HECKE FIELDS OF ANALYTIC FAMILIES
OF MODULAR FORMS

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Fix a prime p, and put $\mathbf{p} = 4$ if $p = 2$ and $\mathbf{p} = p$ otherwise. For a Hecke eigenform $f \in S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ ($p \nmid N, r \geq 0$) and a subfield K of $\mathbb{C}$, the Hecke field $K(f)$ inside $\mathbb{C}$ is generated over K by the eigenvalues $a_n = a(n, f)$ of f for the Hecke operators $T(n)$ for all n. Then $\mathbb{Q}(f)$ is a finite extension of $\mathbb{Q}$ sitting inside the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$. Let $\Gamma = 1 + \mathbf{p}\mathbb{Z}_p$, which is a maximal torsion-free subgroup of $\mathbb{Z}_p^\times$. We choose and fix a generator $\gamma := (1 + \mathbf{p}) \in \Gamma$ so that $\Gamma = \gamma^{\mathbb{Z}_p}$ and identify the Iwasawa algebra $\Lambda = W[[\Gamma]]$ with the power series ring $W[[x]]$ by $\Gamma \ni \gamma \mapsto (1 + x)$ (for a discrete valuation ring W finite flat over $\mathbb{Z}_p$). A p-adic slope 0 analytic family of eigenforms $\mathcal{F} = \{f_P | P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)\}$ is indexed by points of $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$, where $\mathrm{Spec}(\mathbb{I})$ is a finite flat irreducible covering of $\mathrm{Spec}(\Lambda)$. For each $P \in \mathrm{Spec}(\mathbb{I})$, f_P is a p-adic modular form of slope 0 of level Np^∞ for a fixed prime to p-level N. The family is called analytic because $P \mapsto a(n, f_P)$ is a p-adic analytic function on $\mathrm{Spec}(\mathbb{I})$. We call P *arithmetic* of weight $k = k(P) \in \mathbb{Z}$ with character $\varepsilon_P : \Gamma \to \mu_{p^\infty}(\mathbb{C}_p)$ if P contains $(1 + x - \varepsilon_P(\gamma)\gamma^k) \in \Lambda$ and $k(P) \geq 2$. If P is arithmetic, f_P is known to be a p-stabilized classical Hecke eigenform and has Neben character ψ_P whose restriction to Γ is given by ε_P. We write $p^{r(P)}$ for the order of ε_P (then, the classical form f_P has level $Np^{r(P)}\mathbf{p}$). In order to make the introduction succinct, we put off, to Section 1, recalling the theory of analytic families of eigenforms including the definition and necessary properties of CM families. We define the following Hecke fields out of $\mathcal{F}$:

(V) For a fixed level $Np^r\mathbf{p}$ ($0 \leq r \leq \infty$), $\mathbb{Q}_{V,r}(\mathcal{F})$ is the composite of $\mathbb{Q}(f_P)$ for all arithmetic $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $k(P) \geq 2$ and ε_P factoring through Γ/Γ^{p^r}.

(H) For a fixed weight $k \geq 2$, $\mathbb{Q}_{H,k}(\mathcal{F})$ is the composite of $\mathbb{Q}(f_P)$ for all arithmetic $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $k(P) = k$.

Here the composite is taken in the algebraic closure $\overline{\mathbb{Q}}$ inside $\mathbb{C}$. If $r = 0$, periodically in $k(P)$, f_P is old at p associated to a unique new form f_P° of level N prime to p; so, putting $\mathcal{F}^\circ = \{f_P^\circ\}$ for such arithmetic Ps, we can also define $\mathbb{Q}_V(\mathcal{F}^\circ)$ as the composite of $\mathbb{Q}(f_P^\circ)$. Abusing the notation, we put $\mathbb{Q}_{V,-1}(\mathcal{F}) := \mathbb{Q}_V(\mathcal{F}^\circ)$.

Once L. Clozel asked me if (or when) the Hecke field $\mathbb{Q}_{V,r}(\mathcal{F})$ for a finite r is a finite extension of $\mathbb{Q}$. At the time, for a lack of examples, my answer was "probably" that it is finite if and only if the family contains a CM theta series (i.e., a binary theta series) of weight $k(P) \geq 2$. If the family contains a binary theta series of weight $k \geq 2$ of an imaginary quadratic field M, all forms have CM by the same (fixed) imaginary quadratic field M, and hence $\mathbb{Q}_{V,r}(\mathcal{F})$ $(r < \infty)$ is contained in a finite extension of the imaginary quadratic field M (see Corollary 4.2). The following conjecture is anybody's guess (and perhaps Clozel must have had the same thought):

Conjecture. *The following three assertions are equivalent:*
   (1) *The field $\mathbb{Q}_{V,r}(\mathcal{F})$ $(r < \infty)$ is a finite extension of $\mathbb{Q}$.*
   (2) *The family $\mathcal{F}$ contains a theta series of weight $k \geq 2$ of an imaginary quadratic field M.*
   (3) *The family $\mathcal{F}$ is made up of theta series of a fixed imaginary quadratic field M.*

As already mentioned, the assertions (2) and (3) are equivalent; so, hereafter, we only state (2), which implies (3). We tentatively recall that $\mathcal{F}$ has CM (complex multiplication) by M if the above (2) is satisfied (see (CM1–3) in Section 1 for a precise definition of CM families). When $N = 1$ and $r = -1$, Maeda made a conjecture predicting that $d(P) = [\mathbb{Q}(f_P^\circ); \mathbb{Q}] = \dim_{\mathbb{C}} S_{k(P)}(SL_2(\mathbb{Z}))$ and the Galois group of the Galois closure of $\mathbb{Q}(f_P^\circ)_{/\mathbb{Q}}$ is isomorphic to the symmetric group of $d(P)$ letters (see [HM, Conjecture 1.2]; see also Corollary 6.3 and Conjecture 8.1 in the text). Since there is no CM family of prime-to-p level $N = 1$, his conjecture implies our conjecture if $N = 1$.

Write A_P for the abelian variety (defined over $\mathbb{Q}$) associated to f_P at an arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ with $k(P) = 2$ and $r(P) \leq r$ (defined by Shimura [IAT, Theorem 7.14]). Our result towards the conjecture is as follows.

Vertical Theorem. *Suppose $p > 2$ and that we have an arithmetic point $P \in \mathrm{Spec}(\mathbb{I})$ of weight $k(P) = 2$ with $r(P) \leq r$ such that A_P has good ordinary reduction over $\mathbb{Z}_p[\mu_{p^{r+1}}]$. Then the field $\mathbb{Q}_{V,r}(\mathcal{F})$ for a bounded p-power level Np^{r+1} $(r \geq 0)$ is a finite extension of $\mathbb{Q}$ if and only if $\mathcal{F}$ contains a theta series of weight $k \geq 2$ of an imaginary quadratic field.*

The abelian variety A_P has good ordinary reduction at p if one of the following three conditions is satisfied (see Corollary 2.2 in the text and [GME, Section 4.2]):
   • A_P is an elliptic curve with good reduction at p,
   • A_P has good reduction at p and $\mathbb{Q}(f_P)/\mathbb{Q}$ is unramified at p,
   • A_P has good reduction at p and $a(p, f_P)$ generates the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}(f_P) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

The main point of the assumption is the "ordinarity" (not really the good reduction), and we analyze closely the case where A_P has multiplicative reduction at p in Corollary 7.3. By the above facts, for any family $\mathcal{F}$ giving a non-CM elliptic curve of ordinary good reduction at p, $\mathbb{Q}_{V,r}(\mathcal{F})$ is an infinite extension. In the text, we prove a stronger version of the theorem (Theorem 3.2) telling us that any non-CM family $\mathcal{F}$ having an arithmetic P associated to an "ordinary" motive potentially crystalline (at p) has infinite $\mathbb{Q}_{V,r}(\mathcal{F})$; so, if $\mathbb{Q}(f_P^\circ) = \mathbb{Q}$ with $k(P) > 2$, $\mathbb{Q}_{V,r}(\mathcal{F})$ has infinite degree over $\mathbb{Q}$. In particular, for the p-adic family $\mathcal{F}_\Delta$ containing Ramanujan's Δ-function, we have $[\mathbb{Q}_{V,r}(\mathcal{F}_\Delta) : \mathbb{Q}] = \infty$ (though we do not know

if $\mathcal{F}_\Delta$ always has a weight 2 point P with A_P having ordinary good reduction). For a non-CM family $\mathcal{F}$, we conjecture that $\limsup_{P, f_P \in \mathcal{F}}[\mathbb{Q}(f_P) : \mathbb{Q}] = \infty$ for P running over arithmetic points P of bounded level (Conjecture 8.1). If we do not bound the level exponent $r(P)$, $\limsup_{P, f_P \in \mathcal{F}}[K(f_P) : K] = \infty$ for $K = \mathbb{Q}(\mu_{p^\infty})$ as in the following theorem (see also Corollary 6.3).

There is a horizontal version. We have $T(l)^2 - T(l^2) = l \cdot T(l, l)$ for primes l outside the level (for the central action $T(l, l)$ in [IAT, Theorem 3.24]), and $f_P|T(l, l) = \varepsilon_P(l)l^{k-2}f_P$ if $l \in \Gamma$. Thus $\mathbb{Q}_{H,k}(\mathcal{F})$ contains $\mathbb{Q}(\mu_{p^\infty})$. Again, if the family contains a CM theta series of weight $k \geq 2$ of a quadratic field M, all forms have CM by the same imaginary quadratic field M, and hence $\mathbb{Q}_{H,k}(\mathcal{F})$ $(k \geq 2)$ is contained in a finite extension of $M(\mu_{p^\infty})$ (Corollary 4.2). The horizontal theorem is more complete as it does not require the existence of P with ordinary A_P.

Horizontal Theorem. *Suppose $p > 2$. The Hecke field $\mathbb{Q}_{H,k}(\mathcal{F})$ for a fixed weight $k \geq 2$ is a finite extension of $K := \mathbb{Q}(\mu_{p^\infty})$ if and only if $\mathcal{F}$ contains a theta series of weight $k \geq 2$ of an imaginary quadratic field. Moreover, for a non-CM family $\mathcal{F}$, we have $\limsup_{P, f_P \in \mathcal{F}}[K(f_P) : K] = \infty$ for P running over infinitely many arithmetic points P of fixed weight (i.e., $k(P) = k \geq 2$).*

In this paper, we study these assertions and their variants. We will prove a stronger version of the horizontal theorem (Theorem 3.3) which implies not only the horizontal theorem but also the vertical theorem. The proof of the above theorems is based on the elementary finiteness of Weil l-numbers of given weight in $\mathbb{Q}[\mu_{p^\infty}]$ up to multiplication by roots of unity and on a simple analysis of the prime factorization of Weil numbers in Hecke fields. The same finiteness of Weil l-numbers of given weight (up to roots of unity) in the maximal abelian extension of $\mathbb{Q}$ is a nontrivial fact (a conjecture of Robinson proven by Loxton [Lo]; see Lemma 2.3 in the text).

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CONTENTS

1. ANALYTIC FAMILIES OF HECKE EIGENFORMS

Fix a field embedding $\overline{\mathbb{Q}} \stackrel{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ and a positive integer N prime to p. Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$. A p-adic

analytic family $\mathcal{F}$ of modular forms is defined with respect to the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We write $|\alpha|_p$ for the p-adic absolute value (with $|p|_p = 1/p$) induced by i_p. We also fix a field embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ inducing the inclusion on $\overline{\mathbb{Q}} \subset \mathbb{C}$. As a base ring, we take a (sufficiently large) discrete valuation ring $W \subset \overline{\mathbb{Q}}_p$ finite flat over the p-adic integer ring $\mathbb{Z}_p$. Recall that $\mathbf{p} = p$ if p is odd and $\mathbf{p} = 4$ otherwise. Take a Dirichlet character $\psi : (\mathbb{Z}/Np^r\mathbf{p}\mathbb{Z})^\times \to W^\times$ with $(p \nmid N, r \geq 0)$, and consider the space of elliptic cusp forms $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ with character ψ as defined in [IAT, (3.5.4)]. Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra over $\mathbb{Z}[\psi]$ is the subalgebra of the linear endomorphism algebra of $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ generated by Hecke operators $T(n)$:

$$h = \mathbb{Z}[\psi][T(n) | n = 1, 2, \cdots] \subset \operatorname{End}(S_k(\Gamma_0(Np^r\mathbf{p}), \psi)),$$

where $T(n)$ is the Hecke operator as in [IAT, §3.5]. We put $h_{k,\psi} = h_{k,\psi/W} = h \otimes_{\mathbb{Z}[\psi]} W$. When we need to indicate that our $T(l)$ is the Hecke operator of a prime factor l of $Np^r\mathbf{p}$, we write it as $U(l)$, since $T(l)$ acting on a subspace $S_k(\Gamma_0(N'), \psi) \subset S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$ of level N' prime to l does not coincide with $U(l)$ on $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$. The ordinary part $\mathbf{h}_{k,\psi/W} \subset h_{k,\psi/W}$ is the maximal ring direct summand on which $U(p)$ is invertible. We write e for the idempotent of $\mathbf{h}_{k,\psi/W}$, and hence $e = \lim_{n\to\infty} U(p)^{n!}$ under the p-adic topology of $h_{k,\psi/W}$. By the fixed embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, the idempotent e not only acts on the space of modular forms with coefficients in W but also on the classical space $S_k(\Gamma_0(Np^r\mathbf{p}), \psi)$. We write the image of the idempotent as S_k^{ord}.

Fix ψ, and assume now that $\psi_p = \psi|_{\mathbb{Z}_p^\times}$ has conductor at most $\mathbf{p}$ and $\psi(-1) = 1$. Let ω be the modulo $\mathbf{p}$ Teichmüller character (so, if $p = 2$, ω is the unique non-trivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$). Recall that the multiplicative group $\Gamma := 1 + \mathbf{p}\mathbb{Z}_p$ in $\mathbb{Z}_p^\times$ and its topological generator $\gamma = 1 + \mathbf{p}$. Then the Iwasawa algebra $\Lambda = W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$ is identified with the power series ring $W[[x]]$ by a W-algebra isomorphism sending $\gamma \in \Gamma$ to $1 + x$. As constructed in [H86a], [H86b] and [GME], we have a unique 'big' ordinary Hecke algebra $\mathbf{h}$. The algebra $\mathbf{h}$ is characterized by the following two properties (called Control theorems; see [H86a, Theorem 3.1, Corollary 3.2] and [H86b, Theorem 1.2] for $p \geq 5$ and [GME, Theorem 3.2.15 and Corollary 3.2.18] for general p):

(C1) $\mathbf{h}$ is free of finite rank over Λ equipped with $T(n) \in \mathbf{h}$ for all $1 \leq n \in \mathbb{Z}$ (so $U(l)$ for $l | Np$),

(C2) if $k \geq 2$ and $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,

$$\mathbf{h}/(1 + x - \varepsilon(\gamma)\gamma^k)\mathbf{h} \cong \mathbf{h}_{k,\varepsilon\psi_k} \ (\gamma = 1 + p) \text{ for } \psi_k := \psi\omega^{-k},$$

sending $T(n)$ to $T(n)$ (and $U(l)$ to $U(l)$ for $l | Np$).

In the sequel, we often make use of another variable $X = \gamma^{-1}(1 + x) - 1$. We still have $\Lambda = W[[X]]$. The prime ideal $((1 + x) - \gamma)$ is equal to (X); so, if (C2) were valid for $k = 1$, $\mathbf{h}/X\mathbf{h}$ would have been the Hecke algebra of weight 1; so, we call the variable X of Λ the variable *centered* at weight 1.

Let $\operatorname{Spec}(\mathbb{I})$ be a reduced irreducible component $\operatorname{Spec}(\mathbb{I}) \subset \operatorname{Spec}(\mathbf{h})$. Write $a(n)$ for the image of $T(n)$ in $\mathbb{I}$ (so, $a(p)$ is the image of $U(p)$). If a point P of $\operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ kills $(1 + x - \varepsilon(\gamma)\gamma^k) = (1 + X - \varepsilon(\gamma)\gamma^{k-1})$ with $2 \leq k \in \mathbb{Z}$ (i.e.,

$P((1 + x - \varepsilon(\gamma)\gamma^k)) = 0)$, we call it an *arithmetic* point and we write $\varepsilon_P = \varepsilon$, $k(P) = k \geq 2$ and $p^{r(P)}$ for the order of ε_P. If P is arithmetic, by (C2), we have a Hecke eigenform $f_P \in S_k(\Gamma_0(Np^{r(P)}\mathbf{p}), \varepsilon\psi_k)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := P(a(n)) \in \overline{\mathbb{Q}}_p$ for all n. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{f_P | \text{arithemtic } P \in \mathrm{Spec}(\mathbb{I})\}$ of Hecke eigenforms. We define a *p-adic analytic family of slope* 0 (with coefficients in $\mathbb{I}$) to be the family as above of Hecke eigenforms associated to an irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. We call this family slope 0 because $|a_P(p)|_p = 1$ for the p-adic absolute value $|\cdot|_p$ of $\overline{\mathbb{Q}}_p$ (it is also often called an ordinary family). We call this family analytic because the Hecke eigenvalue $a_P(n)$ for $T(n)$ is given by an analytic function $a(n)$ on (the rigid analytic space associated to) the p-profinite formal spectrum $\mathrm{Spf}(\mathbb{I})$. Identify $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{W\text{-alg}}(\mathbb{I}, \overline{\mathbb{Q}}_p)$ so that each element $a \in \mathbb{I}$ gives rise to a "function" $a : \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ whose value at $(P : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is $a_P := P(a) \in \overline{\mathbb{Q}}_p$. Then a is an analytic function of the rigid analytic space associated to $\mathrm{Spf}(\mathbb{I})$. Taking a finite covering $\mathrm{Spec}(\widetilde{\mathbb{I}})$ of $\mathrm{Spec}(\mathbb{I})$ with surjection $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p) \twoheadrightarrow \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, abusing slightly the definition, we may regard the family $\mathcal{F}$ as being indexed by arithmetic points of $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$, where arithmetic points of $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ are made up of the points above arithmetic points of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$. The choice of $\widetilde{\mathbb{I}}$ is often the normalization of $\mathbb{I}$ or the integral closure of $\mathbb{I}$ in a finite extension of the quotient field of $\mathbb{I}$.

Each (reduced) irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$ has a 2-dimensional semi-simple (absolutely irreducible) continuous representation $\rho_{\mathbb{I}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in the quotient field of $\mathbb{I}$ (see [H86b]). The representation $\rho_{\mathbb{I}}$ restricted to the p-decomposition group D_p is reducible with unramified *quotient* character (e.g., [GME, §4.2]). We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D_p. As is well known now (e.g., [GME, §4.2]), $\rho_{\mathbb{I}}$ is unramified outside Np and satisfies, for primes $l \nmid Np$,

(Gal) $\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) = a(l)$, $\rho_{\mathbb{I}}^{ss}([\gamma^s, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} (1+X)^s & 0 \\ 0 & 1 \end{smallmatrix} \right)$, $\rho_{\mathbb{I}}^{ss}([p, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & a(p) \end{smallmatrix} \right)$,

where $\gamma^s = (1 + \mathbf{p})^s = \sum_{n=0}^{\infty} \binom{s}{n} \mathbf{p}^n \in \mathbb{Z}_p^{\times}$ for $s \in \mathbb{Z}_p$ and $[x, \mathbb{Q}_p]$ is the local Artin symbol.

By (Gal) and the Chebotarev density theorem, $\mathrm{Tr}(\rho_{\mathbb{I}})$ has values in $\mathbb{I}$; so, $P \circ \mathrm{Tr}(\rho_{\mathbb{I}}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_p$ ($P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$) gives rise to a pseudo-representation of Wiles (e.g., [MFG, §2.2]). Then by a theorem of Wiles, we can make a unique 2-dimensional semi-simple continuous representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$ unramified outside Np with $\mathrm{Tr}(\rho_P(Frob_l)) = a_P(l)$ for all primes l outside Np (though the construction of ρ_P does not require the technique of pseudo-representation and was known before the invention of the technique; see [MW, §9, Proposition 1]). This is the Galois representation associated to the Hecke eigenform f_P (constructed earlier by Shimura and Deligne) if P is arithmetic (e.g., [GME, §4.2]).

