Evolution equations with fractional-order operators

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1. Introduction

Activities with Anders, always kind, wise and helpful:

The period in Copenhagen. Anders held a position in Copenhagen in the mid 70'ies for 1 1/2 years (until he got a lektor position in Lund), giving lectures on hyperbolic problems and other PDE subjects. This was a temporary use of some some positions that the department wanted to fill permanently at a slow rate (Kalle Andersson also took such a job). He and I did our best to spread the word on (then modern) analysis of PDE.

The Øresund seminar. Anders and I cooperated with Lars Hörmander and, in the start, Johannes Sjöstrand, to run the Danish-Swedish Analysis seminar — the Øresund seminar — which started in the mid 80'ies and provided many interesting visitors to both Lund and Copenhagen.

French collaborations. Another activity we had together was the participation in a steering group with French, Swedish and Danish members, which planned the annual meeting in PDE at Saint-Jean-de-Monts on the coast south of Bretagne in France. This was more a formal and honorary thing, not requiring much administrative work, but some funding, as far as I remember. It went on for a large number of years until taken over fully by people at Ecole Polytechnique in Paris.

1. Heat equations

Let $\Omega \subset \mathbb{R}^n$, and let $1 < q < \infty$. Consider a positive-order operator A in $L_q(\Omega)$, e.g. an elliptic differential operator together with a boundary condition $Bu = 0$; then the heat problem is

$$\begin{aligned} \partial_t u + Au &= f \text{ on } \Omega \times I, \quad I = (0, T) \\ Bu &= 0 \text{ for } t \in I, \\ u &= u_0 \text{ for } t = 0. \end{aligned}$$

Example 1. A is an elliptic diff. op., e.g. $= -\Delta$, B is a diff. op. followed by restriction to $\partial\Omega$.

But more general situations are of interest too:

Example 2. $A = P + G$, $B = T$, where P, G and T belong to the Boutet de Monvel calculus. P is a ps.d.o. of order $m \in \mathbb{N}$, G a singular Green operator of order m, T a suitable trace operator. This situation comes up e.g. when the linearized Navier-Stokes problem is reduced to a truly parabolic form (G.-Solonnikov in the 90'ies.)

Example 3. $A = (-\Delta)^a$ with $0 < a < 1$, the fractional Laplacian — or a ps.d.o. generalization of order $2a$. Here $Bu = 0$ is taken to mean that $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Enters in finance, in differential geometry and physics.

An interesting question is to solve the heat equation in L_q-spaces, $1 < q < \infty$. The operator provided with the boundary condition defines a *realization* $\boldsymbol{A}$ in $L_q(\Omega)$; an unbounded closed densely defined operator. $\boldsymbol{A}$ acts like A, $P+G$ or $(-\Delta)^a$ in the three examples, with domain $D(\boldsymbol{A})$ defined by the boundary condition $Bu = 0$, $Tu = 0$ resp. $\operatorname{supp} u \subset \overline{\Omega}$.
The heat problem (with $u(x,0) = 0$ for simplicity) is then formulated as

$$\partial_t u + \boldsymbol{A}u = f \text{ on } \Omega \times I, \quad u|_{t=0} = 0. \qquad (1)$$

Under suitable hypotheses of strong ellipticity, $\boldsymbol{A}$ has its spectrum in a sectorial region in $\mathbb{C}$ ("keyhole region") $\{|\lambda| \le r\} \cup \{|\operatorname{Im}\lambda| \le c \operatorname{Re}\lambda\}$ opening to the right, so the resolvent set contains a region with $\delta > 0$

$$V_{\delta,K} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid \arg\lambda \in [\pi/2 - \delta, 3\pi/2 + \delta], |\lambda| \ge K\}.$$

Then suitable estimates of the resolvent $(\boldsymbol{A} - \lambda)^{-1}$ on $V_{\delta,K}$ lead to solvability theorems for (1).
Example **1** was treated by Seeley '69 in H^s_q-spaces; recall

$$H^s_q(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_q(\mathbb{R}^n)\},$$

for $s \in \mathbb{R}$, $1 < q < \infty$, $\langle\xi\rangle = (|\xi|^2 + 1)^{\frac{1}{2}}$.

