Tunneling for the $\overline{\partial}$-operator.

Based on joint work with M. Vogel

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1. Introduction

In the early 70:ies Lars Hörmander suggested to Anders and me to extend the theory of Fourier integral operators, developed in his paper in Acta Math. (1971), to the case of complex valued phase functions. Doing so was very interesting and stimulating and I lerned a lot from Anders. (A parallel theory in the framework of Maslov's canonical operator was developed by Maslov, Kucherenko. See also books by Laptev-Safarov-Vassiliev, Combescure-Robert.) This became the main theme for me and has kept me busy through the years. With Anders we returned some 23 years ago to another form of complex phase theory in the study of quantization rules for eigenvalues for semi-classical non-self-adjoint operators in dimension 2. I have had the chance to continue on this and related subjects with Michael Hitrik, a former student of Anders.

I shall present a joint work with Martin Vogel on a model for a general problem of tunneling for non-self-adjoint operators. We will mention at the end some related works for Magnetic Schrödinger and Pauli operators.

Let $P : \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator on a complex Hilbert space $\mathcal{H}$. When P is self-adjoint, the operator norm of the resolvent satisfies $\|(P-z)^{-1}\| = 1/\mathrm{dist}\,(z, \sigma(P))$ for $z \in \mathbf{C}$, where $\sigma(P)$ denotes the spectrum of P and with the convention that the norm is $= +\infty$ for z in the spectrum. In the non-self-adjoint case we may have

$$\|(P-z)^{-1}\| \gg \frac{1}{\mathrm{dist}\,(z, \sigma(P))},$$

and this may be a source of spectral instability, problematic for the numerical computation of eigenvalues, but also of interest for instance in the presence of random perturbations. Under some assumptions the norm of the resolvent is equal to $1/t_{\min}(P-z)$ where $t_{\min}$ denotes the smallest *singular value*, i.e. the smallest eigenvalue of $((P-z)^*(P-z))^{1/2}$. We are therefore interested in the small singular values of $P-z$.

We will work in the semi-classical limit and consider operators of the form

$$P = \sum_{|\alpha|\leq m} a_\alpha(x;h)(hD_x)^\alpha, \ \alpha \in \mathbf{N}^n, \ \ 0 < h \ll 1$$

with $|\alpha| = |\alpha|_{\ell^1}$, $(hD_x)^\alpha = (hD_{x_1})^{\alpha_1}...(hD_{x_n})^{\alpha_n}$, $D_{x_j} = (1/i)\partial_{x_j}$, on a manifold M with $a_\alpha(x;h) = a_\alpha(x;0) + \mathcal{O}(h)$ in C^∞, and semi-classical principal symbol

$$p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x;0)\xi^\alpha \in C^\infty(T^*M)$$

.

Example

$P = -h^2\Delta + V(x)$ *(Schrödinger operator)*, $p(x,\xi) = \xi^2 + V(x)$

If $z \in \mathbf{C}$, $z = p(x,\xi)$ for some $\rho = (x,\xi) \in T^*M$ and

$$\frac{1}{i}\{p - z, \overline{p - z}\}(\rho) > 0, \text{where } \{a,b\} := a'_\xi \cdot b'_x - a'_x \cdot b'_\xi$$

denotes the Poisson bracket of two C^1 functions, then

$$\exists u = u_h \in C_0^\infty(M),\ \|u\|_{L^2} = 1,\ (P-z)u = \mathcal{O}(h^\infty)\ (\text{i.e. } \mathcal{O}_N(h^N), \forall N \geq 0).$$

This implies that $\|(P-z)^{-1}\| \geq 1/\mathcal{O}(h^N)$ for every $N \geq 0$ (when the resolvent is well defined), [Ho60a, Ho60b, Zw01, DeSjZw04].
When M is real analytic and P has analytic coefficients,

$$\exists u \in C_0^\infty(M),\ \text{s.t.} \|u\| = 1,\ (P-z)u = \mathcal{O}(e^{-\frac{1}{C_0 h}}),\ \text{for some fixed } C_0 > 0.$$

In particular, $\|(P-z)^{-1}\| \geq e^{1/\mathcal{O}(h)}$ and $t_{\min}(P-z) = \mathcal{O}(e^{-1/\mathcal{O}(h)})$.
Convention: "$\mathcal{O}(\dots)$" in a denominator always denotes a positive quantity.