A component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{I}^{\times}$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$. We also say that $\mathbb{I}$ has *complex multiplication* if $\mathbb{I}$ is a CM component. In this case, we call the corresponding family $\mathcal{F}$ a CM family (or we say that $\mathcal{F}$ has complex multiplication). If $\mathcal{F}$ is a CM family associated to $\mathbb{I}$ with $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \chi$, then χ is a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which cuts out an imaginary quadratic field M, i.e., $\chi = \left(\frac{M/\mathbb{Q}}{} \right)$. Write $\widetilde{\mathbb{I}}$ for the integral closure of Λ inside the quotient field of $\mathbb{I}$. The following three conditions

are known to be equivalent:

(CM1) $\mathcal{F}$ *has CM with* $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{}\right)$ *($\Leftrightarrow \rho_{\mathbb{I}} \cong \operatorname{Ind}_M^{\mathbb{Q}} \widehat{\lambda}$ for a character $\widehat{\lambda}$ of* $\operatorname{Gal}(\overline{\mathbb{Q}}/M)$ *with values in $\widetilde{\mathbb{I}}^{\times}$).*

(CM2) *For all arithmetic P of $\operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, f_P is a binary theta series of the norm form of $M/\mathbb{Q}$.*

(CM3) *For some arithmetic P of $\operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, f_P is a binary theta series of the norm form of $M/\mathbb{Q}$.*

Indeed, (CM1) is equivalent to $\rho_{\mathbb{I}} \cong \operatorname{Ind}_M^{\mathbb{Q}} \widehat{\lambda}$ for a character $\widehat{\lambda} : \operatorname{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^{\times}$ unramified outside Np (e.g., [MFG, Lemma 2.15]). Since the characteristic polynomial of $\rho_{\mathbb{I}}(\sigma)$ has coefficients in $\mathbb{I}$, its eigenvalues fall in $\widetilde{\mathbb{I}}$; so, the character $\widehat{\lambda}$ has values in $\widetilde{\mathbb{I}}^{\times}$ (see [H86c, Corollary 4.2]). Then by (Gal), $\widehat{\lambda}_P = P \circ \widehat{\lambda} : \operatorname{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^{\times}$ for an arithmetic $P \in \operatorname{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ is a locally algebraic p-adic character, which is the p-adic avatar of a Hecke character $\lambda_P : M_{\mathbb{A}}^{\times}/M^{\times} \to \mathbb{C}^{\times}$ of type A_0 of the quadratic field $M_{/\mathbb{Q}}$. Then by the characterization (Gal) of $\rho_{\mathbb{I}}$, f_P is the theta series with q-expansion $\sum_{\mathfrak{a}} \lambda_P(\mathfrak{a}) q^{N(\mathfrak{a})}$, where $\mathfrak{a}$ runs over all integral ideals of M. By $k(P) \geq 2$ (and (Gal)), M has to be an imaginary quadratic field in which p is split (as the holomorphic binary theta series of real quadratic fields are limited to weight 1; cf. [MFM, §4.8]). This shows that (CM1)$\Rightarrow$(CM2)$\Rightarrow$(CM3). If (CM2) is satisfied, we have an identity

$$\operatorname{Tr}(\rho_{\mathbb{I}}(Frob_l)) = a(l) = \chi(l)a(l) = \operatorname{Tr}(\rho_{\mathbb{I}} \otimes \chi(Frob_l))$$

with $\chi = \left(\frac{M/\mathbb{Q}}{}\right)$ for all primes l outside Np. By Chebotarev density, we have $\operatorname{Tr}(\rho_{\mathbb{I}}) = \operatorname{Tr}(\rho_{\mathbb{I}} \otimes \chi)$, and we get (CM1) from (CM2) as $\rho_{\mathbb{I}}$ is semi-simple. If a component $\operatorname{Spec}(\mathbb{I})$ contains an arithmetic point P with theta series f_P as above of $M/\mathbb{Q}$, either $\mathbb{I}$ is a CM component or otherwise P is in the intersection in $\operatorname{Spec}(\mathbf{h})$ of a component $\operatorname{Spec}(\mathbb{I})$ not having CM by M and another component having CM by M (as all families with CM by M are made up of theta series of M by the construction of CM components in [H86a, §7]). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\operatorname{Spec}(\mathbf{h})$ (i.e., the reduced part of the localization $\mathbf{h}_P$ is étale over Λ_P for any arithmetic point $P \in \operatorname{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$; see [HMI, Proposition 3.78]). Thus (CM3) implies (CM2). We call a binary theta series of the norm form of an imaginary quadratic field a *CM theta series*.

2. Weil numbers

Since $\overline{\mathbb{Q}}$ sits inside $\mathbb{C}$, it has "the" complex conjugation c. For a prime l, a Weil l-number $\alpha \in \overline{\mathbb{Q}}$ of integer weight $k \geq 0$ is defined by the following two properties:

(1) α is an algebraic integer;

(2) $|\alpha^{\sigma}| = l^{k/2}$ for all $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the complex Archimedean absolute value $|\cdot|$.

Note that $\mathbb{Q}(\alpha)$ is in a CM field finite over $\mathbb{Q}$ (e.g., [Ho, Proposition 4]).

Proposition 2.1. *Let α be a Weil p-number of weight k. Write R (resp. R^+) for the integer ring of $\mathbb{Q}(\alpha)$ (resp. of the maximal real subfield of $\mathbb{Q}(\alpha)$). Suppose that one of the following two conditions holds:*

(1) $k = 1$ *and the prime p is unramified in $\mathbb{Q}(\alpha)/\mathbb{Q}$,*

(2) $k \geq 1$ and α generates over $R_p^+ := R^+ \otimes_{\mathbb{Z}} \mathbb{Z}_p$ the integral closure of $\mathbb{Z}_p$ in $R \otimes_{\mathbb{Z}} \mathbb{Q}_p$.

If we have $|\alpha|_p = |i_p(\alpha)|_p = 1$ for the embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ (fixed in the introduction), then the set of all p-adic places of $\mathbb{Q}(\alpha)$ splits into a disjoint union $\Sigma_p \sqcup \Sigma_p^c$ for the complex conjugation c such that $\mathfrak{P}|(\alpha) \Leftrightarrow \mathfrak{P} \in \Sigma_p^c$; indeed, $(\alpha) = \prod_{\mathfrak{P} \in \Sigma_p^c} \mathfrak{P}^{e(\mathfrak{P})k}$ for the ramification index $e(\mathfrak{P})$ of $\mathfrak{P}/p$.

Proof. Write $K = \mathbb{Q}(\alpha)$, and take its Galois closure K^{gal} over $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$. Since K is a CM field, the complex conjugation c is in the center Z of $G = \mathrm{Gal}(K^{gal}/\mathbb{Q})$ (see [IAT, Proposition 5.11]). Since decomposition groups of a prime in G are conjugate to each other, c is either inside all of them or outside all of them. Let $\Sigma := \Sigma_\alpha = \{\sigma : K \hookrightarrow \mathbb{C}_p \big| |\alpha^\sigma|_p = 1\}$ and write Σ_p for the set of prime ideals $\mathfrak{P}$ of K such that $\mathfrak{P}^c = \{\xi \in K \big| |\sigma(\xi)|_p < 1\}$. By assumption, $\Sigma \neq \emptyset$; so, no p-decomposition groups in G contain c. Therefore, any prime factor of p in the maximal totally real field K_+ splits completely in K. Write the prime decomposition of the principal ideal (α) as $(\alpha) = \prod_{\mathfrak{P}|p} \mathfrak{P}^{\varepsilon(\mathfrak{P})}$. Since $\sigma(\alpha)c\sigma(\alpha) = p^k$ implies $\varepsilon(\mathfrak{P}) + \varepsilon(\mathfrak{P}^c) = k$, under the assumption (1), unramifiedness of p tells us that $0 \leq \varepsilon(\mathfrak{P}), \varepsilon(\mathfrak{P}^c) \leq 1$, and hence one of them vanishes. Thus $\{\mathfrak{P}|p\} = \Sigma_p \sqcup \Sigma_p^c$, and $(\alpha) = \prod_{\mathfrak{P} \in \Sigma_p^c} \mathfrak{P}$.

Now assume (2). Decompose $(\alpha) = \prod_{\mathfrak{P}|p} \mathfrak{P}^{\varepsilon(\mathfrak{P})}$. Write $R_{\mathfrak{P}}$ for the $\mathfrak{P}$-adic completion of R. Then for $\mathfrak{p} = \mathfrak{P} \cap R_+$, $R_{\mathfrak{p}} = R_{\mathfrak{P}} \times R_{\mathfrak{P}^c}$. If $\varepsilon(\mathfrak{P}) > 0$ and $\varepsilon(\mathfrak{P}^c) > 0$, $\alpha \equiv c(\alpha) \mod \mathfrak{p}R_{\mathfrak{p}}$ inside $R_{\mathfrak{p}}$, which implies that $R_p^+[\alpha]$ in R_p is contained in the order of conductor $\mathfrak{p}$; so, α cannot span the integral closure R_p of $\mathbb{Z}_p$ in $\mathbb{Q}(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. In other words, under (2), $\varepsilon(\mathfrak{P})\varepsilon(\mathfrak{P}^c) = 0$, which implies that $\{\mathfrak{P}|p\} = \Sigma_p \sqcup \Sigma_p^c$ and $(\alpha) = \prod_{\mathfrak{P} \in \Sigma_p^c} \mathfrak{P}^{e(\mathfrak{P})k}$, as desired. $\square$

Corollary 2.2. *Let the notation and assumption be as in the above proposition. Let V be a discrete valuation ring finite flat over $\mathbb{Z}_p$ with residue field $\mathbb{F}_p$. Let M be a CM field containing $\mathbb{Q}(\alpha)$, and let $A_{/V}$ be a polarized abelian scheme of dimension $[M : \mathbb{Q}]$ with a ring embedding $\theta : M \hookrightarrow \mathrm{End}(A_{/V}) \otimes_{\mathbb{Z}} \mathbb{Q}$ sending the identity to the identity whose image is stable under the Rosati involution. Put $\widetilde{A} = A \times_V \mathbb{F}_p$ (the reduction modulo p). If $F^k - \theta(\alpha) \in \mathrm{End}(\widetilde{A}_{/\mathbb{F}_p})$ has kernel of positive dimension for the relative p-power Frobenius endomorphism F of $A \times_V \mathbb{F}_p$, then $\widetilde{A}$ is an ordinary abelian variety.*

Proof. Take a prime $l \neq p$. The l-adic Tate module $T_l\widetilde{A} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is free of rank 2 over $M_l = M \otimes_{\mathbb{Q}} \mathbb{Q}_l$, and $H(F^k) = 0$ for the polynomial $H(T) := T^2 - aT + p = 0$ $(a = \theta(\alpha + \alpha^c))$ with coefficients in the maximal real subfield of M, as the Rosati involution coincides with complex conjugation c on M. Replacing A by an isogenous abelian scheme, we may assume that $\theta^{-1}(\mathrm{End}(A_{/V}))$ contains the integer ring $\mathfrak{O}$ of M (e.g., [IAT, (7.7.8)]). Thus by the above proposition, $a \mod p$ is a unit in $\mathfrak{O}/p\mathfrak{O}$ and hence $\widetilde{A}$ is an ordinary abelian variety. $\square$

We call two nonzero numbers $a, b \in \overline{\mathbb{Q}}$ equivalent (written as $a \sim b$) if a/b is a root of unity.

Lemma 2.3. *Let K be a finite field extension of $\mathbb{Q}(\mu_{p^\infty})$ inside $\overline{\mathbb{Q}}$. Then for a given prime l and weight $k \geq 0$, there are only finitely many Weil l-numbers of weight k in K up to equivalence. If $K = \mathbb{Q}(\mu_{p^\infty})$ and there is only one prime in $\mathbb{Z}[\mu_{p^\infty}]$ above (l) (for example, if $l = p$), any Weil l-number of weight k is equivalent*

to $(l^*)^{k/2}$ (as long as $(l^*)^{k/2}$ is in K), where $l^* = (-1)^{(l-1)/2}l$ if l is odd, and $l^* = 2$ if $l = 2$.

A result of Loxton confirms that, up to equivalence, there are only finitely many Weil l-numbers of a given weight in the maximal abelian extension $\mathbb{Q}^{ab}$ of $\mathbb{Q}$ (see [Lo, Lemma 7]). Thus the finiteness result of the lemma follows from this analytic result of Loxton. We will give an algebraic proof.

Proof. The decomposition group of each prime l is of finite index in $\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})$ (identifying $\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \cong \mathbb{Z}_p^\times$ by the p-adic cyclotomic character, the decomposition group is generated by l if $l \neq p$, and otherwise if $l = p$, then p is fully ramified in $\mathbb{Q}[\mu_{p^\infty}]$; see [ICF, Chapter 2]); so, there are only finitely many primes $\mathfrak{L}$ of $\mathbb{Z}[\mu_{p^\infty}]$ above (l). Thus for a Weil l-number α of weight k, there are only finitely many possibilities of prime factorization of (α) if $l \neq p$. If $(\alpha) = (\beta)$ for two Weil l-numbers α, β, then α/β is a Weil number of weight 0; so, $\alpha \sim \beta$ by Kronecker's theorem. If there is only one prime over l in $\mathbb{Z}[\mu_{p^\infty}]$, any Weil l-number of weight k is equivalent to $(l^*)^{k/2}$, as long as $(l^*)^{k/2} \in \mathbb{Q}[\mu_{p^\infty}]$. Thus the result follows from this if $K = \mathbb{Q}(\mu_{p^\infty})$.

Let $W(l, k)$ (resp. $W_K(l, k)$) be a complete set of representatives of Weil l-numbers in $\mathbb{Q}(\mu_{p^\infty})$ (resp. in K) of weight k under the equivalence. By the above argument, $W(l, k)$ is a finite set, and we want to prove that $W_K(l, k)$ is finite. Write $d = [K : \mathbb{Q}(\mu_{p^\infty})]$. If $\alpha \in K$ is such a Weil l-number, then $N_{K/\mathbb{Q}(\mu_{p^\infty})}(\alpha)$ is equivalent to a number in $W(l, kd)$. Thus $N_{K/\mathbb{Q}(\mu_{p^\infty})}$ induces a map $N : W_K(l, k) \to W(l, dk)$. Write L for the field generated by elements in $W(l, dk)$. Then $L/\mathbb{Q}$ is a finite abelian extension in $\mathbb{Q}(\mu_{p^\infty})$. Since no prime completely splits in $\mathbb{Q}(\mu_{p^\infty})$, the decomposition subgroup D of l in $\mathrm{Gal}(K/\mathbb{Q})$ is an open subgroup of finite index. Thus there are only finitely many valuations v of K with $v(l) = 1$. Let $\mathcal{V}$ be the set of valuations v of K with $v(l) = 1$, which is a finite set. For $v \in \mathcal{V}$ and $\alpha \in W_K(l, k)$, $v(\alpha) \in [0, k] \cap d^{-1}v(L)$, because $N_{K/\mathbb{Q}(\mu_{p^\infty})}(\alpha)$ is in $W(l, dk)$ up to roots of unity. Let $V := \prod_{v \in \mathcal{V}} \left([0, k] \cap d^{-1}v(L)\right)$, which is a finite set. We have a map $\mathrm{ord}_l : W_K(l, k) \to V$ sending α to $\mathrm{ord}_l(\alpha) = (v(\alpha))_{v \in \mathcal{V}}$. If $\mathrm{ord}_l(\alpha) = \mathrm{ord}_l(\beta)$ $(\alpha, \beta \in W_K(l, k))$, then α/β is an algebraic integer with complex absolute value $|(\alpha/\beta)^\sigma| = 1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; so, by Kronecker's theorem, $\alpha \sim \beta$. Thus ord_l is an injection, proving the finiteness of $W_K(l, k)$. $\square$

Proposition 2.4. *Let G be a finite group and l be a prime. Let $\mathcal{K}_G$ be the set of all Galois extensions of $K := \mathbb{Q}[\mu_{p^\infty}]$ with Galois group G inside $\overline{\mathbb{Q}}$ whose ramification at l over K is tame. Then there are only finitely many Weil l-numbers of a given weight in the set-theoretic union $\bigcup_{L \in \mathcal{K}_G} L$ (in $\overline{\mathbb{Q}}$) up to equivalence.*

The point of the proof is as follows (and we fill in the details after the outline). Write $K_l = K \otimes_\mathbb{Q} \mathbb{Q}_l$. By tameness, there are only finitely many isomorphism classes of K_l-algebras $L_l = L \otimes_\mathbb{Q} \mathbb{Q}_l$ for $L \in \mathcal{K}_G$. Thus we only need to prove finiteness for Weil numbers of a given weight contained in a fixed isomorphism class of L_l. We look at the universal composite $L_l \otimes_{K_l} L_l$, which is a product of fields indexed by l-adic nonequivalent valuations $\mathcal{V}_1, \ldots, \mathcal{V}_n$ normalized so that $\mathcal{V}_i(l) = 1$. Consider a tuple

$$\nu(\alpha) = (\mathcal{V}_1(\alpha \otimes 1), \ldots, \mathcal{V}_n(\alpha \otimes 1), \mathcal{V}_1(1 \otimes \alpha), \ldots, \mathcal{V}_n(1 \otimes \alpha)).$$

If $\alpha \sim \beta$, we have $\nu(\alpha) = \nu(\beta)$. The tuple $\nu(\alpha)$ determines the prime factorization of (α) in any possible composite $K(\alpha, \beta)$; so, if $\nu(\alpha) = \nu(\beta)$, $(\alpha) = (\beta)$ in $K(\alpha, \beta)$;

so, by Kronecker's theorem, $\alpha \sim \beta$. Since there are only finitely many possibilities of $\nu(\alpha)$, there are only finitely many classes.

Proof. Let $\mathcal{V}$ be the set of normalized l-adic valuations of $K := \mathbb{Q}[\mu_{p^\infty}]$. Write $\mathbb{F}_v$ for the residue field of $v \in \mathcal{V}$ inside a fixed algebraic closure $\overline{\mathbb{F}}_l$. If $l \neq p$, then $\mathbb{F}_v$ is an infinite extension of $\mathbb{F}_l$, and otherwise, $\mathbb{F}_v = \mathbb{F}_p$. As we have remarked, $\mathcal{V}$ is a finite set. Let $L \in \mathcal{K}_G$; so, L/K is a finite extension of degree $d = |G|$ tamely ramified at l. We claim that there are only finitely many isomorphism classes of K_l-algebras $L_l = L \otimes_\mathbb{Q} \mathbb{Q}_l$ for L running through $\mathcal{K}_G$. To see this, write $K_l = K \otimes_\mathbb{Q} \mathbb{Q}_l = \prod_{v \in \mathcal{V}} K_v$ for $K_v = \mathbb{Q}_l[\mu_{p^\infty}]$ inside the v-adic completion of K. Let K_v^t/K_v (resp. K_v^{ur}/K_v) be the maximal tamely l-ramified extension (resp. the maximal unramified extension inside K_v^t). Then by class field theory, K_v^t has the l-inertia group isomorphic to $\widehat{\mathbb{Z}}^{(l)}(1)$, where $\widehat{\mathbb{Z}}^{(l)}(1) = \varprojlim_{l \nmid N} \mu_N(\overline{\mathbb{Q}}_l)$ as $\mathrm{Gal}(K_v^{ur}/\mathbb{Q}_p)$-modules. Thus we have

$$(2.1) \qquad \mathrm{Gal}(K_v^t/K_v) \cong \begin{cases} \widehat{\mathbb{Z}}^{(p)} \ltimes \widehat{\mathbb{Z}}^{(l)}(1) & \text{if } l \neq p, \\ \widehat{\mathbb{Z}} \ltimes \widehat{\mathbb{Z}}^{(l)}(1) & \text{if } l = p \end{cases}$$

(a semi-direct product with $\mathrm{Gal}(K_v^t/K_v) \rhd \widehat{\mathbb{Z}}^{(l)}(1)$), since

$$\mathrm{Gal}(K_v^{ur}/K_v) \cong \begin{cases} \mathrm{Gal}(\overline{\mathbb{F}}_l/\mathbb{F}_v) \cong \widehat{\mathbb{Z}}^{(p)} & \text{if } l \neq p, \\ \widehat{\mathbb{Z}} & \text{if } l = p. \end{cases}$$

Thus there are finitely many open subgroups of $\mathrm{Gal}(K_v^t/K_v)$ with index $\leq d$, and hence there are finitely many isomorphism classes of tamely ramified extensions of K_v of degree $\leq d$. Thus the claim follows.

