$

where $S_{\rm v}$ is formed from vertex scattering matrices.

The two relations imply

$$\mathbf{S}_{\rm v} \underbrace{\mathbf{S}_{\rm e}(k) \vec{a}}_{\vec{b}} = \vec{a}$$

and the **secular equation**:

$$\boxed{f_\Gamma(k) := \det \left(\mathbf{S}_{\rm e}(k) - \mathbf{S}_{\rm v} \right) = 0} \qquad (1)$$

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$p_\Gamma(k)$ is a trigonometric polynomial since all entries are exponentials and numbers $\Rightarrow$ the theory of **almost periodic functions**

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Integrate the jump of the logarithmic derivative of $f_\Gamma(k)$ on the real axis $\Rightarrow$ residues at the zeroes.

- LHS: the (real) zeroes of $f_\Gamma(k)$ give delta functions with the supports at k_j.

- RHS:
  - $f_\Gamma(k) = \det \mathbf{S}_\mathrm{v} \det \Big(\mathbf{I} - \mathbf{S}_\mathrm{v} \mathbf{S}_\mathrm{e}(k) \Big)$
  - $\log \det = \mathrm{Tr} \; \log$
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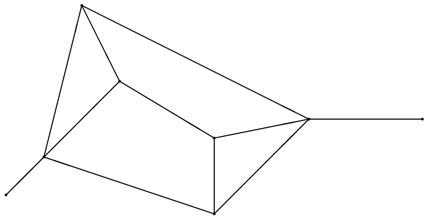

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J.-P. Roth, B. Gutkin, T. Kottos, U. Smilansky, P.K., M. Nowaczyk

$$\underbrace{\sum_{k_n} \left(\delta(k-k_n)+\delta(k+k_n)\right)}_{\text{spectral information}} = \underbrace{\underbrace{(1-\beta_1)}_{=\chi}\delta(k)+\frac{\mathcal{L}}{\pi}+\frac{1}{\pi}\sum_{p\in\mathcal{P}} l(\mathrm{prim}\,(p))S_{\mathrm{v}}(p)\cos k\ell(p)}_{\text{geometric/topologic information}}$$

where

- $\mathcal{L}$ - the total length of the graph;
- χ – Euler characteristic of Γ;
- β_1 – number of independent cycles in Γ;
- $\mathcal{P}$ – the set of closed oriented paths p on Γ;
- $\ell(p)$ – length of the closed path p;
- $S_{\mathrm{v}}(p)$ – product of all vertex scattering coefficients along the path p.

The formula is exact and thus reminds of Selberg trace formula but is close to Gullemin-Melrose formula (for compact Riemannian manifolds):

$$\sum_{\lambda_i\in\mathrm{spec}(\Delta)} \cos\left(\lambda_i\right)^{1/2} t = \sum \frac{T_\gamma^*}{|I-P_\gamma|^{1/2}}\delta(t-T_\gamma)+R(t), \quad R\in L_{1,loc}$$

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$$\boxed{\underbrace{(1+\beta_1)\delta(k) + \sum_{k_n\neq 0}\left(\delta(k-k_n)+\delta(k+k_n)\right)}_{u(k)} = \frac{\mathcal{L}}{\pi} + \frac{1}{\pi}\sum_{p\in\mathcal{P}} l(\mathrm{prim}\,(p)) S_{\mathrm{v}}(p)\cos k\ell(p)}$$

$$\hat{u}(\ell) = 2\mathcal{L}\delta(\ell) + \sum_{p\in\mathcal{P}} \ell(\mathrm{prim}\,(p)) S_{\mathrm{v}}(p)\Big(\delta(\ell-\ell(p)) + \delta(\ell+\ell(p))\Big),$$

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3. Zero sets of secular polynomials

Barra-Gaspard (2000)
The secular function

$$p_\Gamma(k) = \det\Big(\underbrace{\mathbf{S}_{\rm e}(k)}_{\sim\Gamma} - \underbrace{\mathbf{S}_{\rm v}}_{\sim G}\Big)$$

can be written using secular polynomial $P_\Gamma(\mathbf{z})$ in N variables

$$z_n = e^{ik\ell_n}, \quad \ell_n \text{ is the length of the edge } \ell_n.$$

Explicit formula for the secular polynomial:

$$\boxed{P_G(\mathbf{z}) = \det\left(\underbrace{\mathrm{diag}\, \left\{ \begin{pmatrix} 0 & z_n \\ z_n & 0 \end{pmatrix} \right\}_{n=1}^{N}}_{\sim\, \mathbf{S}_{\rm e}} - \mathbf{S}_{\rm v} \right)} \tag{2}$$

G – discrete graph, Γ – metric graph.