The proof in both cases is based on the construction of WKB solutions (for the analytic version see [Sj82, DeSjZw04] and references there to earlier works, in particular to those of Boutet de Monvel–Krée and Sato-Kawai-Kashiwara): Assume that $z = 0$ for simplicity and let $\rho_0 = (x_0, \xi_0)$ satisfy $p(\rho_0) = 0$, $\{p, \overline{p}\}(\rho_0)/i > 0$. Then, as observed by Hörmander ([Ho60a, Ho60b]), $\exists \varphi \in C^\infty(\mathrm{neigh}\,(x_0))$ such that

$$\varphi(x_0) = 0,\ \varphi'_x(x_0) = \xi_0,\ \Im \varphi''_{xx}(x_0) > 0,\ p(x, \varphi'_x(x)) = \mathcal{O}((x - x_0)^\infty),$$

and by solving a sequence of transport equations, we get a Gaussian type quasimode $u(x; h) = a(x; h)e^{i\varphi(x)/h}$ with the symbol

$$a(x; h) \sim a_0(x) + a_1(x)h + \ldots \text{ in } C^\infty(\mathrm{neigh}\,(x_0)),$$

such that $Pu = \mathcal{O}(h^\infty)$, $\|u\| = 1$.

We are interested in the *precise exponential decay* in the analytic case. This leads to a *quantum tunneling problem*, presumably to link a point ρ_0 as above via a path in the complexification of T^*M to points $\rho_1 \in p^{-1}(0)$ where $\{p, \overline{p}\}(\rho_1)/i < 0$.

Tunneling is well understood for some operators like the semi-classical self-adjoint Schrödinger operator, but remains difficult for others like the magnetic Schrödinger operator. We will get fairly complete results for the small singular values of operators that are conjugations of $h\overline{\partial}$ with exponential weights in real dimension 2 under assumptions that allow separation of variables and the use of a Witten complex in dimension 1.

Recall that the singular values of an operator P are the eigenvalues of $(P^*P)^{1/2}$.

2. Results

We will work on $X + iY$, where $X = S^1 = \mathbf{R}/2\pi\mathbf{Z}$ and Y is equal to $\mathbf{R}$ or S^1.
Let $z = x + iy$, $x \in X$, $y \in Y$ and $0 < h \ll 1$.
Let $\varphi = \varphi(y)$ be a real-valued smooth function on $X + iY$ which is independent of x and consider the operator[1]

$$P = 2\, e^{-\varphi/h} \circ hD_{\overline{z}} \circ e^{\varphi/h} = 2(hD_{\overline{z}} + (D_{\overline{z}}\varphi)) = hD_x + h\partial_y + \partial_y\varphi, \quad (1)$$

where $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and $D_{\overline{z}} = \frac{1}{i}\partial_{\overline{z}} = \frac{1}{2}(D_x + iD_y)$. We choose the following two model cases

$$\varphi(y) = \begin{cases} \frac{1}{3}y^3, & \text{when } Y = \mathbf{R}, \\ \sin y, & \text{when } Y = S^1. \end{cases} \quad (2)$$

We equip P with the natural domain, a weighted Sobolev space.

[1] This is basically a Pauli operator. More comments and references on Pauli and magnetic Schrödinger operators at the end of the talk

P has the symbol

$$p(x,y;\xi,\eta) := \xi + i\eta + \partial_y\varphi =: p_\xi(y,\eta), \tag{3}$$

The characteristic set of P is given by

$$p^{-1}(0) = \{(x,y;\xi,\eta) \in T^*(S^1 \times Y);\ \eta = 0,\ \partial_y\varphi = -\xi\}. \tag{4}$$

$$\frac{1}{2i}\{p,\overline{p}\} = \partial_y^2\varphi \ \ (= 4\partial_z\partial_{\overline{z}}\varphi). \tag{5}$$