Consider the set Ξ of all isomorphism classes of K_l-algebras $L \otimes_\mathbb{Q} \mathbb{Q}_l$, where L runs over $\mathcal{K}_G$. Thus Ξ is a finite set. Pick $\mathcal{L} \in \Xi$ and consider the set $\mathcal{K}_\mathcal{L}$ of all $L \in \mathcal{K}_G$ such that $L \otimes_\mathbb{Q} \mathbb{Q}_l \cong \mathcal{L}$ as K-algebras. Then $\mathcal{K}_G = \bigsqcup_{\mathcal{L} \in \Xi} \mathcal{K}_\mathcal{L}$. The set $\mathcal{K}_\mathcal{L}$ is a countable set. Assume $\mathcal{K}_\mathcal{L} \neq \emptyset$. We just order its elements as $L_1, L_2, \ldots$. We fix a K-algebra isomorphism $\iota_j : L_j \otimes_\mathbb{Q} \mathbb{Q}_l \cong \mathcal{L}$. Thus $L_j \subset \mathcal{L}$ for every j. We take the (set-theoretic) union $\Omega_\mathcal{L} := \bigcup_{j=1}^\infty \iota_j(L_j)$ inside $\mathcal{L}$. Since L_j/K is a Galois extension, another choice of ι_j is equal to $\iota_j \circ \sigma$ for $\sigma \in \mathrm{Gal}(L_j/K)$. Thus $\mathrm{Im}(\iota_j : L_j \to \mathcal{L})$ is independent of the choice of ι_j, and we therefore forget about ι_j. Note that $K_l = K \otimes_\mathbb{Q} \mathbb{Q}_l$ is a product of the field K_v indexed by $v \in \mathcal{V}$ such that K_v has a valuation inducing v on K. Similarly $\mathcal{L}$ is a product of finitely many fields with l-adic valuation. Write $\mathcal{L} = \prod_{j=1}^r \mathbb{L}_j$ for a simple component $\mathbb{L}_j$, and write V_j for the valuation giving the l-adic topology of $\mathbb{L}_j$ extending one of the members of $\mathcal{V}$. Write $\mathcal{V}_\mathcal{L} = \{V_1, \ldots, V_r\}$, which is a finite set.

Let $\mathcal{W}(l, k)$ (resp. $\mathcal{W}(l, k, \mathcal{L})$) be the set of Weil l-numbers of weight k in $\bigcup_{L \in \mathcal{K}_G} L$ (resp. in $\Omega_\mathcal{L}$). Thus for each $\alpha \in \mathcal{W}(l, k, \mathcal{L})$, we find a Galois extension L_α/K containing $L(\alpha)$ over K inside $\mathcal{K}_G$ with Galois group G such that $L_\alpha \otimes_\mathbb{Q} \mathbb{Q}_l \cong \mathcal{L}$. Thus we have

$$\mathcal{W}(l, k) = \bigcup_{\mathcal{L} \in \Xi} \mathcal{W}(l, k, \mathcal{L}).$$

The intersection $\mathcal{W}(l, k, \mathcal{L}) \cap \mathcal{W}(l, k, \mathcal{L}')$ for $\mathcal{L}, \mathcal{L}' \in \Xi$ obviously contains all Weil l-numbers of weight k in K (and possibly more). For each $\alpha \in \mathcal{W}(l, k, \mathcal{L})$, we define a tuple of nonnegative positive rational numbers $S_1(\alpha) \in (\mathbb{Q}_{\geq 0})^{\mathcal{V}_\mathcal{L}}$ given by $S_1(\alpha) = (V(\alpha))_{V \in \mathcal{V}_\mathcal{L}}$.

If $l \neq p$, l is unramified in $K/\mathbb{Q}$; so, $V(\alpha) \in |G|^{-1}\mathbb{Z}$ and $0 \leq V(\alpha) \leq k$. Thus

$$\mathcal{S} := \{S_1(\alpha) | \alpha \in \mathcal{W}(l, k, \mathcal{L})\}$$

is a finite set. If $l = p$, by the above lemma, there are only finitely many Weil p-numbers (up to equivalence) in K of a given weight. Thus the set $\mathbb{V} \subset \mathbb{Q}$ of valuations of Weil numbers in K of weight $k|G|$ is a finite set. For each $\alpha \in \mathcal{W}(p, k, \mathcal{L})$, take a Galois extension L_α/K (inside $\mathcal{K}_G$) with $\mathrm{Gal}(L_\alpha/K) = G$ containing α with $L_\alpha \otimes_\mathbb{Q} \mathbb{Q}_l \cong \mathcal{L}$. Then $\sum_{V \in \mathcal{V}_\mathcal{L}} e_V V(\alpha) = v(N_{L_\alpha/K}(\alpha)) \in \mathbb{V}$ for the (tame) ramification index e_V of V over K and $V(\alpha) \in |G|^{-1}\mathbb{V}$. Again,

$$\mathcal{S} := \{S_1(\alpha) | \alpha \in \mathcal{W}(p, k, \mathcal{L})\}$$

is a finite set. If $\alpha/\beta \in \mu_{p^\infty}(K)$, we have $K(\alpha) = K(\beta)$ inside $\mathcal{L}$ and hence $S_1(\alpha) = S_1(\beta)$. By Lemma 2.6 following this proposition, the group of roots of unity in the composite $\mathbf{L}$ of L for all $L \in \mathcal{K}_\mathcal{L}$ in $\overline{\mathbb{Q}}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index; so, $S_1(\alpha) = S_1(\beta)$ up to permutation of valuations in $\mathcal{V}_\mathcal{L}$ if $\alpha \sim \beta$.

We consider for each $s \in \mathcal{S}$ the set $\mathcal{W}(s) = \{\alpha \in \mathcal{W}(l, k, \mathcal{L}) | S_1(\alpha) = s\}$. By definition, we have a decomposition

$$\mathcal{W}(l, k, \mathcal{L}) = \bigsqcup_{s \in \mathcal{S}} \mathcal{W}(s).$$

We want to show that $\mathcal{W}(s)/\mu_{p^\infty}(K)$ is a finite set. Write $\mathcal{O}$ for the integral closure of $\mathbb{Z}_l$ in $\mathcal{L}$ (the l-adic integer ring of $\mathcal{L}$). By definition, for $\alpha, \beta \in \mathcal{W}(s)$, the principal $\mathcal{O}$-ideal (α) is equal to (β), because $S_1(\alpha) = S_1(\beta) = s$ determines the prime factorization of $(\alpha) = (\beta)$ in $\mathcal{L}$. Write $\mathcal{L} \otimes_{K_l} \mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n$ for fields $\mathcal{L}_i$ and write $\mathcal{V}_i$ for the valuation of $\mathcal{L}_i$ normalized so that it extends a member of $\mathcal{V}$. Then the possibilities of $\nu(\alpha) = (\mathcal{V}_i((\alpha \otimes 1), \mathcal{V}_i(1 \otimes \alpha))_i$ for α running over $\mathcal{W}(s)$ are finitely many, and if $\nu(\alpha) = \nu(\beta)$, α/β is a Weil l-number of weight 0, because the prime factorizations of (α) and (β) in $L_\alpha \otimes_K L_\beta$ are equal. Thus α/β is a root of unity by Kronecker's theorem, and hence $\alpha \sim \beta$. As already remarked, $\{\text{roots of unity in } \mathbf{L}\} = \bigcup_{\zeta \in Z} \zeta \cdot \mu_{p^\infty}(K)$ for a finite set Z of roots of unity. Thus $\mathcal{W}(s)/\mu_{p^\infty}(K)$ is finite, which implies that $\mathcal{W}(l, k)/\{\text{roots of unity}\}$ is finite.   □

Corollary 2.5. *Let d be a positive integer. Let $\mathcal{K}_d$ be the set of all finite extensions of $\mathbb{Q}[\mu_{p^\infty}]$ of degree d inside $\overline{\mathbb{Q}}$ whose ramification at l is tame. Then there are only finitely many Weil l-numbers of a given weight in the set-theoretic union $\bigcup_{L \in \mathcal{K}_d} L$ (in $\overline{\mathbb{Q}}$) up to equivalence.*

Proof. The Galois closure L^{gal} of $L \in \mathcal{K}_d$ over $K = \mathbb{Q}[\mu_{p^\infty}]$ has Galois group which is isomorphic to a subgroup of the permutation group $\mathfrak{S}_d$ of d letters. Thus the possible isomorphism classes of $\mathrm{Gal}(L^{gal}/K)$ are finite. Note that l tamely ramifies in L^{gal}/K. Thus applying the above proposition for each $G \subset \mathfrak{S}_d$, we get the desired result.   □

Lemma 2.6. *Let $\mathcal{K}_?$ be one of $\mathcal{K}_\mathcal{L}$, $\mathcal{K}_G$ and $\mathcal{K}_d$. Suppose $\mathcal{K}_? \neq \emptyset$. Then the group of roots of unity in the composite $\mathbf{L}$ of L for $L \in \mathcal{K}_?$ in $\overline{\mathbb{Q}}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index.*

Proof. We first take $\mathcal{K}_? = \mathcal{K}_\mathcal{L}$ or $\mathcal{K}_G$. Let $\mu^{(p)} \subset \overline{\mathbb{Q}}$ be the group of roots of unity of order prime to p. We need to prove that the subgroup $\mu^{(p)} \cap \mathbf{L} \subset \mu^{(p)}$ is finite.

For any finite subset $\mathcal{K} \subset \mathcal{K}_?$, the composite of $L \in \mathcal{K}$ in $\overline{\mathbb{Q}}$ is a Galois extension of K whose Galois group is a subgroup of $\prod_{L \in \mathcal{K}} G$ contained in

$$\{(\sigma_L)_{L \in \mathcal{K}} | \sigma_L \in \mathrm{Gal}(L/K),\ \sigma_L|_{L \cap L'} = \sigma_{L'}|_{L \cap L'} \text{ for all pairs } (L, L') \in \mathcal{K}^2\}.$$

Thus any $\sigma \in \mathrm{Gal}(\mathbf{L}/K)$ has order a factor of $|G|$, so is bounded. Pick any infinite subgroup $\nu \subset \mu^{(p)}$, and decompose it into a product of ℓ-primary subgroups ν_ℓ for primes $\ell \neq p$. Let $\Xi = \{\ell | \nu_\ell \neq 1\}$. If Ξ is finite, we have $\nu \supset \mu_{\ell^\infty}$ for a prime $\ell \in \Xi$, and $\mathrm{Gal}(K(\nu)/K)$ surjects down to $\mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) = \mathbb{Z}_\ell^\times$; so, the elements of $\mathrm{Gal}(K(\nu)/K)$ have unbounded order, and hence $\mathbb{Q}(\nu) \not\subset \mathbf{L}$. If $|\Xi| = \infty$, $\mathrm{Gal}(K(\nu_\ell)/K)$ for odd ℓ is cyclic of order $\geq \ell - 1$, and therefore, elements of $\mathrm{Gal}(K(\nu)/K)$ have unbounded order. Thus the subgroup $\mu^{(p)} \cap \mathbf{L}$ has to be finite. As for $\mathcal{K}_? = \mathcal{K}_d$, the Galois closure of $L \in \mathcal{K}_d$ over K has Galois group G realized inside $\mathfrak{S}_d$, and the same proof works since what we used is only the boundedness of $|G| \leq d!$. $\qquad\square$

3. Theorems and conjectures

Let $f = \sum_{n=1}^\infty a(n, f)q^n \in S_k(\Gamma_0(Np^r \mathbf{p}), \psi)$ be a Hecke eigenform normalized so that $f|T(n) = a(n, f)f$ for all n. We write $f|T(l) = (\alpha_l + \beta_l)f$ and $\alpha_l \beta_l = \psi(l)l^{k-1}$ if $l \nmid Np^{r+1}$ $(\alpha_l, \beta_l \in \overline{\mathbb{Q}})$. If $l | Np^{r+1}$, we put $\beta_l = 0$ and define $\alpha_l \in \overline{\mathbb{Q}}$ by $f|U(l) = \alpha_l f$. Then the Hecke polynomial $H_l(X) = (1 - \alpha_l X)(1 - \beta_l X)$ gives the Euler l-factor of $L(s, f) = \sum_{n=1}^\infty a(n, f)n^{-s}$ after replacing X by l^{-s} and inverting the factor.

Let $\mathcal{F} = \{f_P\}_{P \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)}$ be a p-adic analytic family of p-ordinary Hecke eigen cusp forms of slope 0. We write $\alpha_{l,P}, \beta_{l,P}$ for α_l, β_l for f_P. We have the following two versions of the conjecture:

Conjecture 3.1 (Vertical version).

(Weak form): *The field $\mathbb{Q}_{V,r}(\mathcal{F})$ is a finite extension of $\mathbb{Q}$ for a fixed $r < \infty$ if and only if f_P is a CM theta series for some arithmetic P with $k(P) \geq 2$.*

(Strong form): *Let $\mathcal{A}$ be an infinite set of arithmetic points with bounded level $r(P) \leq r$ for a fixed $r \geq 0$ (so, $\mathrm{Im}(\varepsilon_P) \subset \mu_{p^r}$ for the fixed r). Let $M_{V,\mathcal{A}}(\mathcal{F})$ be the field generated over $\mathbb{Q}$ by $\{\alpha_{p,P}\}_{P \in \mathcal{A}}$, where P runs over all points in $\mathcal{A}$. Then the field $M_{V,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}$ for a fixed $r < \infty$ if and only if f_P is a CM theta series for some arithmetic P with $k(P) \geq 2$.*

Pick a prime l different from p and write $M_{V,\mathcal{A}}^{(l)}(\mathcal{F})$ for the field generated by $\{\alpha_{l,P}, \beta_{l,P}\}_{P \in \mathcal{A}}$, where P runs over all points in $\mathcal{A}$. Then we might speculate that

Vertical l-version of strong form. *The field $M_{V,\mathcal{A}}^{(l)}(\mathcal{F})$ is a finite extension of $\mathbb{Q}$ for a fixed $r < \infty$ if and only if for an arithmetic P with $k(P) \geq 2$, either f_P is a CM theta series or the automorphic representation generated by f_P is square-integrable at l.*

What we can prove is as follows.

Theorem 3.2 (Strong vertical theorem). *Let r be a nonnegative integer. Assume $p > 2$ and that there exists an arithmetic point $P_0 \in \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ with $k(P_0) \geq 2$ such that $\alpha_0 = a(p, f_{P_0})$ is a Weil number and $\Sigma_{\alpha_0} = \{\sigma : \mathbb{Q}(\alpha_0) \hookrightarrow \overline{\mathbb{Q}} | |i_p(\alpha_0^\sigma)| = 1\}$ is a CM type of $\mathbb{Q}(\alpha_0)$. Pick an infinite set $\mathcal{A}$ of arithmetic points P with bounded level $r(P) \leq r$. Then the field $M_{V,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}$ if and only if f_P is a CM theta series for an arithmetic P with $k(P) \geq 2$.*

See Proposition 2.1 for sufficient conditions which guarantee the truth of the assumption of the theorem. Take the rank 2 motive M_0 with coefficients in $\mathbb{Q}(f_{P_0})$ (in [S]). Regarding M_0 as having coefficients in $\mathbb{Q}$, M_0 is potentially crystalline at p and ordinary if and only if Σ_{α_0} is a CM type of $\mathbb{Q}(\alpha_0)$. Here a motive is ordinary if the Newton polygon of Frobenius at p (in our case, the Newton polygon of α_0) coincides with the Hodge polygon. We prove this theorem and the vertical theorem (in the introduction) in Section 8. The horizontal theorem in the introduction follows from the following version (and Corollary 6.3):

Theorem 3.3 (Strong horizontal theorem). *Pick an infinite set $\mathcal{A}$ of arithmetic points P with fixed $k(P) = k \geq 2$. Write $M_{H,\mathcal{A}}(\mathcal{F}) \subset \overline{\mathbb{Q}}$ for the field generated over $\mathbb{Q}(\mu_{p^\infty})$ by $\{\alpha_{p,P}\}_{P \in \mathcal{A}}$, where P runs over all arithmetic points in $\mathcal{A}$. Suppose $p > 2$. Then the field $M_{H,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ if and only if f_P is a CM theta series for an arithmetic P with $k(P) \geq 2$.*

We prove this theorem in Section 6. For a prime $l \neq p$, we may conjecture the l-version of the stronger form in the horizontal case also:

Conjecture 3.4 (Horizontal l-version). *Write $M^{(l)}_{H,\mathcal{A}}(\mathcal{F})$ for the field generated over $\mathbb{Q}(\mu_{p^\infty})$ by $\{\alpha_{l,P}\}_{P \in \mathcal{A}}$. Then $M^{(l)}_{H,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ if and only if for an arithmetic P with $k(P) \geq 2$, either f_P is a CM theta series or the automorphic representation generated by f_P is square-integrable at l.*

One can show that $[M^{(l)}_{H,\mathcal{A}}(\mathcal{F}) : \mathbb{Q}(\mu_{p^\infty})] < \infty$ if f_{P_0} is square-integrable at a prime $l \neq p$ for one arithmetic P_0 as follows: Let $\mathrm{Spec}(\mathbb{I})$ be the irreducible component associated to the family $\mathcal{F}$. If f_{P_0} (or, more precisely, the automorphic representation generated by f_{P_0}) is super-cuspidal at l, then $\rho_{P_0}|_{I_l}$ is absolutely irreducible (by the local Langlands conjecture for $GL(2)$ solved by Kutzko and Carayol; e.g., [Ca]). Then the Galois representation $\rho_{\mathbb{I}}$ is absolutely irreducible over the inertia group I_l and $\rho_{\mathbb{I}}(I_l)$ is finite by (2.1). Therefore, it is rigid; i.e., $\rho_P|_{I_l}$ is independent of points $P \in \mathrm{Spec}(\mathbb{I})$ with characteristic 0 residue field. Thus if f_{P_0} is super-cuspidal for one arithmetic point P_0, then $\rho_{P_0}|_{I_l}$ is irreducible. Therefore, $\rho_{\mathbb{I}}|_{I_l}$ is irreducible, which implies that $\rho_P|_{I_l}$ is irreducible for all P with characteristic 0 residue field. Hence f_P is super-cuspidal for all arithmetic P. Then $\alpha_l = a(l) = 0$ in $\mathbb{I}$; so, $M^{(l)}_{H,\mathcal{A}}(\mathcal{F}) = \mathbb{Q}(\mu_{p^\infty})$ in this case. Suppose that the automorphic representation π_{P_0} generated by f_{P_0} has its local factor at l isomorphic to the Steinberg representation $\sigma(\alpha| \cdot |_l, \alpha)$ for a character $\alpha : \mathbb{Q}_l^\times \to \mathbb{C}^\times$. If α is unramified, f_{P_0} is l-new of l-conductor l, and hence f_P is Steinberg for all arithmetic P because the outside p-conductor and the outside p central character are constant in the family (the old/new theory at the level of families, e.g., [H88, Theorem 3.6]). If α is ramified, taking a global Dirichlet character $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ with $\chi|_{\mathbb{Z}_l^\times} = \alpha^{-1}|_{\mathbb{Z}_l^\times}$ only ramified at l, $\{\pi_P \otimes \chi\}_P$ is associated to another analytic family $\mathcal{F} \otimes \chi$ of slope 0. Replacing $\mathcal{F}$ by $\mathcal{F} \otimes \chi$, we may assume α is unramified; so, if one f_{P_0} is Steinberg at $l \neq p$, all members of $\mathcal{F}$ are Steinberg at $l \neq p$. Using this fact, we prove in Proposition 5.2 that $M^{(l)}_{H,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ if f_{P_0} is Steinberg at $l \neq p$ at one arithmetic P_0. A similar argument works also for the vertical l-version, proving $[M^{(l)}_{V,\mathcal{A}}(\mathcal{F}) : \mathbb{Q}] < \infty$ if f_{P_0} is square-integrable at $l \neq p$ for one arithmetic P_0.