Seeley used pseudodifferential machinery (precluding the Boutet de Monvel calculus), for the case where Ω and the coefficients in A and B are C^∞. There are more recent results assuming less smoothness; e.g. Denk, Hieber and Prüss '03, giving a new point of view and method.

Example **2** was treated in H^s_q-spaces in G.-Solonnikov '91 (for $p = 2$), G.-Kokholm '93 and G. '95, in a smooth setting. Nonsmooth generalizations were introduced by Abels '05.

Example **3** will be discussed in this lecture.

An important problem in L_q is to show **maximal L_q-regularity**, namely that (1) for any $f \in L_q(\Omega \times I)$ has a unique solution $u(x,t)$ satisfying

$$\|\partial_t u\|_{L_q(\Omega\times I)} + \|\boldsymbol{A}u\|_{L_q(\Omega\times I)} \le C\|f\|_{L_q(\Omega\times I)}. \qquad (2)$$

It is obtained in the mentioned treatments of Examples **1** and **2**.

To extend Ex. **1** to nonsmooth cases, there has been developed a functional calculus point of view, through works of Da Prato and Grisvard, Lamberton, Dore and Venni, Clément, Prüss, Hieber, Denk, Weiss, Bourgain and others, to link the question of maximal L_q-regularity with the concept of $\mathcal{R}$-**boundedness**, as explained e.g. in Denk-Hieber-Prüss [DHP03]. It is a kind of "boundedness preserved under signed rearrangement".

Definition 1. *Let* $q \in [1, \infty)$. *Denote by* Z_N *the subset of* $\mathbb{R}^N$
$Z_N = \{(z_1, \dots, z_N) \mid z_j \in \{-1, +1\}$ *for all* $j\}$.
Let X *and* Y *be Banach spaces. Let* $q \in [1, \infty)$. *A subset* $\mathcal{T}$ *of the bounded linear operators* $\mathcal{L}(X, Y)$ *is* $\mathcal{R}$*-bounded if there is a constant* $C \geq 0$ *such that for every choice of* $N \in \mathbb{N}$ *and every choice of* $x_1, \dots, x_N$ *in* X *and* $T_1, \dots, T_N$ *in* $\mathcal{T}$,

$$\Big(\sum_{z \in Z_N} \|\sum_{j=1}^{N} z_j T_j x_j\|_Y^q\Big)^{1/q} \leq C\Big(\sum_{z \in Z_N} \|\sum_{j=1}^{N} z_j x_j\|_X^q\Big)^{1/q}. \qquad (3)$$

(There is an equivalent definition drawing on probability formulations.)

The best constant C, denoted $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$, is called the $\mathcal{R}$-bound of $\mathcal{T}$, and the finiteness for one $q \in [1, \infty)$ implies the finiteness for all other $q \in [1, \infty)$. An $\mathcal{R}$-bounded set is norm-bounded. Finite norm-bounded sets are $\mathcal{R}$-bounded. (3) is trivial when X, Y are Hilbert spaces.

Theorem 2. [DHP03] *Let* $1 < q < \infty$. *Problem* (1) *has maximal* L_q*-regularity on* $I = \mathbb{R}_+$ *if and only if the family* $\{\lambda(\boldsymbol{A} - \lambda)^{-1} \mid \lambda \in V_{\delta,0}\}$ *is* $\mathcal{R}$*-bounded in* $\mathcal{L}(L_q(\Omega))$ *for some* $\delta > 0$.

A very useful result, so much more since the $\mathcal{R}$-boundedness property allows suitable perturbations of $\boldsymbol{A}$.

Proposition 3. 1° *Let* $X = L_q(\Omega)$, *and let* $\boldsymbol{A}$ *satisfy*

$$\|\lambda(\boldsymbol{A}-\lambda)^{-1}\|_{\mathcal{L}(X)} \leq C < \infty \text{ for } \lambda \in V_{\delta,K}. \qquad (4)$$

Let S *be defined on* $D(\boldsymbol{A})$, *satisfying*

$$\|Su\|_X \leq \alpha\|\boldsymbol{A}u\|_X + \beta\|u\|_X \text{ for } u \in D(\boldsymbol{A}). \qquad (5)$$

Then when α *is sufficiently small, there exists* $K_1 \geq K$ *such that* $\boldsymbol{A}+S$ *satisfies an inequality* (4) *on* V_{δ,K_1}.
2°. *Assume in addition that* $\{\lambda(\boldsymbol{A}-\lambda)^{-1} \mid \lambda \in V_{\delta,K}\}$ *is* $\mathcal{R}$*-bounded. Then, for sufficiently small* $\alpha > 0$, *there is a* $K_2 \geq K$ *such that* $\{\lambda(\boldsymbol{A}+S-\lambda)^{-1} \mid \lambda \in V_{\delta,K_2}\}$ *is* $\mathcal{R}$*-bounded.*

Here 1° is a well-known standard result; 2° is proved in [DHP03].