3. Zero sets of secular polynomials

Properties of the secular polynomial

$$\boxed{P_G(\mathbf{z}) = \det \left(\mathrm{diag} \left\{ \begin{pmatrix} 0 & z_n \\ z_n & 0 \end{pmatrix} \right\}_{n=1}^{N} - \mathbf{S}_{\mathrm{v}} \right)}$$

- Second order polynomial in each variable
- Stable and invariant under involution:

$$P_G(1/\mathbf{z}) = z_1^2 z_2^2 .. P(\mathbf{z})$$

- The polynomial P_G is determined by the discrete graph G alone, while the secular function $f_\Gamma(k)$ depends on the edge lengths.
- For any set $\{\ell_n\}_{n=1}^N$ the secular equation

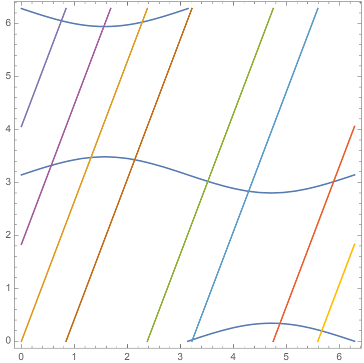
$$P_G(e^{ik\ell_1}, \dots, e^{ik\ell_N}) \equiv f_\Gamma(k) = 0 \tag{3}$$

has only real solutions satisfying Weyl's asymptotics.

3. Zero sets of secular polynomials

Properties of the secular polynomial

- The zero set Z_G for P_G on the torus $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$.
  The spectrum of L_Γ is given by the intersections of the line $k \mapsto (e^{ik\ell_1}, \ldots, e^{ik\ell_N}) \in \mathbb{T}^N$ with Z_G.



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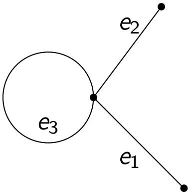

$$P_{(3.4)} = (z_3 - 1)\big(-2z_1^2 z_2^2 z_3 - z_1^2 z_3 - z_2^2 z_3 + z_1^2 + z_2^2 + 2\big)$$

$$L_{(3.4)} = \sin\frac{\varphi_3}{2}\left(2\sin(\varphi_1+\varphi_2+\frac{\varphi_3}{2}) + \sin(\varphi_1-\varphi_2+\frac{\varphi_3}{2}) + \sin(-\varphi_1+\varphi_2+\frac{\varphi_3}{2})\right)$$

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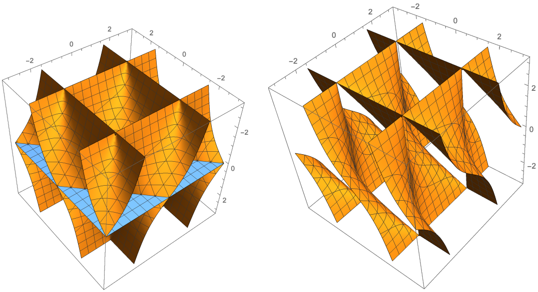

Figure: Zero sets Z and Z^* for $G_{(3.4)}$.

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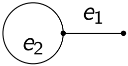
Problem Show that the singular set has codimension 3.