We will prove the following weaker statements in the following section:

Proposition 3.5. (WV) *Let $\widetilde{M}_{V,r}(\mathcal{F})$ be the field generated over $\mathbb{Q}$ by $\{\alpha_{l,P}^2,$ $\beta_{l,P}^2\}_{l,P}$, where P runs over arithmetic points with $\text{Im}(\varepsilon_P) \subset \mu_{p^r}$ for a fixed $r > 0$ and l runs over all primes. Then $\widetilde{M}_{V,r}(\mathcal{F})$ is a finite extension of $\mathbb{Q}$ for a fixed $r < \infty$ if and only if f_P is a CM theta series for some arithmetic P with $k(P) \geq 2$.*

(WH) *Let $\widetilde{M}_{H,k}(\mathcal{F})$ be the field generated over $\mathbb{Q}$ by $\{\alpha_{l,P}^2, \beta_{l,P}^2\}_{l,P}$, where P runs over all arithmetic points with $k(P) = k$ for a fixed $k \geq 2$ and l runs over all primes. The field $\widetilde{M}_{H,k}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$ for a fixed k if and only if f_P is a CM theta series for some arithmetic P with $k(P) \geq 2$.*

The field $\widetilde{M}_{V,r}(\mathcal{F})$ or $\widetilde{M}_{H,k}(\mathcal{F})$ is defined to be generated by the squares $\alpha_{l,P}^2$ and $\beta_{l,P}^2$ for the following reason. Suppose $\mathcal{F}$ has CM. Then, f_P is a theta series of an imaginary quadratic field M with even weight $k = k(P)$; so, $\rho_P = \text{Ind}_M^{\mathbb{Q}} \lambda_P$ for a character $\lambda_P : \text{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^\times$. If l is inert in M, writing $\mathfrak{l}$ for the unique prime of M over l, we have

$$\det(X - \rho_P(Frob_l)) = \det(X - \text{Ind}_M^{\mathbb{Q}} \lambda_P(Frob_l))$$
$$= X^2 - \lambda_P(Frob_{\mathfrak{l}}) = X^2 + \psi_k \varepsilon_P(l) l^{k-1}.$$

Thus $\alpha_l = \pm\sqrt{-\psi_k \varepsilon_P(l) l^{k-1}}$. Since $k - 1$ is odd, $\mathbb{Q}(\alpha_l | l\text{:inert})$ contains infinitely many distinct quadratic extensions $\mathbb{Q}(\sqrt{-l})$ for primes l with $\psi_k \varepsilon_P(l) = 1$. To avoid this, we need to take the square α_l^2 in the vertical case and in the horizontal case if $k(P)$ is even.

4. Hecke fields and their quadratic extensions

The "only if" part of Proposition 3.5 directly follows from the following lemma, though we will give a detailed proof of the proposition after proving the lemma.

Lemma 4.1. *Let $f \in S_k(\Gamma_0(N), \psi)$ be a Hecke eigenform for $k \geq 2$. Let $\mathcal{P}$ be the set of all prime ideals of $\mathbb{Q}(f)$ split over $\mathbb{Q}$ (which has density 1 in the total set of prime ideals of $\mathbb{Q}(f)$). If f does not have CM (that is, it is not a theta series of an imaginary quadratic field), the field generated by the squares of Frobenius eigenvalues α_l and β_l for l running through any density 1 subset $\mathcal{P}_0$ of $\mathcal{P}$ is an infinite extension of $\mathbb{Q}(f)$. Here the density is computed in the set of all primes of $\mathbb{Q}(f)$.*

Proof. For a prime $\mathfrak{l}$ of $\mathbb{Q}(f)$, write $\rho_{\mathfrak{l}}$ for the $\mathfrak{l}$-adic Galois representation of f. Write $O \subset \mathbb{Q}(f)$ for the integer ring of $\mathbb{Q}(f)$. Then we consider the adelic representation $\widehat{\rho} = \prod_{\mathfrak{l}} \rho_{\mathfrak{l}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\widehat{O})$ for $\widehat{O} = \prod_{\mathfrak{l}} O_{\mathfrak{l}}$. By a result of Ribet (see [R]), $\text{Im}(\widehat{\rho})$ contains an open subgroup U of $SL_2(\widehat{\mathbb{Z}})$ for $\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$. Let S_1 be the finite set of primes l such that $U \cap SL_2(\mathbb{Z}_l) \neq SL_2(\mathbb{Z}_l)$ and let $\mathcal{S} \subset \mathcal{P}_0$ be a finite set outside S_1. Since $\mathfrak{l} \in \mathcal{P}_0$ for $\mathfrak{l} \in \mathcal{S}$, we have $O_{\mathfrak{l}} = \mathbb{Z}_l$ (for the residual characteristic $l = l(\mathfrak{l})$ of $\mathfrak{l}$). Let S be the union of the set of residual characteristics of primes in $\mathcal{S}$, prime factors of Np and S_1. Write $\rho_{\mathcal{S}} = \prod_{\mathfrak{l} \in \mathcal{S}} \rho_{\mathfrak{l}}$. Let $O_{\mathcal{S}} = \prod_{\mathfrak{l} \in \mathcal{S}} O_{\mathfrak{l}}$ and put $\mathbb{Q}_{\mathcal{S}}(f) = O_{\mathcal{S}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that $\mathbb{Q}_{\mathcal{S}}(f) = \prod_{\mathfrak{l} \in \mathcal{S}} \mathbb{Q}_{l(\mathfrak{l})}$. We consider $H = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(f))$ which is an open subgroup of finite index in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $\rho_{\mathcal{S}}$ is unramified outside $S \cup \{\infty\}$, we have well defined $\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})$ for the Frobenius element $Frob_{\mathfrak{l}}$

in H for a prime ideal $\mathfrak{l}$ of O outside S. Note that $Frob_{\mathfrak{l}} = Frob_{l(\mathfrak{l})}$ if $O_{\mathfrak{l}} = \mathbb{Z}_{l(\mathfrak{l})}$. Consider the commutative subalgebra $\mathbb{Q}[\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})] \subset M_2(\mathbb{Q}_{\mathcal{S}}(f))$ generated over $\mathbb{Q}$ by $\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})$ for a prime ideal $\mathfrak{l}$ outside S split in $\mathbb{Q}(f)/\mathbb{Q}$. Write $\mathbb{Z}[\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})]$ for the subalgebra of $\mathbb{Q}[\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})]$ generated over $\mathbb{Z}$ by $\rho_{\mathcal{S}}(Frob_{\mathfrak{l}})$. Consider the set $\mathcal{A}$ of tuples of (isomorphism classes of) commutative semi-simple quadratic extensions $A = \{A_{\mathfrak{l}}/\mathbb{F}_{l(\mathfrak{l})}\}_{\mathfrak{l} \in \mathcal{S}}$ indexed by primes in $\mathcal{S}$; so, $A_{\mathfrak{l}}$ is either a quadratic field extension of $\mathbb{F}_{l(\mathfrak{l})}$ or $\mathbb{F}_{l(\mathfrak{l})} \oplus \mathbb{F}_{l(\mathfrak{l})}$. Pick $t_{\mathfrak{l}} \in SL_2(\mathbb{Z}_{l(\mathfrak{l})})$ for each $\mathfrak{l} \in \mathcal{S}$ such that

$$\mathbb{Z}_{l(\mathfrak{l})}[t_{\mathfrak{l}}^2] \otimes_{\mathbb{Z}} \mathbb{F}_{l(\mathfrak{l})} \cong A_{\mathfrak{l}}.$$

Since $\mathcal{P}_0$ has density 1 among primes of $\mathbb{Q}(f)$, by Chebotarev density, we can find a split prime ideal $\mathfrak{l} = \mathfrak{l}(A) \in \mathcal{P}_0$ outside S such that $\rho_{\mathcal{S}}(Frob_{\mathfrak{l}(A)})$ is sufficiently close $\mathcal{S}$-adically to $t_A = (t_{\mathfrak{l}})_{\mathfrak{l} \in \mathcal{S}} \in M_2(\mathbb{Q}_{\mathcal{S}}(f))$ so that $\mathbb{Z}[\rho(Frob_{\mathfrak{l}(A)}^2)] \otimes_{\mathbb{Z}} \mathbb{F}_{l(\mathfrak{l})} \cong A_{\mathfrak{l}}$ for each prime $\mathfrak{l} \in \mathcal{S}$. Note that $\mathbb{Q}_A := \mathbb{Q}[\rho_{\mathcal{S}}(Frob_{\mathfrak{l}(A)}^2)] \cong \mathbb{Q}(\alpha_{\mathfrak{l}(A)}^2)$, and for two distinct $A, A' \in \mathcal{A}$, $\mathbb{Q}_A \not\cong \mathbb{Q}_{A'}$. Thus at least we have found primes $\mathfrak{l}(A)$ indexed by $\mathcal{A}$ such that all quadratic extensions in $\{\mathbb{Q}(\alpha_{\mathfrak{l}(A)}^2)\}_{A \in A}$ of $\mathbb{Q}(f)$ are nonisomorphic. Making $|\mathcal{S}| \to \infty$, we can find an infinite set $\Omega \subset \mathcal{P}_0$ of primes $\mathfrak{l}$ such that $\mathbb{Q}(\alpha_{\mathfrak{l}}^2)$ are distinct for all $\mathfrak{l} \in \Omega$. This shows the result. $\square$

Proof of (WV). Pick f_P in $\mathcal{F}$ without complex multiplication for an arithmetic P. Then by the above lemma, $\widetilde{M}_{V,r}(\mathcal{F}) \supset \mathbb{Q}(\alpha_{\mathfrak{l}}^2|\mathfrak{l} \in \Omega)$ and $\mathbb{Q}(\alpha_{\mathfrak{l}}^2|\mathfrak{l} \in \Omega)$ is an infinite extension of $\mathbb{Q}$, where Ω is as in the above proof of Lemma 4.1.

Suppose now that $\mathcal{F}$ has complex multiplication by an imaginary quadratic extension $M/\mathbb{Q}$ in $\overline{\mathbb{Q}}$ as defined in (CM1). Let $\mathfrak{p}$ be the prime of M induced by $i_p|_M : M \hookrightarrow \overline{\mathbb{Q}}_p$. Then $\rho_{\mathbb{I}} \cong \text{Ind}_M^{\mathbb{Q}} \lambda$ as in (CM1), where $\lambda : \text{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^{\times}$ is a character unramified outside $\mathfrak{p}N$ for the normalization $\text{Spec}(\widetilde{\mathbb{I}})$ of $\text{Spec}(\mathbb{I})$. Write $\mathfrak{O}$ for the integer ring of M. We can forget about finitely many primes $\mathfrak{l}$ in a finite set S, as $\mathbb{Q}(\alpha_{\mathfrak{l}}|\mathfrak{l} \in S)$ is a finite extension of $\mathbb{Q}$. Take S to be the set of ramified primes of $\rho_{\mathbb{I}}$ (so, S includes ramified primes in $M/\mathbb{Q}$). Thus for any prime $\mathfrak{l} \notin S$, it is either split or inert in $M/\mathbb{Q}$. If $(l) = \mathfrak{l}\bar{\mathfrak{l}}$ in M for primes $\mathfrak{l} \neq \bar{\mathfrak{l}}$, α_l is the value of $\lambda_P(Frob_{\mathfrak{l}}) = \lambda(Frob_{\mathfrak{l}}) \mod P$ for a prime $\mathfrak{l}$ in M over l. If $(l) = \mathfrak{l}$ is inert in $M/\mathbb{Q}$, $\alpha_l^2 = \lambda_P(Frob_{\mathfrak{l}})$ as $\det(X - \rho_P(Frob_l)) = X^2 - \lambda_P(Frob_{\mathfrak{l}})$. Let $\mathbb{F}$ be the residue field of $\widetilde{\mathbb{I}}$ (note that $\widetilde{\mathbb{I}}$ is a local ring with maximal ideal $\mathfrak{m}$, because it is finite flat over Λ). Write W for the ring of Witt vectors of $\mathbb{F}$; so, W is a finite flat discrete valuation ring unramified over $\mathbb{Z}_p$. Let $(R, \widetilde{\lambda} : \text{Gal}(\overline{\mathbb{Q}}/M) \to R^{\times})$ be the universal couple with the universal character unramified outside $\mathfrak{p}N$ deforming $(\lambda \mod \mathfrak{m})$ over W. This couple $(R, \widetilde{\lambda})$ is characterized by the following universal property: For any local Artinian W-algebra A with residue field $\mathbb{F}$ and any character $\varphi : \text{Gal}(\overline{\mathbb{Q}}/M) \to A^{\times}$ unramified outside $\mathfrak{p}N$ with $\varphi \mod \mathfrak{m}_A = \lambda \mod \mathfrak{m}$ for the maximal ideal $\mathfrak{m}_A$ of A, there exists a unique W-algebra homomorphism $\iota : R \to A$ such that $\varphi = \iota \circ \widetilde{\lambda}$. The pair (A, φ) is called a deformation of λ (see [M]).

Writing $\widetilde{\Gamma}$ for the p-primary part of $Cl_M(\mathfrak{p}^{\infty}N) = \varprojlim_n Cl_M(\mathfrak{p}^n N)$ (for the ray class groups $Cl_M(\mathfrak{p}^n N)$ modulo $\mathfrak{p}^n N$ of M), we have $R \cong W[[\widetilde{\Gamma}]]$ by class field theory. To see this, we pick a deformation $\varphi : \text{Gal}(\overline{\mathbb{Q}}/M) \to A^{\times}$ of λ unramified outside $\mathfrak{p}N$; thus, A is a local Aritinian W-algebra sharing the residual field with

W and $\varphi \mod \mathfrak{m}_A = \lambda \mod \mathfrak{m}$ for the maximal ideal $\mathfrak{m}_A$ of A. Let λ_0 be the Teichmüller lift of $\lambda \mod \mathfrak{m}$; so, $\varphi' = \varphi\lambda_0^{-1}$ has p-power order. For a prime $l|N$, taking a prime factor $\mathfrak{l}$ of l in M, by class field theory, the image $I_{\mathfrak{l}}^{ab}$ of the inertia group $I_{\mathfrak{l}} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/M)$ in the Galois group of the maximal abelian extension of M over M is isomorphic to the multiplicative group $\mathfrak{O}_{\mathfrak{l}}^{\times}$ of the $\mathfrak{l}$-adic integer ring of $M_{\mathfrak{l}}$. Since φ' has p-power order and $p \neq l$, φ' must be trivial on $1 + \mathfrak{l}\mathfrak{O}_{\mathfrak{l}} \subset \mathfrak{O}_{\mathfrak{l}}^{\times}$. Thus the $\mathfrak{l}$-conductor of φ' is at most $\mathfrak{l}$, and hence $\varphi = \varphi'\lambda_0$ factors through $Cl_M(\mathfrak{p}^{\infty}N)$. Thus φ' factors through the maximal p-profinite quotient $\widetilde{\Gamma}$ and extends to a unique W-algebra homomorphism $\iota = \iota_{\varphi} : W[[\widetilde{\Gamma}]] \to A$ such that $\iota|_{\widetilde{\Gamma}} = \varphi'$. Since $\widetilde{\Gamma}$ is the maximal p-profinite quotient of $Cl_M(\mathfrak{p}^{\infty}N)$, by class field theory, we have the corresponding subfield $\widetilde{M}$ of the ray class field modulo $\mathfrak{p}^{\infty}N$ with $\mathrm{Gal}(\widetilde{M}/M) \cong \widetilde{\Gamma}$ by the Artin symbol. Writing the inclusion $\widetilde{\Gamma} \subset W[[\widetilde{\Gamma}]]$ as $\gamma \mapsto [\gamma]$ and identifying $\mathrm{Gal}(\widetilde{M}/M) = \widetilde{\Gamma}$, define a character $\widetilde{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to W[[\widetilde{\Gamma}]]$ by $\widetilde{\lambda}(\sigma) = \lambda_0(\sigma)[\sigma|_{\widetilde{M}}]$. Then by our construction $\iota \circ \widetilde{\lambda} = \varphi$; so, $(W[[\widetilde{\Gamma}]], \widetilde{\lambda})$ satisfies the universal property of $(R, \widetilde{\lambda})$ for deformations φ of λ.

Let $\Delta \subset \widetilde{\Gamma}$ be the maximal torsion subgroup of $\widetilde{\Gamma}$. Then Δ is finite, and we put $\Gamma_{tf} = \widetilde{\Gamma}/\Delta$. We fix a splitting $\widetilde{\Gamma} \cong \Delta \times \Gamma_{tf}$; so, $R \cong W[[\Gamma_{tf}]][\Delta]$. Since Γ_{tf} is a p-profinite cyclic group isomorphic to the additive group $\mathbb{Z}_p$, $W[[\Gamma_{tf}]]$ is isomorphic to a one-variable power series ring $W[[T]]$. Thus $\mathrm{Spec}(R)$ is equidimensional reduced, and its irreducible components are isomorphic to $\mathrm{Spec}(W[[T]])$.

By universality, we have a W-algebra homomorphism $\pi : R \to \widetilde{\mathbb{I}}$ such that $\pi \circ \widetilde{\lambda} = \lambda$. The image $\mathbb{I}'$ of π is the $\mathbb{I}$-subalgebra of $\widetilde{\mathbb{I}}$ topologically generated by $\lambda(\sigma)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Since $\mathrm{Spec}(\mathbb{I}')$ and $\mathrm{Spec}(R)$ has equal dimension, $\mathrm{Spec}(\mathbb{I}')$ is isomorphic to one of the irreducible components of $\mathrm{Spec}(R)$; so, π induces $\mathrm{Spec}(\mathbb{I}') \cong \mathrm{Spec}(W[[T]]) = \mathrm{Spec}(W[[\Gamma_{tf}]])$. Since $\mathrm{Spec}(\widetilde{\mathbb{I}})$ is normalization of $\mathrm{Spec}(\mathbb{I})$ and $\mathrm{Spec}(\mathbb{I}') \cong \mathrm{Spec}(W[[\Gamma_{tf}]]$ is regular, we have $\widetilde{\mathbb{I}} = \mathbb{I}' \cong W[[\Gamma_{tf}]]$.