Note that $\mathcal{R}$-boundedness of $\{\lambda(\boldsymbol{A}-\lambda)^{-1} \mid \lambda \in V_{\delta,K}\}$ implies that when $\mu > K$, $\mathcal{R}$-boundedness holds for $\{\lambda(\boldsymbol{A}+\mu-\lambda)^{-1} \mid \lambda \in V_{\delta',0}\}$ for some $\delta' > 0$. Then the shifted operator $\boldsymbol{A}+\mu$ has maximal L_q-regularity on $\mathbb{R}_+$, and $\boldsymbol{A}$ itself has it on finite intervals $I = (0,T)$.

Now to Example **3**, where P is of **fractional order**:

$$\begin{aligned} \partial_t u + Pu &= f \text{ on } \Omega \times I, \\ u &= 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\ u|_{t=0} &= 0. \end{aligned} \tag{6}$$

Here $P = (-\Delta)^a$ with symbol $|\xi|^{2a}$, or is more generally a ps.d.o. of order $2a$ $(0 < a < 1)$ with special properties.
Recall that the ps.d.o. P with symbol $p(x, \xi)$ is defined by use of the Fourier transform $\mathcal{F}$: $u(x) \mapsto (\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$, as

$$(Pu)(x) = \mathcal{F}^{-1}_{\xi\to x}\big(p(x,\xi)(\mathcal{F}u)(\xi)\big) = \operatorname{Op}(p)u.$$

Our current hypotheses are: $p(x, \xi)$ is C^τ in x (some $\tau > 2a$) and C^∞ in ξ, satisfying

$$\|D^\alpha_\xi p(\cdot, \xi)\|_{C^\tau(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{2a-|\alpha|} \text{ for } \xi \in \mathbb{R}^n,\ \alpha \in \mathbb{N}_0^n.$$

Moreover, it satisfies for $|\xi| \ge 1$:
(i) p is **classical**, i.e., $p \sim \sum_{j\in\mathbb{N}_0} p_j$ with $p_j(x, t\xi) = t^{2a-j} p_j(x, \xi)$.
(ii) p is **strongly elliptic:** $\operatorname{Re} p_0(x, \xi) \ge c|\xi|^{2a}$ with $c > 0$.
(iii) p is **even**, $p_j(x, -\xi) = (-1)^j p_j(x, \xi)$, all j.

Along with $H^s_q(\mathbb{R}^n) = \{u \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_q(\mathbb{R}^n)\}$, define

$$\overline{H}^s_q(\Omega) = r^+ H^s_q(\mathbb{R}^n), \qquad \dot{H}^s_q(\overline{\Omega}) = \{u \in H^s_q(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}.$$

Here r^+ denotes restriction to Ω; e^+ will indicate extension by 0 from Ω to $\mathbb{R}^n$. (The dot and overline notation stems from Hörmander '85.) For $q = 2$, the index q is omitted.

Let Ω be bounded and $C^{1+\tau}$ with $\tau > 2a$, let $1 < q < \infty$, let P satisfy (i)–(iii) (G. '15 for $\tau = \infty$, Abels-G. '23 for $\tau < \infty$). The Dirichlet realization P_D in $L_q(\Omega)$, acting like r^+P on $\dot{H}^a_q(\overline{\Omega})$, has the domain

$$D(P_D) = \{u \in \dot{H}^a_q(\overline{\Omega}) \mid r^+ Pu \in L_q(\Omega)\} = H^{a(2a)}_q(\overline{\Omega}),$$

where the space $H^{a(2a)}_q(\overline{\Omega})$ is a so-called *a-transmission space*. It is defined in local coordinates from the definition for $\Omega = \mathbb{R}^n_+$ by

$$H^{a(2a)}_q(\overline{\mathbb{R}}^n_+) = \operatorname{Op}((\langle\xi'\rangle + i\xi_n)^{-a}) e^+ \overline{H}^a_q(\mathbb{R}^n_+).$$