4. The first explicit crystalline measure



Take

$$P(z_1, z_2) = \cancel{(z_1 - 1)}\left(1 - \frac{1}{3}z_1 + \frac{1}{3}z_2^2 - z_1 z_2^2\right)$$

Stability:

$$P(\mathbf{z}) = 0 \Leftrightarrow \frac{z_1 - 3}{1 - 3z_1} = z_2^2.$$

Invariance under involution:

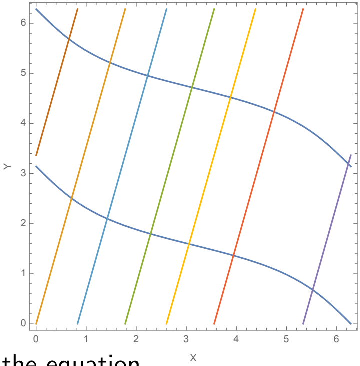
$$z_1 z_2^2 P(1/\mathbf{z}) = P(\mathbf{z}).$$

The polynomials are treated projectively.

4. The first explicit crystalline measure

The zero set on the torus: $z_1 = e^{ix}, z_2 = e^{iy}$, $(x, y) \in [0, 2\pi] \times [0, 2\pi]$

$$L(x, y) = 3\sin(\frac{x}{2} + y) + \sin(\frac{x}{2} - y) = 0$$



The spectrum given by the equation

$$f(k) = 3\sin\left((\frac{\ell_1}{2} + \ell_2)k\right) + \sin\left((\frac{\ell_1}{2} - \ell_2)k\right) = 0 \Rightarrow \{k_j\}$$

is clearly **uniformly discrete**.

4. The first explicit crystalline measure

The summation formula takes the form

$$\sum_{\gamma_j} h(k_j) = (\ell_1 + 2\ell_2)\,\hat{h}(0) - \sum_{\mathbf{n}=(n_1,n_2)\in\mathbb{Z}_+^2} c(n_1, 2n_2)\Big(\hat{h}(n_1\ell_1 + 2n_2\ell_2) + \hat{h}(-(n_1\ell_1 + 2n_2\ell_2))\Big),$$

with

$$c(n_1, 2n_2) = -(n_1\ell_1 + 2n_2 b_2) \sum_{\substack{k_1,k_2,k_3\in\mathbb{N}\cup\{0\}\\ k_1+k_3=n_1\\ k_2+k_3=n_2}} \frac{(k_1+k_2+k_3-1)!}{k_1!\;k_2!\;k_3!}\frac{(-1)^{k_2}}{3^{k_1+k_2}}.$$

Important observations:

- $n_1\ell_1 + 2n_2\xi_2$ - the lengths of periodic orbits on the lasso graph
- $c(n_1, n_2)$ - products of scattering coefficients on the lasso graph.

4. The first explicit crystalline measure

This crystalline measure is **not** a combination of Poisson formulas

Theorem 1. *Every uniformly discrete finite combination of Dirac combs is periodic.*

Obviously, the function $f(k)$ is **not** periodic, provided ℓ_1 and ℓ_2 are rationally independent. Hence its zero set is not periodic. The measure is not a finite sum of Dirac combs.

Mathematics is simple, one should only understand why (P. Sarnak)

Constructed measure resolves the following questions

- a positive crystalline measure which is not a generalised Dirac comb (question by Y. Meyer);
  In fact idempotent.
- a positive Fourier quasicrystal for which every arithmetic progression meets the support in a finite set (question by Lev–Olevskii);
  Not only μ and $\hat{\mu}$ are tempered, but also $|\mu|$ and $|\hat{\mu}|$.
- Fourier quasicrystal for which the support (that is Λ) is a Delone (Delaunay) set, but the spectrum (that is S) is not (question by Y. Meyer and Lev–Olevskii);
- a discrete set (that is Λ) which is a Bohr almost periodic Delone (Delaunay) set, but is not an ideal crystal (question by J. Lagarias).

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Question

Consider the trigonometric equation:

$$\boxed{3 \sin Ak + \sin k = 0, \quad A \notin \mathbb{Q}}$$

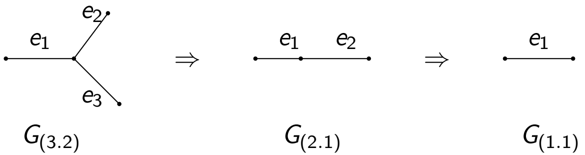
Prove that

$$\dim_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}} \{ k_n \} = \infty$$

The question was asked to ca 50 mathematicians.