The inclusion $\mathbb{Z}_p^{\times} \hookrightarrow \mathfrak{O}_{\mathfrak{p}}^{\times}$ induces an inclusion $\Gamma \to Cl_M(\mathfrak{p}^{\infty}N)$. Composing this inclusion with the projection: $Cl_M(\mathfrak{p}^{\infty}N) \twoheadrightarrow \Gamma_{tf}$, we regard Γ as a subgroup of $\widetilde{\Gamma}_{tf}$ of finite index. This $W[[\Gamma]]$-algebra structure of $\mathbb{I} = W[[\Gamma_{tf}]]$ is equal to that coming from the inclusion $W[[\Gamma]] \hookrightarrow \mathbb{I}$ given by $\gamma \mapsto (1+X)$ for the variable X not x. This is because a Hecke character of infinity type $k-1$ gives rise to its theta series of weight k. Then for an arithmetic point P with $r(P) \leq r$, $\lambda_P = P \circ \lambda$ has infinity type $k(P) - 1$; that is, $\lambda_P(\alpha) = \alpha^{k(P)-1}$ for $\alpha \in M$ congruent to 1 modulo Np^r. Thus for the class number h of M, taking a generator α of $\mathfrak{l}^h$, we have $\lambda_P(\mathfrak{l}) = \alpha^{(1/h)(k(P)-1)}\zeta$ for $\zeta \in \mu_{p^r h}$. In other words, choosing a complete representative set $\{\mathfrak{a}_j\}_{j=1,\dots,h}$ of ideal classes of M, taking a generator α_j of $\mathfrak{a}_j^h$ and writing $K_r = \mathbb{Q}(\mu_{p^r h})[\alpha_j^{1/h}|j=1,\dots,h]$, we find that $\mathbb{Q}(\alpha_l^2) \subset K_r$, which is a finite extension of $\mathbb{Q}$ independent of P (as long as $r(P) \leq r$); so, $\widetilde{M}_{V,r} \subset K_r$. This finishes the proof. $\qquad\square$

Proof of (WH). By the above proof of (WV), if $\mathcal{F}$ has complex multiplication, $\widetilde{M}_{H,r}(\mathcal{F}) \subset K_{\infty} = \bigcup_r K_r = \mathbb{Q}(\mu_{p^{\infty}}, \mu_h)[\alpha_j^{1/h}|j=1,\dots,h]$, which is a finite extension of $\mathbb{Q}(\mu_{p^{\infty}})$.

In the proof of Lemma 4.1, for a non-CM form f, we constructed an infinite set Ω of primes such that $\{\mathbb{Q}(f)(\alpha_{l(\mathfrak{l})}^2)\}_{\mathfrak{l}\in\Omega}$ is an infinite set of distinct quadratic extensions of $\mathbb{Q}(f)$. Since $L := \mathbb{Q}(f)[\mu_{p^{\infty}}]$ has only finitely many distinct quadratic

extensions of $\mathbb{Q}(f)$ inside, the extension $L(\alpha_{l(\mathfrak{l})}^2|\mathfrak{l} \in \Omega)/L$ still has infinite dimension over $\mathbb{Q}[\mu_{p^\infty}]$. Thus the result follows, taking $f = f_P$ for an arithmetic member f_P ($k(P) = k$) of a non-CM family $\mathcal{F}$. $\qquad\square$

We have the following corollary to the above proofs of (WV) and (WH):

Corollary 4.2. *Let the notation be as in the previous section. If $\mathcal{F}$ is a CM family, then*

$$[M_{H,\mathcal{A}}(\mathcal{F}) : \mathbb{Q}(\mu_{p^\infty})] < \infty \quad and \quad [M_{V,\mathcal{A}}(\mathcal{F}) : \mathbb{Q}] < \infty,$$

where $\mathcal{A}$ is an infinite set of arithmetic points for a fixed weight $k \geq 2$ in the horizontal case and of bounded p-power level $p^r\mathbf{p}$ in the vertical case.

Proof. The necessity of taking the square α_l^2 in the above proof of (WV) and (WH) comes from the inertness of the prime l in M (and also the fact that we vary the primes l in an infinite set). Since p is split in M, we have $\alpha_{p,P} = \lambda_P(Frob_{\mathfrak{p}})$ for a factor $\mathfrak{p}$ of p in M, and under the notation of the proof of (WV), we find that $M_{V,\mathcal{A}}(\mathcal{F}) \subset K_r$, since the prime p splits in M. The horizontal case can be proven similarly, because $M_{H,\mathcal{A}}(\mathcal{F}) \subset K_\infty$. $\qquad\square$

We add one more lemma:

Lemma 4.3. *Let $\mathcal{F}$ be a slope 0 p-adic analytic family of Hecke eigenforms with coefficients in $\mathbb{I}$. Then we have:*
   (1) *Fix $0 \leq r < \infty$. Then the degree $[\mathbb{Q}(f_P) : \mathbb{Q}(a(p, f_P))]$ for arithmetic P with $r(P) \leq r$ is bounded independently of P.*
   (2) *Let $K = \mathbb{Q}(\mu_{p^\infty})$ and fix $k \geq 2$. Then the degree $[K(f_P) : K(a(p, f_P))]$ for arithmetic P with $k(P) = k$ is bounded independently of P.*

Proof. Since the proof is basically the same, we prove (1). We prove that the index $[L(f_P) : L(a(p, f_P))]$ is bounded independently of P for $L = \mathbb{Q}(\mu_{p^r})$. Let $\Gamma_r = \Gamma_1(N\mathbf{p}) \cap \Gamma_0(p^r\mathbf{p})$. Then by the control theorem as in (C1–2) in Section 1, $\dim_{\mathbb{C}_p} S_k^{ord}(\Gamma_r, \varepsilon; \mathbb{C}_p)$ is a constant d independent of k and r, where $S_k(\Gamma_r, \varepsilon; A)$ is the space of modular forms with coefficients in A with character $\varepsilon : \Gamma/\Gamma^{p^r} \to A^\times$. Since ε has values in $\mu_{p^r}(\mathbb{C}_p)$, if $\sigma \in \mathrm{Aut}(\mathbb{C}_p/L(a(p, f_P)))$, then f_P^σ is another Hecke eigenform within the same space $S_k^{ord}(\Gamma_r, \varepsilon_P; \mathbb{C}_p)$. Thus

$$[L(f_P) : L(a(p, f_P))] = \#\{f_P^\sigma|\sigma \in \mathrm{Aut}(\mathbb{C}_p/L(a(p, f_P)))\} \leq d,$$

as desired. $\qquad\square$

5. Results towards the strong horizontal theorem

We start with

Lemma 5.1. *Let $W \subset \overline{\mathbb{Q}}_p$ be a valuation ring finite flat over $\mathbb{Z}_p$ with quotient field F. Let $\Phi(T) \in W[[T]]$, and suppose that there is an infinite subset $\Omega \subset \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ such that $\Phi(\zeta - 1) \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ for all $\zeta \in \Omega$. Then there exists $\zeta_0 \in \mu_{p^\infty}(W)$ and $s \in \mathbb{Z}_p$ such that $\zeta_0^{-1}\Phi(T) = (1 + T)^s = \sum_{n=0}^\infty \binom{s}{n}T^n$.*

We shall give two proofs of the lemma. The second more elementary one is due to Kiran Kedlaya.

First proof. We use the following lemma of Chai (see [Ch1, Theorem 4.2], [Ch2, Theorem 6.6] and [H10, Theorem 3.6]), which in the simplest case can be stated as

Rigidity Lemma. *Let Z be an integral connected formal subscheme Z of codimension 1 containing the identity of $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_{m/\overline{\mathbb{F}}_p} = \mathrm{Spf}(\widehat{\overline{\mathbb{F}}_p[t,t^{-1}]}) \times \mathrm{Spf}(\widehat{\overline{\mathbb{F}}_p[t',t'^{-1}]})$. If Z is stable under the diagonal action $(t,t') \mapsto (t^z, t'^z)$ for all z in an open subgroup U of $\mathbb{Z}_p^\times$, then Z is a formal subtorus of codimension 1.*

Making the variable change $T \mapsto \zeta_1^{-1}(T+1) - 1$ for a $\zeta_1 \in \Omega$ (replacing W by its finite extension if necessary), we may replace Ω by $\zeta_1^{-1}\Omega \ni 1$; so, rewriting $\zeta_1^{-1}\Omega$ as Ω, we may assume that $1 \in \Omega$. Then $\Phi(0) = \zeta_0 \in \mu_{p^\infty}$. Thus again replacing Φ by $\zeta_0^{-1}\Phi$, we may assume that $\Phi(0) = 1$.

For $\sigma \in \mathrm{Gal}(F(\mu_{p^\infty})/F)$ with the quotient field F of W, $\Phi(\zeta^\sigma - 1) = \Phi(\zeta - 1)^\sigma$. Writing $\phi(\zeta) = \Phi(\zeta - 1)$, the above identity yields $\phi(\zeta^\sigma) = \phi(\zeta)^\sigma$. Identifying $\mathrm{Gal}(F(\mu_{p^\infty})/F)$ with an open subgroup Γ of $\mathbb{Z}_p^\times$ and writing $\sigma_z \in \mathrm{Gal}(F(\mu_{p^\infty})/F)$ for the element corresponding to $z \in \Gamma$, we find that

$$\Phi \circ z(\zeta - 1) = \Phi(\zeta^z - 1) = \Phi(\zeta^{\sigma_z} - 1) = \Phi(\zeta - 1)^{\sigma_z} = z \circ \Phi(\zeta - 1).$$

We find that $z \circ \phi = \phi \circ z$ is valid on the Zariski dense subset Ω of $\mathrm{Spec}(W[[T]])$; so, ϕ as a scheme endomorphism of $\widehat{\mathbb{G}}_m$ commutes with the action of z.

Regard $W[[T]]$ as the affine ring of the formal torus $\widehat{\mathbb{G}}_{m/W} = \mathrm{Spf}(\widehat{W[t,t^{-1}]})$ (so that T is given by $t - 1$). Note that $z \in \mathbb{Z}_p^\times$ acts on $\widehat{\mathbb{G}}_m$ as a group automorphism induced by a W-bialgebra automorphism of $W[[T]]$ sending $t = (1+T) \mapsto t^z = (1+T)^z$. Take $\phi \in \mathrm{End}_{W\text{-SCH}}(\widehat{\mathbb{G}}_m)$ sending 1 to 1. Put $\widehat{\mathbf{G}} := \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_{m/W}$. We consider the graph Γ_ϕ of ϕ, which is an irreducible formal subscheme $\Gamma_\phi \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ smooth over W. Writing the variable on $\widehat{\mathbf{G}}$ as (T, T'), Γ_ϕ is the closed formal subscheme defined by the principal ideal $(t' - \phi(t))$. If $\phi \circ z = z \circ \phi$ for all z in an open subgroup U of $\mathrm{Aut}_{gp}(\widehat{\mathbb{G}}_m) = \mathbb{Z}_p^\times$, then Γ_ϕ is stable under the diagonal action of U on $\widehat{\mathbf{G}}$. Then, extending the scalar from W to the composite $\widetilde{W}$ of W and the ring $W(\overline{\mathbb{F}}_p)$ of Witt vectors with coefficients in the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, by the rigidity lemma, we find that the reduction modulo $\mathfrak{m}_{\widetilde{W}}$ of Γ_ϕ is a formal subtorus of $\widehat{\mathbf{G}}_{/\overline{\mathbb{F}}_p}$.

We regard Φ as an endomorphism ϕ of $W[[T]]$ sending $1 + T$ to $\Phi(T)$. In other words, $f(T)^\phi = f(\Phi(T) - 1)$. Applying the above argument to the morphism ϕ, we find $\Phi(T) \equiv (1+T)^s \mod \mathfrak{m}_{\widetilde{W}}$ for some $s \in \mathbb{Z}_p$. The quotient $\Phi_1(T) = \Phi(T)/(1+T)^s \in W[[T]]$ again satisfies $\Phi_1(t^z) = \Phi_1(t)^z$ for $z \in U$ and $\Phi_1(0) = 1$. Then by the same argument as above, $\Phi_1(T) \equiv (1+T)^{s_1} \mod \mathfrak{m}_{\widetilde{W}}$ for some $s_1 \in \mathbb{Z}_p$. On the other hand, $(1+T)^{s_1} \equiv \Phi_1(T) = 1 \mod \mathfrak{m}_{\widetilde{W}}$ in $\overline{\mathbb{F}}_p[[T]]$, which shows that $s_1 = 0$. Since $\Phi(\zeta - 1) \in \mu_{p^\infty}$ for $\zeta \in \Omega$, $\Phi_1(\zeta - 1) \equiv 1 \mod \mathfrak{m}_W$. Since there are only finitely many p-power roots of unity congruent to 1 modulo $\mathfrak{m}_{\widetilde{W}}$, Φ_1 is constant. Since $\Phi_1(0) = 1$, we get $\Phi_1 = 1$, and $\Phi(T) = (1+T)^s$, as desired. This fact is mentioned in [Ch2, Remark 6.6.1 (iv) after Theorem 6.6].

Second proof (Kiran Kedlaya). As we have remarked above, we may assume that $1 \in \Omega$ and $\Phi(0) = 1$. Note that $t = 1 \Leftrightarrow T = 0$. Thus we are trying to show that $\Phi(T) = (1+T)^s$ for some $s \in \mathbb{Z}_p$. In this proof, the residue field $\mathbb{F}$ of W is a finite extension of $\mathbb{F}_p$.

Write the valuation of W as v (and use the same symbol v for an extension of v to $W[\mu_{p^\infty}]$). Normalize v so that $v(p) = 1$. Suppose that $\Phi(T) \notin W$ (nonconstant). Write $\Phi(T) - 1 = \sum_{i=1}^\infty a_i T^i$. Since W is a discrete valuation ring, there is a least

index $j > 0$ for which $v(a_j)$ is minimized. For ϵ sufficiently small, if $v(\tau) = \epsilon$, then $v(\Phi(\tau) - 1) = v(a_j) + j\epsilon$. In particular, for ζ a p-power root of unity, taking $\tau = \zeta - 1$, we have $v(\zeta - 1) = p^{-m}/(p-1)$ for some nonnegative integer m, so we have infinitely many relations of the form $jp^{-m}/(p-1) + v(a_j) = p^{-n}/(p-1)$. Then, we have $m \to \infty \Rightarrow n \to \infty$ (by continuity and nonconstancy of $\tau \mapsto \Phi(\tau)$); so, taking limits under $m \to \infty$ yields $v(a_j) = 0$. Also, j must be a power of p, say $j = p^h$, and for m large we have $n = m - h$.

Since $v(a_j) = 0$, $a_j \mod \mathfrak{m}_W$ is in $\mathbb{F}^\times$. For the moment, assume $\mathbb{F} = \mathbb{F}_p$. That is, a_j reduces to an integer b_0 coprime to p in the residue field of W. We can thus replace $\Phi(T)$ by $\Phi_1(T)$ defined by $\Phi(T) = \Phi_1(T) \times (1 + T)^s$ for some s (namely $s = b_0 j = b_0 p^{h_0}$ for $h_0 := h$) so as to increase the least index j for which $v(a_j) = 0$. Indeed, writing $\Phi(T) = \sum_{n=0}^{j} a_n T^n + T^{j+1} f(T)$ with $f(T) \in W[[T]]$, we have

$$\sum_{n=0}^{j} a_n T^n \equiv 1 + b_0 T^{p^{h_0}} \equiv (1 + T^{p^{h_0}})^{b_0} \equiv (1 + T)^s \mod (\mathfrak{m}_W + (T^{j+1})).$$

We have $\Phi_1(T) \equiv 1 + T^{j+1} f(T)(1 + T)^{-s} \equiv 1 \mod (\mathfrak{m}_W + (T^{j+1}))$. Thus if we write j_1 for the j for this new Φ_1, then $j_1 > j$, and $j_1 = p^{h_1}$ with $h_1 > h_0$ and $a_{j_1} \equiv b_1 \mod \mathfrak{m}_W$ for $b_1 \in \mathbb{Z}$. Repeating this, for $s = \sum_{k=0}^{\infty} b_k p^{h_k} \in \mathbb{Z}_p$,

$$\frac{\Phi(T)}{(1 + T)^s} - 1 = \sum_{n=1}^{\infty} a_n T^n$$

no longer has a least j with minimal $v(a_j)$; so, $\Phi(T)/(1 + T)^s = 1$, and we get $\Phi(T) = (1 + T)^s$.

Suppose now that $\mathbb{F} \neq \mathbb{F}_p$. We have the Frobenius automorphism ϕ fixing $\mathbb{Z}_p[\mu_{p^\infty}] \subset W[\mu_{p^\infty}]$. Letting ϕ acts on power series by $(\sum_n a_n T^n)^\phi = \sum_n a_n^\phi T^n$, we find $\Phi^\phi(t^\phi) = \Phi(t)^\phi$. Since $\Phi(\zeta - 1)$ is a p-power root of unity for ζ in an infinite set $\Omega \subset \mu_{p^\infty}$, we have $\Phi^\phi(\zeta - 1) = \Phi^\phi(\zeta^\phi - 1) = \Phi(\zeta - 1)^\phi = \Phi(\zeta - 1)$. Since $\Omega \subset \widehat{\mathbb{G}}_m$ is Zariski dense, we find that $\Phi^\phi = \Phi$, which shows that $\Phi \in W^\phi[[T]]$ for the subring W^ϕ fixed by ϕ. Note that the residue field of W^ϕ is $\mathbb{F}_p$, and the earlier argument applies to $\Phi \in W^\phi[[T]]$. $\square$

Take a prime l with $\alpha_{l,P} \neq 0$ for some P. If $l | Np$, put $A = a(l)$ (the image of $U(l)$ in $\mathbb{I}$), and otherwise, fix a root A of $\det(T - \rho_{\mathbb{I}}(Frob_l)) = 0$. Replacing $\mathbb{I}$ by its finite extension, we assume that $A \in \mathbb{I}$. Recall that $A_P = P(A)$. Let $X = \gamma^{-1}(1+x) - 1$ be the variable of $\Lambda = W[[\Gamma]]$ centered at weight 1. Let Q be the quotient field of Λ and fix its algebraic closure $\overline{Q}$. We embed $\mathbb{I}$ into $\overline{Q}$ and regard it as a subring of $\overline{Q}$. Take the p^n-th root X^{1/p^n} of X in $\overline{Q}$ and consider $W[\mu_{p^n}][[X]][X^{1/p^n}] \subset \overline{Q}$ which is independent of the choice of X^{1/p^n}.

Recall, as explained after Conjecture 3.4, that if a prime l is a factor of N (so $l \neq p$) and if f_P (or more precisely the automorphic representation generated by f_P) is Steinberg (resp. super-cuspidal) at l for an arithmetic point P, then all members of $\mathcal{F}$ are Steinberg (resp. super-cuspidal) at l. Since $\alpha_{l,P} \neq 0$ for some P, f_P is not super-cuspidal at l for any arithmetic P.

Proposition 5.2. *Let the notation be as above and write $K := \mathbb{Q}[\mu_{p^\infty}]$ and $L_P = K(A_P)$ for each arithmetic point P with $k(P) = k$. Fix a rational prime l. Suppose that there exists an infinite set $\mathcal{A}$ of arithmetic points with $k(P) = k \geq 2$ satisfying*

one of the following two conditions:

(i) *L_P/K is a finite extension of bounded degree independently of $P \in \mathcal{A}$, and in L_P/K, the prime l is at worst tamely ramified for all $P \in \mathcal{A}$.*

(ii) *The composite of L_P for all $P \in \mathcal{A}$ is a finite extension of $\mathbb{Q}[\mu_{p^\infty}]$.*

Then we have $A \in W[\mu_{p^n}][[X]][(1+X)^{1/p^n}] \cap \mathbb{I}$ in $\overline{\mathbb{Q}}$ for $0 \le n \in \mathbb{Z}$, and we have the following two possibilities:

(1) *There exist a Weil l-number α_1 of weight 1 and a root of unity ζ_0 such that $A_P = \alpha_{l,P} = \zeta_0 \langle \alpha_1 \rangle^{k(P)-1}$; in other words, $A(X) = \zeta_0(1+X)^s$ for $s = \frac{\log_p(\alpha_1)}{\log_p(\gamma)}$ with the Iwasawa p-adic logarithm $\log_p$. This is the case where the automorphic representation generated by f_P for an arithmetic point P with $k(P) > 2$ or with $\varepsilon_P \ne 1$ is in the principal series at l.*

(2) *We have $l \ne p$, and there exists a root of unity ζ_0 such that $A(X) = \zeta_0 l^{-1/2}(1+X)^s$ for $s = \frac{\log_p(l)}{2\log_p(\gamma)}$. This is the case where the automorphic representation generated by f_P for an arithmetic points P is Steinberg at $l \ne p$.*

Proof. We give a proof assuming (i), since the proof of the other case is almost identical using Lemma 2.3 in place of Corollary 2.5. By Corollary 2.5, we have only a finite number of Weil l-numbers of weight k in $\bigcup_{P \in \mathcal{A}} L_P$ up to multiplication by roots of unity, and hence A_P for $P \in \mathcal{A}$ hits one of such Weil l-numbers α of weight $k - 1$ infinitely many times, up to roots of unity, unless the automorphic representation generated by f_P is Steinberg at $l \ne p$. If f_{P_0} is Steinberg at $l \ne p$ for one arithmetic P_0, f_P is Steinberg for all arithmetic P. This fact, as already explained, follows either from the Λ-adic version of the theory of new forms (cf. [H88, Theorem 3.6]) or from the rigidity of local Galois deformations at l in characteristic 0 (cf. [HMI, Theorem 3.75]) or from the fact that we can shift the family $\mathcal{F}$ to automorphic forms on a quaternion algebra ramified at l by the Jacquet-Langlands correspondence (cf. [H00, Section 7]). Thus $A_P = l^{-1/2}l^{(k(P)-1)/2}$ up to roots of unity for all $P \in \mathcal{A}$ (see [MFM, Theorem 4.6.17]). In this case, we put $\alpha = l^{(k-2)/2}$.