Here $H^{a(2a)}_q(\overline{\Omega}) = \dot{H}^{2a}_q(\overline{\Omega})$ if $a < 1/q$; generally $H^{a(2a)}_q(\overline{\Omega}) \subset \dot{H}^{a+1/q}_q(\overline{\Omega})$ $\cap H^{2a}_{q,loc}(\Omega)$ and carries a singularity $\operatorname{dist}(x, \partial\Omega)^a$. We shall apply the heat equation theory to $\boldsymbol{A} = P_D$. The domain is denoted for short

$$H^{a(2a)}_q(\overline{\Omega}) = D_q(\overline{\Omega}).$$

For $q = 2$ it is easy to show, by methods going back to Lions and Magenes '68:

Theorem 4. [G. '18 for $\tau = \infty$, G. '23 for finite $\tau > 2a$.] *For any $f \in L_2(\Omega \times I)$, there is a unique solution $u(x,t) \in \overline{C}^0(\bar{I}; L_2(\Omega))$; it satisfies:*

$$u \in L_2(I; D_2(\overline{\Omega})) \cap \overline{H}^1(I; L_2(\Omega)).$$

There are also results with higher regularity, that we omit here.

Other works have mostly been concerned with $(-\Delta)^a$ and x-independent generalizations. There are results on Schauder estimates and Hölder properties, by e.g. Felsinger and Kassmann '13, Chang-Lara and Davila '14, Jin and Xiong '15; and more precise results on regularity in anisotropic Hölder spaces by Fernandez-Real and Ros-Oton '17, Ros-Oton and Vivas '18. For $P = (-\Delta)^a$, Leonori, Peral, Primo and Soria '15 showed $L_q(I; L_r(\Omega))$ estimates; Biccari, Warma and Zuazua '18 $L_q(I; B^{2a}_{q,r,loc}(\Omega))$-estimates, Choi, Kim and Ryu '23 weighted L_q-estimates. There are results on $\mathbb{R}^n$ with x-dependence by Dong, Jung and Kim '23.

We showed an optimal L_q-result in '18 under an extra hypothesis:

(iv) p is x-**independent**, **real** and homogeneous for $\xi \neq 0$.

Theorem 5. *Assume* (iv) *in addition to* (i)–(iii). *Then when* $1 < q < \infty$, (6) *has for any* $f \in L_q(\Omega \times I)$*, a unique solution* $u(x,t) \in \overline{C}^0(I; L_q(\Omega))$*; it satisfies:*

$$u \in L_q(I; D_q(\overline{\Omega})) \cap \overline{H}^1_q(I; L_q(\Omega)).$$

This is maximal L_q-regularity.
Proved for $\tau = \infty$ in '18, extended to finite τ in '23. The proof uses that the sesquilinear form obtained by closure on $\dot{H}^a(\overline{\Omega})$ of

$$s(u,v) = \int_\Omega Pu\,\bar{v}\,dx, \quad u,v \in C_0^\infty(\Omega), \qquad (7)$$

is for real u, v a so-called Dirichlet form, as in books of Davies '89, Fukushima, Oshima and Takeda '94. Then P_D is what is called sub-Markovian, and by a result of Lamberton '87, the heat problem (6) has maximal L_q-regularity.

Currently, I have been trying for a long time to weaken hypothesis (iv) — to extend the result to suitable variable-coefficient operators, by perturbation and localization arguments. Lately, I have had a cooperation with Helmut Abels on this, and we have just recently managed to show:

Theorem 6. *Let Ω be bounded with $C^{1+\tau}$-boundary, $\tau > 2a$, and let $1 < q < \infty$. Besides our hypotheses* (i)–(iii), *assume that the principal symbol $p_0(x_0, \xi)$ is real positive at each* **boundary point** *$x_0 \in \partial\Omega$. Then there are constants $\delta > 0$, $K \geq 0$ such that $\{\lambda(P_D - \lambda)^{-1} \mid \lambda \in V_{\delta,K}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L_q(\Omega))$.*