5. Reducibility of secular polynomials

Graph's contraction $\Gamma/_{e_j}$ - deletion of the edge e_j in Γ.



$$
\begin{array}{rrcl}
 & P_{(3.2)}(z_1, z_2, z_3) & = & 3z_1^2 z_2^2 z_3^2 + \left(z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2\right) - \left(z_1^2 + z_2^2 + z_3^2\right) - 3 \\
\xrightarrow{z_3 \to 1} & P_{(2.1)}(z_1, z_2) & = & z_1^2 z_2^2 - 1 \\
\xrightarrow{z_2 \to 1} & P_{(1.1)}(z_1) & = & z_1^2 - 1
\end{array}
$$

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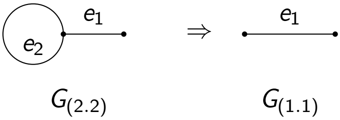

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The polynomials are treated **projectively**.

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$$P_{G/_{e_j}}(z_1, \dots, \cancel{z_j}, \dots, z_N) = \lim_{z_j \to 1} P_G(z_1, \dots, z_N).$$

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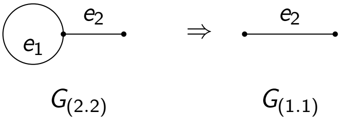

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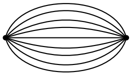
5. Reducibility of secular polynomials
Colin de Verdiére's conjecture

Theorem 4. *The secular polynomial for G is reducible iff the metric graph is symmetric for any choice of the edge lengths:*

- *G has loops;*
- *G is a watermelon graph $\mathbf{W}_N$:*



Moreover, if the secular polynomial is reducible, then

- *if G has loops*

$$P_G(\mathbf{z}) = \Big(\prod_{e_n \text{ is a loop in } G} (z_n - 1) \Big) Q_G(\mathbf{z}),$$

the product is over the loop edges, Q_G is irreducible;

- *if G is a watermelon $\mathbf{W}_N$*

$$P_G = P^s_{\mathbf{W}_N} P^a_{\mathbf{W}_N}$$

$P^s_{\mathbf{W}_N}$ and $P^a_{\mathbf{W}_N}$ are irreducible, first order in each variable.

Proof by reducing graphs to elementary graphs on $2, 3, 4, 5, 6$ edges and watermelon graphs and their closest relatives.

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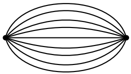

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6. Lang's GM conjecture (theorem)

Theorem 6. Lang's conjecture *(P. Sarnak, P. Liardet, W. Schmidt, M. Laurent, J-H. Evertse ...)*

- $V \subset (\mathbb{C}^*)^N$ *- an algebraic subvariety given by the zero set of Laurent polynomial;*
- Γ *- finitely generated subgroup of rank* r *of the torus* $T \subset (\mathbb{C}^*)^N$ *considered as a group under coordinatewise product;*
- $\overline{\Gamma}$ *- the division group of* Γ

$$\overline{\Gamma} = \left\{ z \in T : z^m \in \Gamma \text{ for some } m \geq 1 \right\}.$$

Then there exists finitely many translates of (may be low dimensional) subtori $T_1, T_2, \dots, T_\nu$ **contained** *in* V *such that*

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \cup \cdots \cup T_\nu)$$

and

$$\nu \leq \Big(C(V) \Big)^r,$$

where $C(V)$ *is an effectively computable constant.*

6. Lang's GM conjecture (theorem)

How to apply this to quantum graphs?

We want to prove that: $\boxed{\dim_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}\{k_n\} = \infty}$

Assume the opposite $\dim_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}\{k_n\} < \infty$, *i.e.*

$$k_n = \alpha_1^n k_1 + \alpha_2^n k_2 + \dots + \alpha_{n_0}^n k_{n_0}, \quad \alpha_j^n \in \mathbb{Q}$$

for a certain n_0 and arbitrary n.