Suppose for the moment that $\mathbb{I} = W[[X]]$ for a discrete valuation ring W finite flat over $\mathbb{Z}_p$ (for the variable X centered at weight 1); so, $n = 0$. After a variable change $X \mapsto Y = \gamma^{1-k}(1+X) - 1$, we have $A(Y)|_{Y=0} = A(X)|_{X=\gamma^{k-1}-1}$. Note that $|\alpha|_p = 1$. Let $\Omega_1 = \{\varepsilon_P(\gamma)|P \in \mathcal{A}\}$, which is an infinite set in $\mu_{p^\infty}(K)$. Let $\Phi_1(Y) := \alpha^{-1}A(Y) = A(\gamma^{1-k}(1+X) - 1) \in W[[Y]]$ and $\mathbf{L}$ be the composite of L_P for P running through $\mathcal{A}$. The subset Ω_2 of Ω_1 made up of $\zeta \in \Omega_1$ such that $\Phi_1(\zeta - 1)$ is a root of unity in $\mathbf{L}$ is an infinite set. By Lemma 2.6, the group of roots of unity of $\mathbf{L}$ contains $\mu_{p^\infty}(K)$ as a subgroup of finite index, and we find an infinite subset $\Omega \subset \Omega_2$ and a root of unity ζ_1 such that $\{\Phi_1(\zeta - 1)|\zeta \in \Omega\} \subset \zeta_1 \mu_{p^\infty}(K)$. Then $\Phi = \zeta_1^{-1}\Phi_1$ satisfies the assumption of Lemma 5.1, and for a root of unity ζ, we have $A(Y) = \zeta\alpha(1+Y)^{s_1}$ for $s_1 \in \mathbb{Z}_p$, and $A(X) = \zeta\alpha(\gamma^{1-k}(1+X))^{s_1}$. Let $X = \zeta'\gamma^{k'-1} - 1$ for $\zeta' \in \mu_{p^\infty}(K)$. Then $A(\zeta'\gamma^{k'-1} - 1) = \zeta\alpha(\zeta'\gamma^{-k+k'})^{s_1}$, which is equal either to a Weil l-number of weight $k' - 1$ (the case of principal series at l) or to a root of unity ζ'' times $l^{(k'-2)/2}$ (the case of Steinberg representation at l). In the latter case, taking the logarithm of $\zeta\alpha(\gamma^{-k+k'})^{s_1} = \zeta''l^{(k'-2)/2}$, we find

$$s_1 = \frac{\log_p(l)}{2\log_p(\gamma)}.$$

This shows that $l = p$ is impossible, since f_P generates an automorphic representation principal at p if $k(P) > 2$ by p-ordinarity. In the former case, take $k' > 2$. Then

$$\alpha_1 := \frac{A(\gamma^{k'} - 1)}{A(\gamma^{k'-1} - 1)} = \frac{\zeta\alpha(\zeta'\gamma^{-k+k'+1})^{s_1}}{\zeta\alpha(\zeta'\gamma^{-k+k'})^{s_1}} = \gamma^{s_1},$$

which is an algebraic number α_1 independent of k'. Note that for $k' > 2$, α_1 is a ratio of Weil l-numbers of weight $k' - 1$ and k', and hence α_1 is not a root of unity. Thus we have $s_1 = \frac{\log(\alpha_1)}{\log_p(\gamma)}$. We now equate

$$\zeta\zeta_\alpha \gamma^{\log_p(\alpha)/\log_p(\gamma)}(\gamma^{1-k}(1+X))^{\log_p(\alpha_1)/\log_p(\gamma)} = \zeta_0(1+X)^{\log_p(\alpha_1)/\log_p(\gamma)},$$

where $\alpha = \zeta_\alpha \gamma^{\log_p(\alpha)/\log_p(\gamma)}$ for roots ζ_α and ζ_0 of unity. By putting $X = 0$, we get

$$\zeta\zeta_\alpha \gamma^{\log_p(\alpha)/\log_p(\gamma)+(1-k)(\log_p(\alpha_1)/\log_p(\gamma))} = \zeta_0,$$

which shows that

$$\zeta_0 = \zeta\zeta_\alpha \quad \text{and} \quad (k-1)s_1 = \log_p(\alpha)/\log_p(\gamma).$$

We conclude that $\alpha_1 = \sqrt[k-1]{\langle \alpha \rangle}$ for $\langle \alpha \rangle = \alpha\zeta_\alpha^{-1}$, which is a Weil l-number of weight 1.

We now assume that $A \in W[[X]][(1+X)^{1/p^n}]$. Since

$$\mathrm{Spf}(W[[X]][(1+X)^{1/p^n}]) \cong \widehat{\mathbb{G}}_m \xrightarrow{t \mapsto t^{p^n}} \widehat{\mathbb{G}}_m = \mathrm{Spf}(W[[X]]),$$

by applying the same argument as above to $W[[X]][(1+X)^{1/p^n}]$, we get $A(X) = \zeta_0(1+X)^{s_1}$ for $s_1 = \log_p(\alpha_1)/\log_p(\gamma)$, where $\alpha_1 = \sqrt[k-1]{\langle \alpha \rangle}$ for $\langle \alpha \rangle = \alpha\zeta_\alpha^{-1}$.

We thus need to show that $A \in W[\mu_{p^n}][[X]][(1+X)^{1/p^n}]$ for sufficiently large n, and then the result follows from the above argument. Again we make the variable change $X \mapsto Y$ that we have already done. Replacing A by $\alpha^{-1}A$ for a suitable Weil l-number α of weight k (up to $\mu_{p^\infty}(\overline{\mathbb{Q}}_p)$), we may assume that there exists an infinite set $\mathcal{A}_0 \subset \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ such that $P \cap \Lambda = (1 + Y - \zeta_P)$ for $\zeta_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ and $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ for all $P \in \mathcal{A}_0$. By another variable change $(1+Y) \mapsto \zeta(1+Y)$ for a suitable $\zeta \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ (as explained in the beginning of the first proof of Lemma 5.1), we may further assume that we have $P_0 \in \mathcal{A}_0$ with $\zeta_{P_0} = 1$ and $A_{P_0} = 1$ (specifying α well in $\alpha \cdot \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$). We now write $\mathbb{J}$ for the subalgebra of $\mathbb{I}$ topologically generated by A over $\Lambda = W[[Y]]$. Then we have $\mathbb{J} = \Lambda[A] \subset \mathbb{I}$. Replacing W by its finite extension, we may assume that W is integrally closed in $\mathbb{J}$. Since A is a unit in $\mathbb{I}$, we may embed the irreducible formal scheme $\mathrm{Spf}(\mathbb{J})$ into $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m = \mathrm{Spf}(W[t, \widehat{t^{-1}, t'}, t'^{-1}])$ by the surjective W-algebra homomorphism $\pi : W[t, \widehat{t^{-1}, t'}, t'^{-1}] \twoheadrightarrow \mathbb{J}$ sending (t, t') to $(1 + Y, A)$. Write $Z \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ for the image of $\mathrm{Spf}(\mathbb{J})$. Thus we are identifying Λ with $W\widehat{[t, t^{-1}]}$ by $t \leftrightarrow 1 + Y$. Then $P_0 \in Z$ is the identity element of $(\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m)(\overline{\mathbb{Q}}_p)$. Since A is integral over Λ, it is a root of a monic polynomial $\Phi(t') = \Phi(t, t') = t'^d + a_1(t)t'^{d-1} + \cdots + a_d(t) \in \Lambda[t']$ irreducible over the quotient field Q of Λ, and we have $\mathbb{J} \cong \Lambda[t']/(\Phi(t, t'))$. Thus $\mathbb{J}$ is free of rank d over Λ; so, $\pi : Z \to \widehat{\mathbb{G}}_m = \mathrm{Spf}(\Lambda)$ is a finite flat morphism of degree d. We let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ act on Λ by $\sum_{n=0}^{\infty} a_n Y^n \mapsto \sum_{n=0}^{\infty} a_n^\sigma Y^n$ and on $\Lambda[t']$ by $\sum_j A_j(Y)t'^j \mapsto \sum_j A_j^\sigma(Y)t'^j$ for $A_j(Y) \in \Lambda$. Note that $\Phi(\zeta_P, A_P) = 0$ for $P \in \mathcal{A}_0$. Since $A_P \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$, $A_P^\sigma = A_P^{\chi(\sigma)}$ for the p-adic cyclotomic character $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$. Since W is a discrete valuation ring, for its quotient field

F, the image of χ on $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is an open subgroup U of $\mathbb{Z}_p^\times$. Thus we have $\Phi^\sigma(\zeta_P^{\chi(\sigma)}, A_P^{\chi(\sigma)}) = \Phi(\zeta_P, A_P)^\sigma = 0$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, $\Phi^\sigma = \Phi$. Thus we get

$$\Phi(\zeta_P^{\chi(\sigma)}, A_P^{\chi(\sigma)}) = \Phi(\zeta_P, A_P)^\sigma = 0 \;\; \text{for all } P \in \mathcal{A}_0.$$

In other words, for the integral closed formal subscheme $Z_s \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ defined by $\Phi(t^s, t'^s) = 0$, we have $\mathcal{A}_0 \subset Z \cap Z_s$ if $s \in U$. Since Z and Z_s are finite flat over Λ and $\mathcal{A}_0$ is an infinite set, we conclude that $Z = Z_s$. Thus $Z \subset \widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$ is stable under the diagonal action $(t, t') \mapsto (t^s, t'^s)$ for $s \in U$. We may assume that $U = 1 + p^r \mathbb{Z}_p$ for $r > 0$. Since Z is flat of relative dimension 1 over W, replacing W by its finite extension if necessary, we find in Z a W-point $(t_0, t_0') \in \widehat{\mathbb{G}}_m^2(W)$ of infinite order (indeed, $(t_0, t_0') = (Q, A(Q))$ for an arithmetic point Q with $k(Q) \neq k$ does the job). Thus we have an infinite set $\Xi = \left\{ (t_0^s, t_0'^s) \big| s \in U \right\} = \left\{ (t_0 t_0^{p^r u}, t_0' t_0'^{p^r u}) \big| u \in p^n \mathbb{Z}_p \right\}$ inside Z. By translation $\tau : (t, t') \mapsto (t t_0^{-1}, t' t_0'^{-1})$, we find that $\tau(Z)$ contains

$$\Xi_0 = \left\{ (t_0^{p^r u}, t_0'^{p^r u}) \big| u \in p^n \mathbb{Z}_p \right\}.$$

Since Z is integral of codimension 1 finite flat over $\mathrm{Spf}(\Lambda)$, $\tau(Z)$ is the Zariski closure of the infinite subgroup Ξ_0 of $\widehat{\mathbb{G}}_m^2$. Then $\tau(Z)$ must be a formal subgroup of $\widehat{\mathbb{G}}_m^2$ of codimension 1, and Z is a coset $(t_0, t_0')\tau(Z)$ in $\widehat{\mathbb{G}}_m \times \widehat{\mathbb{G}}_m$. Since Z contains the identity P_0, we must have $\tau(Z) = Z$. Since $\pi : Z \to \mathrm{Spf}(\Lambda) = \widehat{\mathbb{G}}_m$ is finite flat of degree d, $\pi : Z \to \widehat{\mathbb{G}}_m$ is an isogeny; so, Z is a formal multiplicative group and is a formal subtorus of $\widehat{\mathbb{G}}_m^2$. In particular, we obtain $Z/\mathrm{Ker}(p^n : Z \to Z) \overset{\pi}{=} \widehat{\mathbb{G}}_m$ and hence $d = p^n$ for $n \geq 0$. Thus

$$\mathbb{J} = \Lambda[A] \subset W[\mu_{p^n}][[Y]][(1 + Y)^{p^{-n}}] = W[\mu_{p^n}][[X]][(1 + X)^{p^{-n}}],$$

as desired. This finishes the proof. $\qquad\square$

Corollary 5.3. *If $a(p, f_P)$ for an arithmetic point P with $k(P) = 2$ is a root of unity (that is, the automorphic representation generated by f_P is Steinberg at p), then the extensions $M_{H,\mathcal{A}}(\mathcal{F})$ and $\mathbb{Q}_{H,k}(\mathcal{F})$ are infinite extensions of $\mathbb{Q}(\mu_{p^\infty})$ for any integer $k \geq 2$.*

Proof. Since $\mathbb{Q}_{H,k}(\mathcal{F}) \supset M_{H,\mathcal{A}}(\mathcal{F})$, we prove infiniteness of $[M_{H,\mathcal{A}}(\mathcal{F}) : \mathbb{Q}(\mu_{p^\infty})]$. If $M_{H,\mathcal{A}}(\mathcal{F})$ is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$, we are in Case (1) of the above proposition for $l = p$. Then $a(p, f_P)$ for a weight 2 arithmetic point P is a weight 1 Weil p-number that is not a root of unity. Thus if we have one arithmetic point with a root of unity $a(p, f_P)$, then $M_{H,\mathcal{A}}(\mathcal{F})$ has to be an infinite extension of $\mathbb{Q}(\mu_{p^\infty})$. $\qquad\square$

6. Proof of the strong horizontal theorem

Let X be the variable of $W[[\Gamma]]$ centered at weight 1. We start with a couple of preliminary results. Consider the endomorphism $\sigma_s : (1 + X) \mapsto (1 + X)^s = \sum_{n=0}^\infty \binom{s}{n} X^n$ of a power series ring $B[[X]]$ over B. Take $B = W$ and write $\sigma = \sigma_2$. Let G be a group.

Lemma 6.1. *Let the notation be as above. Let A be an integral domain over $W[[X]]$ of characteristic different from 2 with quotient field F. Assume that the endomorphism σ_2 on $W[[X]]$ extends to an endomorphism σ of the field F. Let $\rho : G \to GL_2(F)$ be a representation, and put $\rho^\sigma := \sigma \circ \rho$. Let $H \subset G$ be a*

subgroup of G, and put $\rho_H := \rho|_H$. *Suppose we have a subgroup H of G such that* $\mathrm{Tr}(\rho_H^\sigma) = \mathrm{Tr}(\rho_H^2)$. *Then ρ_H is not absolutely irreducible.*

Here $\rho_H^2 : H \to GL_2(F)$ is the map sending h to the square of $\rho_H(h)$.

Proof. We suppose that ρ_H is absolutely irreducible over F, and we try to get absurdity. We have the identity $\mathrm{Tr}(\rho_H^\sigma) = \mathrm{Tr}(\rho_H^2) = \mathrm{Tr}(\rho_H^{sym\otimes2}) - \det(\rho_H)$ for the symmetric second tensor representation $\rho_H^{sym\otimes2}$ of ρ_H. Over F, by absolute irreducibility, we have the identity of semi-simplification: $(\rho_H^{sym\otimes2})^{ss} \cong \rho_H^\sigma \oplus \det(\rho_H)$. Tensoring $\det(\rho)^{-1}$ and writing $Ad(\rho_H)$ for $\rho_H^{sym\otimes2} \otimes \det(\rho_H)^{-1}$, we get

$$Ad(\rho_H)^{ss} \cong (\rho_H^\sigma \otimes \det(\rho)^{-1}) \oplus \mathbf{1}.$$

The representation $Ad(\rho_H)$ can be realized on the space

$$V := \{x \in \mathrm{End}_F(F^2) | \mathrm{Tr}(x) = 0\}$$

with the action $Ad(\rho_H)(h)x = \rho_H(h)x\rho_H(h)^{-1}$. Since ρ_H^σ is absolutely irreducible, we have either $\mathbf{1}$ as a subrepresentation of $Ad(\rho_H)$ or a quotient of $Ad(\rho_H)$. Since $Ad(\rho_H)$ is self-dual under the trace pairing $\langle x, y \rangle = \mathrm{Tr}(xy)$ for $x, y \in V$ (as A is not of characteristic 2), in the latter case of having $\mathbf{1}$ as a quotient, by duality, $\mathbf{1}$ is also a subrepresentation of $Ad(\rho_H)$. Thus we have $\mathbf{1} \hookrightarrow Ad(\rho_H)$ as H-modules. In other words, we have a nontrivial element $0 \neq \phi \in \mathrm{End}_{A[H]}(\rho_H)$ such that $\mathrm{Tr}(\phi) = 0$. Since ρ_H is absolutely irreducible, ϕ has to be a scalar multiplication by $z \in A^\times$ by Schur's lemma; so, $\mathrm{Tr}(\phi) = 2z \neq 0$, a contradiction (unless A has characteristic 2). $\square$

Suppose $p > 2$. Let Q be the quotient field of $W[[X]]$; so, $\sigma := \sigma_2$ extends to a field automorphism of Q uniquely, which we denote again by σ. Then by the Steinitz theorem (e.g., [BAL, V.4.2]), σ extends to an automorphism of an algebraic closure $\overline{Q}$ of Q. Let $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Taking $A = \overline{Q}$ and embedding $\mathbb{I}$ into $\overline{Q}$ as the quotient field of $\mathbb{I}$ is a finite extension of Q, we can apply the above lemma to $\rho_\mathbb{I}$. Now we want to make a choice of the subgroup H. Let I be an open ideal of the profinite ring $\mathbb{I}$ and put

$$H = H_I = \left\{ h \in G \big| \mathrm{Tr}(\rho_\mathbb{I}(h\tau)) \equiv \mathrm{Tr}(\rho_\mathbb{I}(\tau)) \equiv \mathrm{Tr}(\rho_\mathbb{I}(\tau h)) \mod I \text{ for all } \tau \in G \right\}.$$

This group is the stabilizer of the trace map modulo I (i.e., the kernel of the pseudo-representation associated to $\mathrm{Tr}(\rho)$; see [MFG, §2.2]); so, it is a normal subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and $\mathrm{Tr}(h) \mod I$ for $h \in H$ only depends on the coset $hH = Hh$. Since I is an open ideal of $\mathbb{I}$, $\mathrm{Tr}(\rho_\mathbb{I}) \mod I$ has values in the finite ring $\mathbb{I}/I$; so, its stabilizer H_I is an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $\rho_\mathbb{I}$ has values in $GL_2(\mathbb{I})$ with absolutely irreducible $\rho_\mathbb{I} \mod \mathfrak{m}_\mathbb{I}$, then H_I is equal to $\mathrm{Ker}(\rho_\mathbb{I} \mod I)$ (by the theory of pseudo-representation; see [MFG, §2.1.7 and §2.2.1]). By the above trace identity: $\mathrm{Tr}(\rho_\mathbb{I}(h\tau)) \equiv \mathrm{Tr}(\rho_\mathbb{I}(\tau)) \equiv \mathrm{Tr}(\rho_\mathbb{I}(\tau h)) \mod I$, taking $\tau = h^n$, we get $\mathrm{Tr}(\rho_\mathbb{I}(h^{n+1})) \equiv \mathrm{Tr}(\rho_\mathbb{I}(h^n)) \mod I$ for all $n \in \mathbb{Z}$, which implies $\mathrm{Tr}(\rho_\mathbb{I}(h^n)) \equiv 2 \mod I$ (for any $n \in \mathbb{Z}$) if $h \in H_I$. Since $2 \cdot \det(\rho_\mathbb{I}(h)) = \mathrm{Tr}(\rho_\mathbb{I}(h^2)) - \mathrm{Tr}(\rho_\mathbb{I}(h))^2$, as $p > 2$, we have $\det(T - \rho_\mathbb{I}(h)) \equiv (T - 1)^2 \mod I$ if $h \in H_I$; so, the eigenvalues α of $\rho_\mathbb{I}(h)$ satisfy $(\alpha - 1)^2 \equiv 0 \mod I$ as long as $h \in H_I$ and $p \neq 2$. We write $\rho_I = \rho_\mathbb{I}|_{H_I}$.