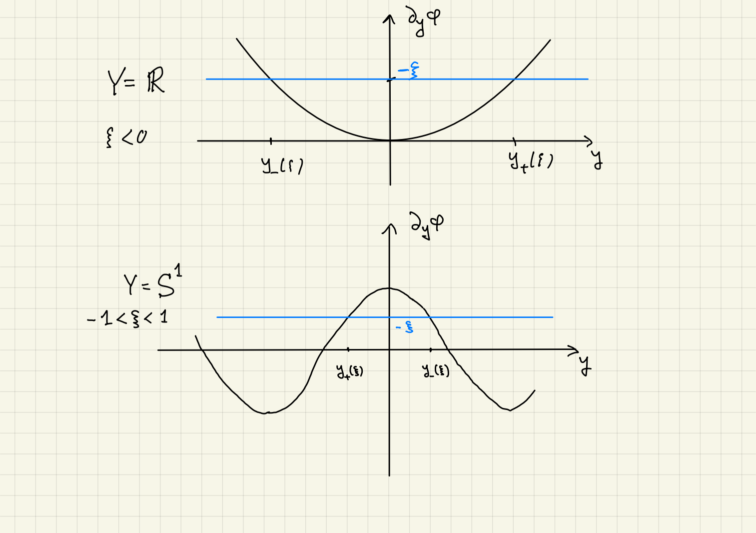
Let

$$\Sigma_\pm = \{(x,y;\xi,\eta) \in p^{-1}(0);\ \pm\frac{1}{i}\{p,\overline{p}\}(x,y;\xi,\eta) > 0\}. \tag{6}$$

More explicitly, $\Sigma_\pm$ is given by

$$\eta = 0,\ \xi \in \begin{cases}]-\infty,0[, \text{ when } Y = \mathbf{R}, \\ ]-1,1[, \text{ when } Y = S^1, \end{cases} \quad y = y_\pm(\xi), \text{ where}$$

$\partial_y \phi(y_\pm(\xi)) = -\xi$:



The submanifolds $\Sigma_\pm$ are symplectic.
Using Fourier series in x we see that P has the orthogonal direct sum decomposition

$$P = \bigoplus_{\xi \in h\mathbf{Z}} P_\xi, \quad P_\xi = \xi + h\partial_y + \partial_y \varphi \text{ on } Y. \tag{7}$$

The set of singular values of P is the union over $h\mathbf{Z}$ of the sets of singular values of P_ξ.
We equip the operator P_ξ with its natural domain, a weighted semiclassical Sobolev space. Adjoint:

$$P_\xi^* = -h\partial_y + \xi + \partial_y \varphi. \tag{8}$$

We show that the spectrum of $P_\xi^* P_\xi$, $\xi \in h\mathbf{Z}$, is purely discrete and equal to the spectrum of $P_\xi P_\xi^*$. We will denote the singular values of P_ξ (i.e. the eigenvalues of $(P_\xi^* P_\xi)^{1/2}$) by

$$0 \leq t_0(\xi) \leq t_1(\xi) \leq \cdots \to +\infty. \tag{9}$$

Theorem

We define φ *as in* (2) *and* $P_\xi = h\partial_y + \xi + \partial_y\varphi$ *as in* (7).
1. Let $Y = \mathbf{R}$. *For every* $C_0 > 0$, *there exists a constant* $C > 0$ *such that if* $-C_0 h^{2/3} \leq \xi$, *then the smallest singular value* $t_0(\xi)$ *of* P_ξ *satisfies*

$$t_0(P_\xi) \geq \frac{1}{C}(|\xi| + h^{2/3}).$$

2. There is a similar statement when $Y = \mathbf{R}/2\pi\mathbf{Z}$, *valid for* $|\xi| \geq 1 - C_0 h^{2/3}$.

Theorem

Assume $Y = \mathbf{R}$ for simplicity. For $\xi < 0$, let y_+, $y_- \in Y$ be the two solutions of the equation $\partial_y \varphi(y) = -\xi$, labelled so that $\pm\partial_y^2\varphi(y_\pm) > 0$. Let d denote the Lithner-Agmon distance on Y for the metric $(\xi + \partial_y\varphi(y))^2 dy^2$ and define the action $\mathcal{D}(S_0) :=]-\infty, 0[\ni \xi \mapsto S_0(\xi) = d(y_+(\xi), y_-(\xi)) \in]0, +\infty[$. Then, uniformly for ξ varying in any compact h-independent subset of $]-\infty, 0[$, the smallest singular value of P_ξ satisfies for any $\varepsilon > 0$

$$t_0(\xi) = h^{\frac{1}{2}} \left(\frac{|\{p_\xi, \overline{p}_\xi\}(y_+, 0)|^{1/4} |\{p_\xi, \overline{p}_\xi\}(y_-, 0)|^{1/4}}{(4\pi)^{1/2}} + \mathcal{O}(h) \right) e^{-S_0/h}.$$

We will later view S_0 as a function on the symplectic manifold Σ_+, cf. (6).