It follows that

$$e^{ik_n \ell_j} = \left(e^{ik_1 \ell_j}\right)^{\alpha_1^n} \left(e^{ik_2 \ell_j}\right)^{\alpha_2^n} \dots \left(e^{ik_{n_0} \ell_j}\right)^{\alpha_{n_0}^n}$$

in other words, all $(e^{ik_n \ell_1}, \dots, e^{ik_n \ell_N})$ belong to the division group for the multiplicative group Γ generated by

$$(e^{ik_i \ell_1}, e^{ik_i \ell_2}, \dots, e^{ik_i \ell_N}), \quad i = 1, 2, \dots, n_0.$$

Multiplication is carried coordinate-wise.

It feels like Liardet's theorem was proven especially to serve our purposes!

6. Lang's GM conjecture (theorem)

Hypertori in the zero set

Hypertori

$$z_1^{\alpha_1} z_2^{\alpha_2} \dots z_N^{\alpha_N} = q z_1^{\beta_1} z_2^{\beta_2} \dots z_N^{\beta_N}$$
$$\alpha_j \beta_j = 0 \quad \text{and} \quad |q| = 1.$$

Quantum graphs have hypertori $z_n - 1$ corresponding to loops in G.

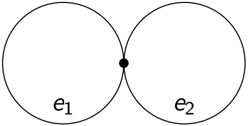
No other hypertori are contained in the zero manifold – follows essentially from the factorisation of the secular polynomial (Colin de Verdiére's conjecture) and Hilbert's Nullstellensatz.

Only hypertori determine arithmetic sequences in the spectrum for rationally independent edge lengths, lower dimensional tori are not "dangerous."

Conclusion:

$$\left.\begin{array}{l}\text{Edge lengths are rationally independent} \\ P_G \textbf{ is not} \text{ a product of hypertoric factors}\end{array}\right\} \Rightarrow \dim_{\mathbb{Q}} \{k_n\} = \infty.$$

Interval, single loop and figure eight graph (the only exceptional cases):



$$P_{(1.1)} = (z_1 - 1)(z_1 + 1) - \text{interval}$$
$$P_{(1.2)} = (z_1 - 1)^2 - \text{loop}$$
$$P_{(2.4)} = (z_1 - 1)(z_2 - 1)(z_1 z_2 - 1)$$

7. Arithmetic structure of the spectrum

Theorem 7.

- - *If Γ is a segment or a single loop, then*

    $$\mathrm{Spec}\,(\Gamma) = L_1(\Gamma)$$

  - *If Γ is a figure eight graph, then*

    $$\mathrm{Spec}\,(\Gamma) = L_1(\Gamma) \cup L_2(\Gamma) \cup L_3(\Gamma)$$

- *If all edge lengths are* **pairwise rationally dependent**, *then*

  $$\mathrm{Spec}\,(\Gamma) = L_1(\Gamma) \cup L_2(\Gamma) \cup \cdots \cup L_m(\Gamma),$$

- *If edge lengths are* **rationally independent** *and G is not exceptional, then*

  $$\mathrm{Spec}\,(\Gamma) = \underbrace{L_1(\Gamma) \cup L_2(\Gamma) \cup \cdots \cup L_\nu(\Gamma)}_{\textit{coming from loops}} \cup \underbrace{N(\Gamma)}_{\neq \emptyset}$$

  $N(\Gamma)$ contains no finite arithmetic progressions of length more than $c(G)$,

  $$\dim_{\mathbb{Q}} \mathrm{Spec}\,(\Gamma) = \infty.$$

  $\nu =$ the number of loops in G

8. Stable polynomials: summation formula

Multivariate polynomial $P(\mathbf{z})$ is $\mathbb{D}$ stable iff $P(\mathbf{z}) \neq 0$ for $\mathbf{z} = (z_1, \dots, z_n)$: $|z_j| < 1$.