Lemma 6.2. *Let the notation be as above. Let $\mathcal{F}$ be a p-adic analytic family of slope 0 with coefficients in $\mathbb{I}$ with Galois representation $\rho := \rho_\mathbb{I} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{Q})$. Write σ for an extension of σ_2 on $W[[X]]$ to the algebraic closure of the field of*

fractions of $\mathbb{I}$. *If* $\mathrm{Tr}(\rho_I^\sigma) = \mathrm{Tr}(\rho_I^2)$ *for an open ideal* $I \subsetneq \mathbb{I}$, *then there exists an imaginary quadratic extension* $M/\mathbb{Q}$ *and a character* $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^\times$ *such that* $\rho \cong \mathrm{Ind}_M^{\mathbb{Q}} \varphi$ *over* $\overline{Q}$.

Proof. By Lemma 6.1 applied to $H = H_I$ and $\rho_I = \rho|_{H_I}$, ρ_I is reducible. Write $\rho_I = \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$, extending the scalar $\mathbb{I}$ if necessary. Since ρ extends ρ_I, $g \mapsto \rho_I^h(g) := \rho_I(hgh^{-1}) = \rho(h)\rho_I(g)\rho(h)^{-1}$ is equivalent to ρ_I for all $h \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\{\delta, \epsilon\}$ by inner conjugation:

$$(6.1) \qquad \left(\begin{smallmatrix} \epsilon^h & * \\ 0 & \delta^h \end{smallmatrix}\right) = \rho(h) \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right) \rho(h)^{-1}.$$

Let $\Delta \subset G$ be the stabilizer of δ. Then $M = \overline{\mathbb{Q}}^\Delta$ is at most a quadratic extension of $\mathbb{Q}$. If $\delta \neq \epsilon$ or ρ_I is nonsemisimple, by (6.1), $\rho_\Delta = \rho|_\Delta$ has to be upper triangular. Thus the two characters extend to $\delta, \epsilon : \Delta \to \mathbb{I}^\times$, and $\rho_\Delta^{ss} = \delta \oplus \epsilon$. Since ρ is absolutely irreducible (over the quotient field Q of $\mathbb{I}$), $[G : \Delta] = 2$ and by Frobenius reciprocity, $\rho \cong \mathrm{Ind}_M^{\mathbb{Q}} \delta \cong \mathrm{Ind}_M^{\mathbb{Q}} \epsilon$. As already remarked in (CM1) in Section 1, this can happen only for imaginary M for Galois representations associated to a p-adic analytic family of modular forms (of slope 0).

Suppose that $\delta = \epsilon$. Let D be the decomposition group at p. Then $\rho_D := \rho|_D$ is reducible with two distinct diagonal characters (by (Gal)): one trivial and another ε giving the $W[[X]]$-algebra structure (centered at weight 1) of $\mathbb{I}$ (see (Gal)). In particular, ε is of infinite order. Thus we may assume that δ restricted to D is trivial and ϵ restricted to D has infinite order agreeing with ε. Then it is impossible to have $\delta = \epsilon$, because $\mathbb{I}/I$ is a finite ring. $\square$

Proof of Theorem 3.3. Let $K := \mathbb{Q}(\mu_{p^\infty})$ and $L_P = K(\alpha_{l,P})$ for a prime l. By Corollary 4.2, we need to prove that $[M_{H,\mathcal{A}}(\mathcal{F}) : K] < \infty \Rightarrow \mathcal{F}$ has CM. Suppose $[M_{H,\mathcal{A}}(\mathcal{F}) : K] < \infty$. For each arithmetic P with $k(P) = k$, by Lemma 4.3,

$$[K(f_P) : K(a(p, f_P))] < d$$

for a positive integer d independent of P. Thus $[L_P : K] < 2d[M_{H,\mathcal{A}}(\mathcal{F}) : K]$ for each prime l. Therefore, any odd prime $l > 2d[M_{H,\mathcal{A}}(\mathcal{F}) : K]$ is at most tamely ramified in L_P/K. Take such an odd prime $l > 2d[M_{H,\mathcal{A}}(\mathcal{F}) : K]$ prime to Np. Let $\rho = \rho_{\mathbb{I}}$ be the Galois representation associated to $\mathcal{F}$. Thus by Proposition 5.2, we have $\mathrm{Tr}(\rho(Frob_l)) = \zeta(1+X)^a + \zeta'(1+X)^{a'}$ for two roots of unity ζ, ζ' and $a, a' \in \mathbb{Q}_p$ with denominator bounded independent of l (i.e., $ap^B \in \mathbb{Z}_p$ and $a'p^B \in \mathbb{Z}_p$ for an integer $B \geq 0$ independent of l).

Take an arithmetic $P_0 \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ to see that the order of ζ is bounded independent of l. Let α be a root of $\det(X - \rho_{P_0}(Frob_l)) = 0$ in $\overline{\mathbb{Q}}_p$. Then $[\mathbb{Q}(f_{P_0}, \alpha) : \mathbb{Q}(f_{P_0})] \leq 2$. Write $m = [\mathbb{Q}(f_{P_0}) : \mathbb{Q}]$, $\zeta = \zeta_p \zeta^{(p)}$ with $\zeta_p \in \mu_{p^\infty}$ and $\zeta^{(p)}$ of order prime to p. Since $(1+X)^s \equiv 1 \mod \mathfrak{m}_{\mathbb{I}}$, the order of $\zeta^{(p)}$ is bounded by the degree of the residue fields modulo p of the integer ring of $\mathbb{Q}_p(f_{P_0}, \alpha)$. Thus the order of $\zeta^{(p)}$ is bounded by p^{2m}. Note that $P_0((1+X)^a) = (1+X)^a \mod P_0$ is in a finite extension L of $\mathbb{Q}_p$ depending only on the denominator p^n of a. For example, if P_0 contains $(1 + X) - \gamma^k$, $L \subset \mathbb{Q}_p[\gamma^a] = \mathbb{Q}_p[\sqrt[p^n]{(1+p)}] \subset \mathbb{Q}_p[\sqrt[p^B]{(1+p)}]$ for $\gamma = 1 + p$. We have $\zeta_p \in L[\zeta_p]$ whose degree is bounded by $2m[L : \mathbb{Q}_p]$; so, the order of ζ_p is also bounded independent of l. Replacing W by its finite extension, we may assume that all such roots of unity are in W.

Let $I = \mathfrak{m}_{\mathbb{I}}^{2N} + (X)$ for a sufficiently large N and F be the fixed field of H_I. Then $\mathbb{I}/I \cong W/\mathfrak{m}_W^{2N}$. We have $\mathrm{Tr}(\rho(Frob_{\mathfrak{l}})) = \zeta^f(1+X)^{fa} + \zeta'^f(1+X)^{fa'} \equiv \zeta^f + \zeta'^f$ mod I, and $\det(T - \rho(Frob_{\mathfrak{l}})) \equiv (T-1)^2$ mod I for a prime $\mathfrak{l}|l$ of F of residual degree f. So $(\zeta^f - 1)^2 \equiv 0$ mod $\mathfrak{m}_W^{2N}$. Since any square 0 element in $W/\mathfrak{m}_W^{2N}$ is contained in $\mathfrak{m}_W^N/\mathfrak{m}_W^{2N}$, we have $\zeta^f \equiv 1$ mod $\mathfrak{m}_W^N$; so, taking N large, we may assume that $\zeta^f = \zeta'^f = 1$. This shows that $\mathrm{Tr}(\sigma_s(\rho(Frob_{\mathfrak{l}}))) = \mathrm{Tr}(\rho(Frob_{\mathfrak{l}})^s)$ for all $0 \neq s \in \mathbb{Z}_p$. Thus by the Chebotarev density theorem, we get $\mathrm{Tr}(\sigma_s(\rho_I)) = \mathrm{Tr}(\rho_I^s)$ over $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. In particular, we have $\mathrm{Tr}(\sigma_2(\rho_I)) = \mathrm{Tr}(\rho_I^2)$. Then by Lemma 6.2, $\rho = \mathrm{Ind}_M^{\mathbb{Q}} \varphi$ for an imaginary quadratic field M. Thus $\mathcal{F}$ has to be a CM family. $\square$

Here is an obvious corollary of the above proof.

Corollary 6.3. *Let* $K := \mathbb{Q}[\mu_{p^\infty}]$ *and* $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ *be an infinite set of arithmetic points* P *with fixed weight* $k(P) = k \geq 2$. *Unless* $\mathcal{F}$ *has complex multiplication,* $\limsup_{P \in \mathcal{A}}[K(a(p, f_P)) : K] = \infty$.

Indeed, if $\limsup_P[K(a(p, f_P)) : K] < \infty$, the index $[L_P : K]$ $(P \in \mathcal{A})$ is bounded for $A \in \mathbb{I}$ as in Proposition 5.2. We can still apply the above proof and conclude that $\mathcal{F}$ has complex multiplication. We also conclude that $\limsup_P[K(f_P) : K] = \infty$ as in the horizontal theorem in the introduction.

7. Results towards vertical conjectures

We start with

Lemma 7.1. *Let* $f \in S_2(\Gamma_0(Np^{r+1}), \psi)$ $(p \nmid N)$ *be a* p-*ordinary Hecke eigenform with exact level* Np^{r+1} *for* $r \geq 0$. *Then,* f *is old at* p *if and only if* $r = 0$, $\psi|_{(\mathbb{Z}/p\mathbb{Z})^\times} = 1$ *and* f *is associated to a Hecke eigenform of level* N. *Suppose either that the* p-*component* $\psi_p := \psi|_{\mathbb{Z}_p^\times}$ *is nontrivial or that* f *is old at* p. *Then* $\mathbb{Q}(f)$ *is a CM field, and* $a(p, f)$ *is a Weil* p-*number of weight* 1. *Let* R *(resp.* R^+*) be the integer ring of* $\mathbb{Q}(a(p, f))$ *(resp. the maximal real subfield of* $\mathbb{Q}(a(p, f))$*). If either* p *is unramified in* $\mathbb{Q}(a(p, f))/\mathbb{Q}$ *or* $a(p, f)$ *generates* R_p *over* R_p^+*, there exists a* p-*ordinary CM-type* $\Sigma_p = \{\mathfrak{P}|p\}$ *of* $\mathbb{Q}(f)$ *such that* $(a(p, f)) = \prod_{\mathfrak{P} \in \Sigma_p^c} \mathfrak{P}^{e(\mathfrak{P})}$ *for the ramification index* $e(\mathfrak{P})$ *of* $\mathfrak{P}/(p)$.

Proof. Since f is ordinary at p, the p-component π_p of the automorphic representation generated by f is either in principal series $\pi(\alpha, \beta)$ or a Steinberg representation $\sigma(|\cdot|_p^{-1}\alpha, \alpha)$ for an unramified character α of $\mathbb{Q}_p^\times$ (with $\alpha(p) = a(p, f)$). As is well known, f is old if and only if β is unramified (so, f comes from a primitive new form of prime-to-p level). By the congruence relation (proved by Eichler, Shimura, Igusa and Deligne/Rapoport; see [IAT] and [GME, Section 4.2]), $a(p, f)$ is a Weil p-number of weight 1. Then the rest of the assertions follow from Proposition 2.1. $\square$

Write $M_{V,r}(\mathcal{F}) = M_{V, \mathcal{A}_r}(\mathcal{F})$ for the set $\mathcal{A}_r$ of all arithmetic points of $\mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ with $r(P) \leq r$.

Proposition 7.2. *Let* $\mathcal{F} = \{f_P\}_{P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)}$ *be a* p-*adic analytic family of classical* p-*ordinary Hecke eigenforms and* $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ *be an infinite set of arithmetic points* P *with* $r(P) \leq r$ *for a fixed* $r \geq 0$ *such that* $M_{V, \mathcal{A}}(\mathcal{F})$ *is a finite extension over* $\mathbb{Q}$. *Assume that for* $P_0 \in \mathcal{A}$,

(1) $\alpha_0 = a_{P_0}(p)$ *is a Weil* p-*number of weight* $k(P_0) - 1$,

(2) $\Sigma_{\alpha_0} = \{\sigma : \mathbb{Q}(\alpha_0) \hookrightarrow \overline{\mathbb{Q}} \,|\, |i_p(\alpha_0^\sigma)| = 1\}$ is a CM type of $\mathbb{Q}(\alpha_0)$.

Then there exist a Weil p-number α of weight 1 with $|i_p(\alpha)|_p = 1$ such that $a(p)_P = \zeta \langle \alpha \rangle^{k(P)-1}$ for a root of unity ζ for all arithmetic P with $k(P) \geq 2$, where $\langle \alpha \rangle = \exp_p(\log_p(i_p(\alpha)))$ for the Iwasawa p-adic logarithm $\log_p$. In particular, $M_{V,r}(\mathcal{F})$ is a finite CM extension of $M_{V,\mathcal{A}}(\mathcal{F})$.

Proof. In order to give a simple sketch of the proof, suppose first that $M := M_{V,\mathcal{A}}(\mathcal{F})$ is an imaginary quadratic field. Thus Σ_{α_0} is made of a single element $\iota = i_p|_M$, and for each k, there exists at most one Weil number $\alpha_k \in M$ of weight k (up to roots of unity in M) such that $|\alpha_k|_p = 1$. In M, $(\alpha_k) = \overline{\mathfrak{p}}^k$ for the prime ideal $\mathfrak{p}$ of M corresponding to $i_p|_M$. Fix such a k. Taking a k-th root $\alpha = \sqrt[k]{\alpha_k}$, we have $\alpha_l = \alpha^l$ up to roots of unity for all l as $(\alpha_l) = \overline{\mathfrak{p}}^l$.

Since $\mathcal{A}$ is an infinite set, there exists an infinite sequence $P_1, P_2, \ldots, P_n, \ldots$ in $\mathcal{A}$ with increasing weight $k(P_1) < k(P_2) < \cdots$ such that $(a_{P_j}(p)) = \overline{\mathfrak{p}}^{k(P_j)-1}$ for all $j > 0$. Put

$$\langle \alpha \rangle = \exp_p\left(\frac{1}{k(P_0)-1} \log_p(a(p, f_{P_0}))\right) = \exp_p(\log_p(\alpha)).$$

Since $(a_{P_j}(p)) = \overline{\mathfrak{p}}^{(k(P_j)-1)}$, $a_{P_j}(p)/\langle \alpha \rangle^{k(P_j)-1}$ is a Weil number of weight 0; that is, it is an algebraic integer with all its conjugates having absolute value 1. Then by Kronecker's theorem, we find $a_{P_j}(p) = \zeta_{P_j} \langle \alpha \rangle^{k(P_j)-1}$ for a root of unity ζ_{P_j}. Note that $\langle \alpha \rangle$ is contained in a finite extension M'/M. Since there are finitely many roots of unity in M', we have only finitely many possibilities for ζ_{P_j}. Therefore, replacing $\{P_j\}_j$ by its subsequence, we find an infinite sequence $P_1, P_2, \ldots, P_n, \ldots$ of increasing weights such that $a_{P_j}(p) = \zeta \langle \alpha \rangle^{k(P_j)-1}$ for all $j = 1, 2, \ldots$ for a fixed root of unity ζ. We have a power series $\Phi_\alpha(X) \in W[[X]][(1+X)^{1/p^n}]$ (for a suitable $n \geq 0$) with coefficients in a discrete valuation ring W finite flat over $\mathbb{Z}_p$ such that $\Phi_\alpha(\gamma^k - 1) = \zeta \langle \alpha \rangle^k$ for all integers k. Indeed, $\Phi_\alpha(X) = \zeta(1+X)^s$ for $s = \log_p(\alpha)/\log_p(\gamma)$. Since M is an imaginary quadratic field in which p splits into $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, we have $M_\mathfrak{p} = \mathbb{Q}_p$ and hence $\log_p(\alpha) = \frac{\log(a_{P_0}(p))}{k(P_0)-1} \in M_\mathfrak{p} = \mathbb{Q}_p$. Thus $s \in \mathbb{Q}_p$, and p^n is the denominator of s. Since $\mathcal{F}$ is an ordinary family, there exists an element $A \in \mathbb{I}$ such that $a(p, f_P) = (A \mod P)$ for all height 1 primes P of $\mathbb{I}$ containing $(1 + X - \gamma^{k(P)})$. Thus we find $A \equiv \Phi_\alpha \mod P_j$ for infinitely many distinct primes P_j; so, $A = \Phi_\alpha$, as desired.

We now treat the general case where M may not be an imaginary quadratic field. Let $K \subset \overline{\mathbb{Q}}$ be a number field with integer ring O. Consider $O \otimes_\mathbb{Z} K$. Then $O \otimes_\mathbb{Z} K$ is a product of fields $\sigma(O)K \subset \overline{\mathbb{Q}}$ indexed by a set of embeddings $S := \{\sigma : O \hookrightarrow \overline{\mathbb{Q}}\}$. Take the base ring W containing O. Then $\mathbb{I} \otimes_\mathbb{Z} K$ contains $O \otimes_\mathbb{Z} K$, and $\mathbb{I} \otimes_\mathbb{Z} K$ decomposes accordingly: $\mathbb{I} \otimes_\mathbb{Z} K = \prod_{\sigma \in S} \mathbb{I}_\sigma$. Regard $\mathbb{I} \otimes_\mathbb{Z} K$ as a K-algebra from the right factor (and K is embedded in $\overline{\mathbb{Q}}_p$ by i_p). Note that $\mathbb{I} \otimes_\mathbb{Z} K = \mathbb{I} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \otimes_\mathbb{Z} K = \mathbb{I} \otimes_{\mathbb{Z}_p} K_p$ for $K_p = \mathbb{Z}_p \otimes_\mathbb{Z} K$. For an arithmetic prime P, we have $\mathbb{Z}[f_P] := \mathbb{Z}[a_P(n) | n = 1, 2, \ldots] \subset \mathbb{I}/P$. Then $\mathbb{Z}[f_P] \otimes_\mathbb{Z} K \subset \mathbb{I}/P \otimes_\mathbb{Z} K$ as K is $\mathbb{Z}$-flat. On the other hand,

$$\mathbb{Z}[f_P] \otimes_\mathbb{Z} K = \mathbb{Q}(f_P) \otimes_\mathbb{Z} K \cong \prod_{\tau : \mathbb{Q}(f_P) \hookrightarrow \overline{\mathbb{Q}}_p} i_p(\tau(\mathbb{Q}(f_P))K).$$

The composite $\tau(\mathbb{Q}(f_P))K$ is taken in $\overline{\mathbb{Q}}_p$ by sending it by i_p inside $\overline{\mathbb{Q}}_p$. For some τ (for example, complex conjugation $\tau = c$), we may have $|i_p(\tau(a_P(p)))|_p < 1$.