In the case $Y = S^1$ there is a similar statement for ξ varying in a compact subset of $]-1,1[$ now with a possible cancellation in the leading terms for $\xi \approx 0$. (Indeed $t_0(0) = 0$!)
Furthermore, we can extend the result to regions $-C \leq \xi \leq -h^{2/3}$ when $Y = \mathbf{R}$ and similarly when $Y = S^1$.
We show that the second smallest singular value is well above the region of interest.

We also study the distribution of the values $t_0(\xi)$ and hence of the small singular values of P and obtain a form of Weyl asymptotics.

Theorem

Let P *be* (1), *let* Σ_+ *be as in* (6) *and recall* (5). *Let* S_0 *be as in Theorem 2.2. Let* $C_0 > 0$ *be large enough and let*

$$C_0 h \leq \frac{\delta^{3/2}}{\log \delta^{-1}}, \quad \delta > 0.$$

Then, for $0 < a < b$ *with* $b \asymp 1$ *and* $a \asymp \delta^{3/2}$,

$$\# \left(\mathrm{Spec}(\sqrt{P^* P} \cap [e^{-b/h}, e^{-a/h}] \right) = \frac{1}{2\pi h} \int_{S_0^{-1}([a,b])} \sigma_{|_{\Sigma_+}} + \mathcal{O}(1) \frac{\log \delta^{-1}}{\sqrt{\delta}}.$$

Here $\sigma_{|_{\Sigma_+}}$ *denotes the symplectic 2-form on* Σ_+ *and* S_0 *is viewed as a function on* Σ_+.

3. About the proofs, Witten Laplacians

Putting

$$f(y,\xi) = y\xi + \varphi(y), \tag{10}$$

we have (7), (8) in the form

$$\begin{aligned} P_\xi &= h\partial_y + \partial_y f = e^{-f/h} h\partial_y e^{f/h}, \\ P_\xi^* &= -h\partial_y + \partial_y f = -e^{f/h} h\partial_y e^{-f/h} \end{aligned} \tag{11}$$

Cf. the Witten complex. We will use [HeSj85]. $f(\cdot,\xi)$ is in general multi-valued when $Y = S^1$, but $\partial_y f$ is well-defined.

$$\begin{aligned} Q_+ &:= P_\xi^* P_\xi = -h^2\partial_y^2 + (\partial_y f)^2 - h\partial_y^2 f, \\ Q_- &:= P_\xi P_\xi^* = -h^2\partial_y^2 + (\partial_y f)^2 + h\partial_y^2 f \end{aligned} \tag{12}$$

can be viewed as the Witten Laplacians for 0 and 1 forms respectively. They are semi-classical Schrödinger operators with potential $V_0 + \mathcal{O}(h)$, where

$$V_0(y,\xi) = (\partial_y f)^2 \geq 0. \tag{13}$$

We now restrict ξ to a compact h-independent subset

$$\text{of }]-\infty, 0[, \quad \text{when } Y = \mathbf{R}, \text{ and of }]-1,1[, \quad \text{when } Y = S^1. \tag{14}$$

$V_0 \geq 0$, with equality precisely at y_+, y_-.
We check that

$$d(y_\pm, y) = \pm(f(y) - f(y_\pm)), \quad y \in \mathrm{neigh}\,(y_\pm). \tag{15}$$

$$Q_{\pm} \geq 0,$$

,

$$Q_{+}(e^{-(f-f(y_{+}))/h}) = 0, \text{ near } y_{+},$$

$$Q_{-}(e^{(f-f(y_{-})/h}) = 0, \text{ near } y_{-}$$

Also (12) implies that

$$Q_{\pm} \geq h/C \text{ near } y_{\mp}.$$

By standard arguments, if $C > 0$ is large enough, then for $h > 0$ small enough, Q_{+} has a unique eigenvalue λ_{+} in $[0, h/C[$ which is simple and in addition $\lambda_{+}(h) = \mathcal{O}(e^{-1/(Ch)})$. Similarly Q_{-} has a unique eigenvalue $\lambda_{-}(h) \in [0, h/C[$ which is simple, $= \mathcal{O}(e^{-1/(Ch)})$, and