Definition. *Two multivariate polynomials P, Q are said to form stable pair if*

1. *both polynomials P and Q are $\mathbb{D}$-stable;*
2. *there exist an integer-valued vector $\boldsymbol{\ell} = (\ell_1, \ell_2, \dots, \ell_n) \in \mathbb{N}^n$ and a constant η such that P and Q satisfy the functional equation*

$$Q(\mathbf{z}) = \eta\, z_1^{\ell_1} z_2^{\ell_2} \dots z_n^{\ell_n} P(1/\mathbf{z}); \tag{4}$$

3. *the normalization condition*

$$P(\vec{0}) = Q(\vec{0}) = 1$$

is fulfilled.

8. Stable polynomials: summation formula

Dirichlet series

Let

$b_1, b_2, \ldots, b_n \geq 1$ – arbitrary positive real numbers,

$\xi_j = \ln b_j > 0, \; j = 1, 2, \ldots, n.$

Entire functions of order 1

$$F(s) = P(b_1^{-s}, b_2^{-s}, \ldots, b_n^{-s}) = 1 + \sum_{\mathbf{m} \in M_P} a_P(\mathbf{m})(\mathbf{b}^{\mathbf{m}})^{-s}$$

$$\Rightarrow \frac{F'(s)}{F(s)} = -\sum_{\mathbf{k} \in \mathbb{Z}_+^n} (\boldsymbol{\xi} \cdot \mathbf{k}) c_P(\mathbf{k}) e^{-(\boldsymbol{\xi} \cdot \mathbf{k})s}.$$

Stable pair $\Rightarrow$ all zeroes of F are on the imaginary axis.

Integrating the logarithmic derivative one obtains summation formula à la Poisson.

9. Improvements

- **Yves Meyer: several alternative approaches**
  - curved model sets
  - inner functions in several variables
  - linear recurrence relations on lattices
  - almost periodic perturbations of lattices
- **Olevskii-Ulanovskii**: trigonometric polynomials
  Every Fourier quasicrystal with unit (integer) masses is given by an exponential polynomial

$$f(k) = \sum_{1 \leq j \leq N} c_j e^{2\pi i \gamma_j k}, \quad n \in \mathbb{N}, c_j \in \mathbb{C}, \gamma_j \in \mathbb{R} \text{ with only } \textbf{real} \text{ zeroes.}$$

Equivalent to **construction via stable polynomials**.
This follows directly from the fact that for every trig polynomial with real coefficients, it holds:

$$\frac{f(k)}{f^*(k)} = e^{-i\theta} e^{-i\omega k}$$

  - $P(\mathbf{z})$ leading to $f(k)$ can be chosen invariant under involution
  - stability of $P(\mathbf{z})$ comes from the fact that $\log P(\mathbf{z})$ can be defined on the torus.

Using amoebas: Alon-Cohen-Vinzant

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10. Perspectives

Multidimensional crystalline measures:

- Two-dimensional crystalline measures
  - Yves Meyer
    - starting from two-dimensional measures having one-dimensional character
    - more two-dimensional measures
  - M. de Courcy Ireland + P.K.
    - as intersections between families of curves
  - L.Alon, M.Kummer, P.K., C.Vinzant
    - via Lee-Yang varieties

11. New question

Consider the trigonometric equation:

$$\boxed{3 \sin Ak + \sin k = 0, \quad A \notin \mathbb{Q}}$$

Prove that for any $\alpha \in \mathbb{R}$ only finitely many pairs satisfy

$$k_i / k_j = \alpha$$

Alternative formulation:
For any $A \notin \mathbb{Q}$ and $\alpha \in \mathbb{R}$, prove that the trigonometric polynomials

$$\begin{aligned} P(x) &= 3 \sin Ak + \sin k \\ Q(x) &= 3 \sin A\alpha k + \sin \alpha k \end{aligned}$$

have at most finitely many common zeroes.

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More about spectral theory of metric graphs and Schrödinger operators on metric graphs

P. Kurasov **Spectral Geometry of Graphs**, 2023, almost 700 pages.

Congratulations to Anders!

Short history

2005: J.P. Kahane: not surprising (thanks Jan Boman)
"Playing with Poisson formulas in several dimensions gives a lot of formulas on the line."

2018: Y. Meyer: lecture at Institute Mittag-Leffler on crystalline measures, constructed an explicit example assuming Riemann hypothesis.