Let us give more details why this strange phenomenon: $|i_p(\tau(a_P(p)))|_p < 1$ could occur. Suppose $K/\mathbb{Q}$ is a Galois extension with $O \subset W$. Then writing $V = K \cap W$ (the valuation ring corresponding to $i_p : K \hookrightarrow \overline{\mathbb{Q}}_p$), we have

$$V \otimes_{\mathbb{Z}} V \subset \prod_{\sigma \in \mathrm{Gal}(K/\mathbb{Q})} \sigma(V)V.$$

Let e_σ for the idempotent of $\sigma(V)V$. Writing $D_V \subset \mathrm{Gal}(K/\mathbb{Q})$ for the decomposition subgroup of V, unless $\sigma \in D_V$ (i.e., $\sigma(V) = V$), $\sigma(V)V = K$. Since V is embedded in $\mathbf{h}_{k(P),\psi_P}$, we regard $e_\sigma \in \mathbf{h}_{k(P),\psi_P} \otimes_{\mathbb{Z}} V$. Since $U(p)$ is invertible in $\mathbf{h}_{k(P),\psi_P}$, the image of $e_\sigma(U(p) \otimes 1)$ is invertible in $K = \sigma(V)V$, but that does not mean that $e_\sigma(U(p) \otimes 1)$ is a p-adic unit. Define

$$E_P = \lim_{n \to \infty} (U(p) \otimes 1)^{n!}$$

under the p-adic topology $\mathcal{T}_P$ of $\mathbf{h}_{k(P),\psi_P} \otimes_{\mathbb{Z}} V$ inducing the natural topology on $1 \otimes V \subset \mathbf{h}_{k(P),\psi_P} \otimes_{\mathbb{Z}} V$. Then E_P is orthogonal to e_σ if $e_\sigma(U(p) \otimes 1)$ is p-adically nilpotent under the p-adic topology $\mathcal{T}_P$ of $\mathbf{h}_{k(P),\psi_P} \otimes_{\mathbb{Z}} V$. The idempotent $e_P = \lim_{n \to \infty} U(p)^{n!}$ in $h_{k(P),\psi_P}$ (for $\psi_P = \psi_{k(P)}\varepsilon_P$) is only defined over $\overline{\mathbb{Q}}$; so, e_P may not commute with some σ. In other words, we could have $e_P \otimes 1 \neq E_P$, and

$$E_P = \sum_{\tau : |i_p(\tau(a_P(p)))|_p = 1} e_\tau.$$

We can embed $\mathbf{h}$ into $\prod_{P \in \mathcal{A}} \mathbf{h}_{k(P),\psi_P} \subset \prod_{P \in \mathcal{A}} h_{k(P),\psi_P}$ for any infinite set $\mathcal{A}$ of arithmetic points P of $W[[T]]$ sending $T(n)$ to diagonal $T(n)$ in the product of the right-hand side. The tensor product $\mathbf{h} \otimes_{\mathbb{Z}} K$ is embedded in $\prod_P (h_{k(P),\psi_P} \otimes_{\mathbb{Z}} K)$. We write $E = \prod_P E_P$, which is an idempotent of $\prod_P (h_{k(P),\psi_P} \otimes_{\mathbb{Z}} K)$ but may not be in $(\prod_P h_{k(P),\psi_P}) \otimes_{\mathbb{Z}} K$. The closure $\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K}$ of $\mathbf{h} \otimes_{\mathbb{Z}} K$ inside $\prod_P (h_{k(P),\psi_P} \otimes_{\mathbb{Z}} K)$ contains E (as $E = \lim_{n \to \infty} U(p)^{n!} \otimes 1$), and $E(\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K})$ is free of finite rank over $W[[T]][\frac{1}{p}]$ (though $\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K}$ could be huge). Each irreducible component of $E(\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K})$ gives rise to another p-adic analytic family of slope 0.

Pick an arithmetic point P, and write $\alpha = a_P(p)$. Take an irreducible component $\mathrm{Spec}(\mathbb{I}_\sigma^\circ)$ of $\mathrm{Spec}(\mathbb{I}_\sigma) \cap \mathrm{Spec}(E(\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K}))$. Let P_τ be a factor of $P \otimes_{\mathbb{Z}} K \subset \mathbb{I} \otimes_{\mathbb{Z}} K = \prod_{\sigma \in S} \mathbb{I}_\sigma$ corresponding to $\mathbb{I}_\sigma^\circ$. Regarding $P_\tau : \mathbb{I}_\sigma^\circ \to \overline{\mathbb{Q}}_p$, we have $P_\tau(\alpha) = \tau(\alpha)$ and $f_{P_\tau} = f_P^\tau$. Since $\mathbb{I}_\sigma \subset E(\widehat{\mathbf{h} \otimes_{\mathbb{Z}} K})$, we have $|\tau(\alpha)|_p = 1$. The image $a_\sigma(p)$ of $a(p) \otimes 1$ in $\mathbb{I}_\sigma$ modulo P_τ gives the p-adic unit $\tau(a_P(p))$; so, $a_\sigma(p)$ is a unit in the integral closure of $W[[T]]$ in $\mathbb{I}_\sigma$.

Here is a more down-to-earth approach/proof of the fact that $\mathbb{I}_\sigma^\circ$ above gives rise to another analytic family $\mathcal{F}_\sigma$ containing f_P^τ. Start with another arithmetic $(Q : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, but regarding Q as a point of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, $\mathbb{I}/Q$ has a unique embedding $\mathbb{I}/Q \subset \overline{\mathbb{Q}}_p$ induced by $Q : \mathbb{I} \to \overline{\mathbb{Q}}_p$. Then $\mathbb{I}_\sigma^\circ/Q_{\tau'} \subset \mathbb{I}/Q \otimes_{\mathbb{Z}} K$ for the corresponding $Q_{\tau'} \in \mathrm{Spec}(\mathbb{I}_\sigma^\circ)(\overline{\mathbb{Q}}_p)$. Indeed, tensoring K with the exact sequence $\mathrm{Ker}(Q) \hookrightarrow \mathbb{I} \twoheadrightarrow \mathrm{Im}(Q)$, we get another exact sequence:

$$\mathrm{Ker}(Q) \otimes_{\mathbb{Z}} K \hookrightarrow \prod_\sigma \mathbb{I}_\sigma \twoheadrightarrow \mathrm{Im}(Q) \otimes_{\mathbb{Z}} K,$$

and $\mathrm{Im}(Q) \otimes_{\mathbb{Z}} K$ contains $\sigma(K)K$ canonically and τ' coincides with σ on $K \cap \mathbb{Q}(f_Q)$ and induces $\tau' = Q_{\tau'}|_{\overline{\mathbb{Q}} \cap W} : \overline{\mathbb{Q}} \cap W \hookrightarrow \overline{\mathbb{Q}}_p$. Then we have $f_{Q_{\tau'}} = f_Q^{\tau'}$, which is a classical modular form. It is slope 0 with respect to i_p (i.e., with respect to the

product topology $\prod_P \mathcal{T}_P$) because of $E \cdot \mathbb{I}_\sigma^\circ = \mathbb{I}_\sigma^\circ$. Thus $\mathcal{F}_\sigma = \{f_Q^{\tau'}\}_Q$ is another slope 0 family. We rewrite $\sigma_{Q,\sigma}$ for τ'. Let $\pi_\sigma : \mathbb{I} \otimes_{\mathbb{Z}} K \to \mathbb{I}_\sigma^\circ$ be the projection. We have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{I}/Q & \xrightarrow{\ \pi_\sigma\ } & \mathbb{I}_\mathfrak{p}^\circ/Q_{\tau'} \\
\big\uparrow{\scriptstyle\cup} & & \big\uparrow{\scriptstyle\cup} \\
\widehat{K} & \xrightarrow[\tau'=\sigma_{Q,\sigma}]{} & \widehat{\sigma(K)K},
\end{array}
$$

where $\widehat{K}$ is the closure of K in $\mathbb{I}/Q \otimes_{\mathbb{Z}} \mathbb{Q}_p$ and $\widehat{\sigma(K)K}$ is the closure of $\sigma(K)K$ in $\mathbb{I}_\sigma^\circ/Q_\tau$.

Take K to be the maximal real subfield of M (to avoid complex conjugation c with $|a_P(p)^c|_p < 1$). Let Σ be the CM type of M given by

$$
\Sigma_0 = \left\{ \sigma : M \to \overline{\mathbb{Q}} \big| \sigma|_{\mathbb{Q}(\alpha_0)} \in \Sigma_{\alpha_0} \right\}.
$$

Take the starting P to be P_0. Then the set I of p-adic places of K is in bijection to $\Sigma_{0,p}$, and $\sigma_{P_0,\sigma}|_M \in \Sigma_0$. By the assumption (2), any prime $\mathfrak{p}|p$ in K splits as $\mathfrak{p} = \mathfrak{P}\overline{\mathfrak{P}}$ in M and $M_\mathfrak{P} = K_\mathfrak{p} = M_{\overline{\mathfrak{P}}}$; so, $M \subset \widehat{K}$ noncanonically. Since Σ_0 is a CM type of M and since $\{K \hookrightarrow \sigma(K)K | \sigma \in \Sigma_0\}$ covers all conjugates of K inside $\overline{\mathbb{Q}}$, for any $\sigma \neq \sigma'$ in I, we find $P \in \mathcal{A}$ such that $\sigma_{P,\sigma}(\alpha_{p,P}) \neq \sigma_{P,\sigma'}(\alpha_{p,P})$ as M is generated by $\{\alpha_{p,P}\}_{P\in\mathcal{A}}$. On the other hand, $|i_p(\sigma_{P,\sigma}(\alpha_{p,P_0}))|_p = 1$ for all $\sigma \in \Sigma_0$. Since any two given components of $\mathbf{h}$ do not intersect each other at any of arithmetic points (see [HMI, Proposition 3.78]), we have at least $|I|$ distinct families: $\{\mathcal{F}_\sigma\}_{\sigma\in I}$. In other words, the set Σ_Q of p-adic places of M induced by $\{\sigma_{Q,\sigma}\}_{\sigma\in I}$ for $Q \in \mathcal{A}$ is a p-adic CM type of M.

Since there are only finitely many p-adic CM types of M, replacing $\mathcal{A}$ by an infinite subset, we may assume that Σ_P is identical (independent of P) for all $P \in \mathcal{A}$, and write Σ_p for the set of prime ideals/places over p in K induced by $i_p \circ \sigma$ for $\sigma \in \Sigma_P$. This forces $(a_P(p)) = \prod_{\mathfrak{p}\in\Sigma_p^c} \mathfrak{p}^{e(\mathfrak{p})(k(P)-1)}$ for the absolute ramification index $e(\mathfrak{p})$ of $\mathfrak{p}/(p)$.

As before we choose an infinite sequence in $\mathcal{A}$,

$$
P_1, P_2, \ldots, P_n, \ldots,
$$

with increasing weight $k(P_1) < k(P_2) < \cdots$ such that

$$
(a_{P_j}(p)) = \prod_{\mathfrak{p}\in\Sigma_p^c} \mathfrak{p}^{e(\mathfrak{p})(k(P_j)-1)}
$$

for all $j > 0$. Then $a_{P_j}(p)/\langle\alpha\rangle^{k(P_j)-1}$ is a Weil number of weight 0; that is, it is an algebraic integer with all its conjugates having absolute value 1. Then by Kronecker's theorem, we find $a_{P_j}(p) = \zeta_{P_j}\langle\alpha\rangle^{k(P_j)-1}$ for a root of unity ζ_{P_j}. Note that $\langle\alpha\rangle$ is contained in a finite extension K'/K. Since there are finitely many roots of unity in K', we have only finitely many possibilities for ζ_{P_j}. Therefore, replacing $\{P_j\}_j$ by its subsequence, we find an infinite sequence $P_1, P_2, \ldots, P_n, \ldots$ of increasing weights such that $a_{P_j}(p) = \zeta\langle\alpha\rangle^{k(P_j)-1}$ for all $j = 1, 2, \ldots$ for a fixed root of unity ζ. By the same argument as before, we conclude $A = \Phi_\alpha$, as desired. $\qquad\square$

Remark 7.1. Let F be the quotient field of W. In the above proof, when M is not an imaginary quadratic field, Φ_α is a convergent power series with coefficients in F convergent over a closed disk D_ϵ of radius $0 < \epsilon \leq 1$ centered at an accumulation

point P_∞ of $\{P_j | j = 1, 2, \dots\}$ (a priori, Φ_α is not necessarily in $W[[X]]$ as we do not know if $i_p(\alpha) \in \mathbb{Z}_p^\times$). Since $r(P_j) \le r$ and the P_js for all j are inside the closed disk of radius $1 - (1/p^r(p-1))$ centered at the origin, we have an accumulation point P_∞. Replacing $\{P_j\}_j$ by an infinite subsequence, we may assume that $\{P_j\}_j \subset D_\epsilon$ and $\lim_{j\to\infty} P_j = P_\infty$ in D_ϵ. Thus Φ_α is in the power series ring $R = F\{\{t\}\}$ convergent over D_ϵ which is an affinoid ring over F. Here $t = (1+X) - P_\infty(1+X)$, and $\mathbb{I}$ is naturally embedded in $F\{\{t\}\}$ regarding $a \in \mathbb{I}$ as an analytic function on D_ϵ. By the rigid analytic version of the Weierstrass preparation theorem (see [NAA, §5.2.2 and §6.1.5]), $\zeta\Phi_\alpha - A$ has finitely many zeros in the closed disk of radius ϵ (i.e., the rigid analytic space of R), and hence $A = \Phi_\alpha$ as $A_{P_j} = \Phi_\alpha(P_j)$ for infinitely many j. This then shows that Φ_α is in the finite extension $\mathbb{I}/\Lambda$, and hence $\Phi_\alpha = \zeta(1+X)^s$ with $s = \frac{\log_p(\alpha)}{\log_p(\gamma)}$, which has to be in $\mathbb{Q}_p^\times$ (by the argument in the last paragraph of the proof of Proposition 5.2).

The following corollary shows that $M_{V,r}(\mathcal{F})$ $(r < \infty)$ is an infinite extension if it is Steinberg (that is, its weight 2 member has abelian variety multiplicative at p; see [DR, §V.3] and [La, Theorem 7.5]):

Corollary 7.3. *Let $\mathcal{F}$ be a p-adic analytic family of classical p-ordinary Hecke eigenforms satisfying the condition* (2) *of the above proposition. If $\mathcal{F}$ contains a member f_2 such that $|a(p, f_2)| = 1$, then $M_{V,r}(\mathcal{F})$ for finite $r > 0$ is an infinite extension of $\mathbb{Q}$. More generally, if we find two arithmetic points P with $r(P) \le r$ such that in the maximal totally real subfield of $\mathbb{Q}(a(p, f_P))/\mathbb{Q}$ the prime (p) decomposes differently, then $M_{V,r}(\mathcal{F})$ is an infinite extension of $\mathbb{Q}$.*

Proof. By the above theorem, if $K = M_{V,r}(\mathcal{F})$ is a finite extension, then $a(p, f_2)$ has to be a Weil number of weight 1, which is not the case if f_2 has multiplicative reduction. If $M_{V,r}(\mathcal{F})$ is a finite extension, then $\mathbb{Q}(a(p, f_P))$ is a subfield of $M_{V,r}(\mathcal{F})$; so, the splitting of the prime p in the real subfield of $M_{V,r}(\mathcal{F})$ is the same in $M_{V,r}(\mathcal{F})$. Then the second assertion is obvious.   □

8. Proof of vertical theorems

We first prove Theorem 3.2. Suppose that $K := M_{V,\mathcal{A}}(\mathcal{F})$ is a finite extension and assume the existence of an arithmetic point P_0 as in the theorem. Therefore the assumption (2) of Proposition 7.2 is met. By Proposition 7.2 (and the remark following its proof), we find a Weil number α of weight 1 and a power series $\Phi_\alpha(X) \in W[\mu_{p^n}][[X]][(1+X)^{1/p^n}]$ such that $a_P(p) = \Phi_\alpha(\varepsilon_P(\gamma)\gamma^{k(P)-1} - 1) = \zeta(\varepsilon_P(\gamma))^{\log_p(\alpha)/\log_p(\gamma)}\langle\alpha\rangle^{k(P)-1}$ for all arithmetic P, where ζ is a root of unity independent of P. Then, for any infinite set $\mathcal{B}$ of arithmetic points of weight 2, we find $M_{H,\mathcal{B}}(\mathcal{F}) \subset \mathbb{Q}(\mu_{p^\infty(p-1)})(\zeta, \alpha)$, which is a finite extension of $\mathbb{Q}(\mu_{p^\infty})$. Then by the strong horizontal theorem, $\mathcal{F}$ has complex multiplication. The converse follows from Corollary 4.2. This finishes the proof of Theorem 3.2.

We now prove the vertical theorem in the introduction. We want to apply Theorem 3.2 since $M_{V,r}(\mathcal{F}) \subset \mathbb{Q}_{V,r}(\mathcal{F})$. Thus we only need to verify the assumption (2) of Proposition 7.2 to apply Theorem 3.2. Pick P of weight 2 such that A_P has good ordinary reduction modulo $\mathfrak{m}$ for the maximal ideal $\mathfrak{m}$ of $\mathbb{Z}_p[\mu_{p^{r+1}}]$. Since $\widetilde{A} = A_P \times_{\mathbb{Z}_p[\mu_{p^r}]} \mathbb{F}_p$ is ordinary, by a theorem of Tate (cf. [ABV, Section 22]), $\widetilde{A}$ has complex multiplication by $\mathbb{Q}(F) \subset \text{End}(\widetilde{A}/\mathbb{F}_p)$ for the p-power relative Frobenius

map $F : \widetilde{A} \to \widetilde{A}$. Note that by the congruence relation (cf. [IAT, Theorem 7.9] or [GME, Section 4.2]), F satisfies the characteristic polynomial

$$T^2 - a_p T + p = (T - F)(T - V)$$

for the dual V of F and $a_p = \alpha + \alpha^c$ over $\mathbb{Q}(f_P) \subset \mathrm{End}(\widetilde{A}/\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus $\widetilde{A}$ is isogenous to $\widetilde{B}^m$ ($m = 2[\mathbb{Q}(f_P) : \mathbb{Q}(\alpha)]$) for an ordinary abelian variety $\widetilde{B}_{/\overline{\mathbb{F}}_p}$ with complex multiplication by $\mathbb{Q}(\alpha)$ with α giving F on $\widetilde{B}$ (see [Ch, Lemmas 3 and 6]). Take the canonical lift $B_{/W(\overline{\mathbb{F}}_p)}$ of $\widetilde{B}$ over the ring of Witt vectors $W(\overline{\mathbb{F}}_p)$ with coefficients in $\overline{\mathbb{F}}_p$ (e.g., [K]). Then the CM type of B is given by the action of $\mathbb{Q}(\alpha)$ on $Lie(B)$, which is exactly $\Sigma = \{\sigma \big| |\sigma(\alpha)|_p = 1\}$. Thus the assumption (2) of Proposition 7.2 is satisfied by $P_0 = P$, and therefore, Theorem 3.2 implies the vertical theorem.

We could make the following conjecture, which is a vertical version of Corollary 6.3:

Conjecture 8.1. *Let $\mathcal{A} \subset \mathrm{Spec}(\mathbb{I})(\mathbb{C}_p)$ be an infinite set of arithmetic points P with bounded level $r(P) \leq r$. Suppose that $\mathcal{F}$ does not have complex multiplication. Then we have*

$$\limsup_{P \in \mathcal{A}}[\mathbb{Q}(a(p, f_P)) : \mathbb{Q}] = \infty \quad and \quad \limsup_{P \in \mathcal{A}}[\mathbb{Q}(a(p, f_P^\circ)) : \mathbb{Q}] = \infty.$$

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