$$\lambda_{+} = \lambda_{-}.$$

Let e_+, e_- be the corresponding normalized eigenvectors: $Q_\pm e_\pm = \lambda_\pm e_\pm$, and let $F_\pm = \mathbf{C} e_\pm$ denote the associated 1-dimensional eigenspaces. From the intertwining properties,

$$P_\xi Q_+ = Q_- P_\xi, \quad Q_+ P_\xi^* = P_\xi^* Q_-,$$

we see that

$$P_\xi : F_+ \to F_-, \quad P_\xi^* : F_- \to F_+.$$

We have

$$P_\xi e_+ = m_+ e_-, \quad P_\xi^* e_- = m_- e_+, \;\; m_\pm \in \mathbf{C},$$

$$m_+ = (P_\xi e_+ | e_-) = (e_+ | P_\xi^* e_-) = \overline{m}_-. \tag{16}$$

The lowest eigenvalue λ_+ of $Q_+ = P_\xi^* P_\xi$ is given by

$$\lambda_+ = (P_\xi^* P_\xi e_+ | e_+) = (P_\xi e_+ | P_\xi e_+) = |m_+|^2. \tag{17}$$

We show that e_+, e_- are well approximated by

$$u_\pm(y;h) = a_\pm(h)\chi_\pm(y) e^{\mp(f(y) - f(y_\pm))/h}, \tag{18}$$

where

$$a_\pm(h) \sim h^{-1/4}(a_0^\pm + a_-^\pm h + \ldots) \text{ with } \; a_0^\pm > 0 \tag{19}$$

is a normalization factor such that $\|u_\pm\| = 1$. $\chi_\pm$ are suitable cutoffs to large closed interval neighborhoods of $y_\pm$, not containing $y_\mp$.

Further work with exponentially weighted estimates (following [HeSj85]), leads to

$$m_+ = (P_\xi u_+|u_-) + \widetilde{\mathcal{O}}_\eta(e^{-\frac{3}{h}S_0}). \tag{20}$$

We get

$$(P_\xi u_+|u_-) = ha_+(h)\overline{a_-(h)}e^{-S_0/h},$$

hence

$$m_+ = ha_+(h)\overline{a_-(h)}e^{-S_0/h} + \widetilde{\mathcal{O}}_\eta(e^{-3S_0/h}). \tag{21}$$

When $Y = S^1$, we get two terms with opposite signs that can be in competition when $\xi \approx 0$.

4. Related works, prospects

Our results are related to several recent works about tunneling for Schrödinger operators and especially Pauli operators with magnetic fields even though our motivations have been different.
Fournais, Morin, Raymond [FoMoRa23] study the exponentiall small splitting of eigenvalues for the magnetic Schrödinger operator with two magnetic wells with Coulomb type symmetries. Helffer, Kachmar Sundqvist [HeKaSu23] make a similar study and show braidstructure for the lowest eigenvalues in the presence of a triangular symmetry. An earlier related result was obtained by Fefferman, Shapiro, Weinstein [FeShWe22]. See also [HeKa22]

Recent works on the Pauli operator are even closer. The operator

$$\begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}^2 \tag{22}$$

with P as in (1) is a Pauli operator[2]. The works [EkKoPo16, HeSu17, HeSu17a, HeKoSu17, BaTrRaSt18] deal mainly with the bottom of the spectrum. In particular [HeKoSu17] consider the case of magnetic fields ($\Delta\phi$) that change sign.

A possible next step is to deal with (1) when $\phi(x,y)$ depends on both variables x, y. An important step might be to extend the known description of the asymptotic Bergman kernel $K_B(z,w)$ when $\Delta\phi > 0$ to the case when $\Delta\phi$ changes sign along a curve γ . We believe that this can be done for x,y on opposite sides of γ for $\Delta\phi(y) > 0$, $\Delta\phi(x) < 0$ when roughly $\mathrm{dist}\,(x,\gamma) < \mathrm{dist}\,(y,\gamma)$).

[2] We thank J.P. Solovej and M. P. Sundqvist for remarks prompting us to add the exponent 2 in (22).

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