The proof involves a comparison, at each boundary point $x_0 \in \partial\Omega$, of P with the constant-coefficient operator $\overline{P} = \mathrm{Op}(p_0(x_0, \xi))$ in an auxiliary bounded domain Σ coinciding with Ω in a small ball around x_0, where perturbation estimates and blow-up techniques can be applied.
This leads to the desired heat equation result:

Theorem 7. *Hypotheses as in Theorem* 6*. Then for any $f \in L_q(\Omega \times I)$, the heat equation* (6) *has a unique solution $u(x,t) \in \overline{C}^0(\overline{I}; L_q(\Omega))$ satisfying*

$$u \in L_q(I; D_q(\overline{\Omega})) \cap \overline{H}_q^1(I; L_q(\Omega)).$$

This is a first result on maximal L_q-regularity for variable-coefficient nonselfadjoint ps.d.o. boundary problems of fractional order.

4. Nonhomogeneous problems

Nonhomogeneous boundary problems can also be considered. There is a local nonzero Dirichlet boundary condition associated with P, namely the assignment of $\gamma_0(u/d^{a-1})$; here $d(x) = \operatorname{dist}(x, \partial\Omega)$. The problem

$$Pu = f \text{ in } \Omega, \quad \gamma_0(u/d^{a-1}) = \varphi, \quad \operatorname{supp} u \subset \overline{\Omega}, \qquad (8)$$

had good solvability properties for given $f \in L_q(\Omega)$, $\varphi \in B_q^{a+1-1/q}(\partial\Omega)$, when u is sought in the $(a-1)$-transmission space $H_q^{(a-1)(2a)}(\overline{\Omega})$. This is a larger space than $D_q(\overline{\Omega}) = H_q^{a(2a)}(\overline{\Omega})$, satisfying

$$H_q^{a(2a)}(\overline{\Omega}) = \{u \in H_q^{(a-1)(2a)}(\overline{\Omega}) \mid \gamma_0(u/d^{a-1}) = 0\}.$$

So the case $\varphi = 0$ in (8) is the homogeneous Dirichlet problem.
One has that $H_q^{(a-1)(2a)}(\overline{\Omega}) \subset L_q(\Omega)$ when $q < \frac{1}{1-a}$. We assume this for the nonhomogeneous heat problem:

$$\begin{aligned} \partial_t u + Pu &= f \text{ on } \Omega \times I, \\ \gamma_0(u/d^{a-1}) &= \psi \text{ on } \partial\Omega \times I, \\ u &= 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \qquad (9) \\ u\big|_{t=0} &= 0. \end{aligned}$$

Here we can show:

Theorem 8. *In addition to the hypotheses of Theorem* 6, *assume that* $\tau > 2a+1$ *and* $q < \frac{1}{1-a}$. *Then* (9) *has for* $f \in L_q(\Omega \times I)$, $\psi \in L_q(I; B_q^{a+1-1/q}(\partial\Omega)) \cap \dot{H}_q^1(\overline{I}; B_q^{\varepsilon}(\partial\Omega))$ *a unique solution* $u(x,t)$ *satisfying*

$$u \in L_q(I; H_q^{(a-1)(2a)}(\overline{\Omega})) \cap \overline{H}_q^1(I; L_q(\Omega)).$$

Let us finally mention that one can also use the resolvent estimates (just in uniform norms) to show results in other function spaces. For example, by a strategy of Amann '97:

Theorem 9. *Hypotheses as in Theorem* 6. *Let* s *be noninteger* > 0. *For any* $f \in \dot{C}^s(\overline{\mathbb{R}}_+; L_q(\Omega))$ *there is a unique solution* $u \in \dot{C}^s(\overline{\mathbb{R}}_+; D_q(\overline{\Omega}))$, *and there holds*

$$f(x,t) \in \dot{C}^s(\overline{\mathbb{R}}_+; L_q(\Omega)) \iff u(x,t) \in \dot{C}^s(\overline{\mathbb{R}}_+; D_q(\overline{\Omega})) \cap \dot{C}^{s+1}(\overline{\mathbb{R}}_+; L_q(\Omega)).$$

Here $\dot{C}^s(\overline{\mathbb{R}}_+; X)$ stands for functions in $C^s(\mathbb{R}; X)$ vanishing on $\mathbb{R}_-$.

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Dear Anders!

Congratulations with the 80 years!