Global Hecke equivariance

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Lecture 8: We describe how to extend the Weil representation $\mathbf{w}$ from $\mathrm{SL}_2(F_{\mathbb{A}}) \times \mathrm{O}_{D_\sigma}(\mathbb{A})$ to $(g,h) \in \mathrm{GL}_2(F_{\mathbb{A}}) \times B_{\mathbb{A}}^{\times}$ so that $\theta(\phi)(g,h) = \sum_{\alpha\in D_\sigma}(L(h)\mathbf{w}(g)\phi)(\alpha)$ is kept automorphic. The idea is that $g \in GL_2(F_{\mathfrak{p}_F})$ for primes $\mathfrak{p}_F$ splitting in E together with $\mathrm{GL}_2(F)$ generates almost the entire group $\mathrm{GL}_2(F_{\mathbb{A}})$. We prove that $\theta(\phi)(g,h)|\mathbb{T}^*_{\mathfrak{p}} = \theta(\phi)(g,h)|\mathbb{T}^+_{\mathfrak{p}_F}$ for a prime $\mathfrak{p}$ split in E/F as long as $\phi = \phi_{\mathfrak{p}_F}\phi^{(\mathfrak{p}_F)}$ with $\phi_{\mathfrak{p}_F}$ given by characteristic function of $M_2(O_{F_{\mathfrak{p}}})$. Details are in §4.7.5.

§**0. A set of density 1 split primes.** Let $\mathbf{S}$ be a set of split primes $\mathfrak{p}|\mathfrak{p}_F$ of E for each prime $\mathfrak{p}_F$ of F with $D_{\sigma,E_{\mathfrak{p}_F}} \cong M_2(E_\mathfrak{p})$. Suppose

(S) *The set* $\mathbf{S}$ *has Chebotarev density* 1 *over* E.

Choose an inert prime $\mathfrak{p}'$ with $B_{\mathfrak{p}'} \cong M_2(E'_\mathfrak{p})$ and put $\mathbf{S}' = \mathbf{S} \sqcup \{\mathfrak{p}'\}$. Let $\mathbb{G} \subset \mathrm{GL}_2(F_\mathbb{A})$ (resp. $\mathbb{G}' \subset \mathrm{GL}_2(F_\mathbb{A})$) generated by $\mathrm{SL}_2(F_\mathbb{A})$, $\mathrm{GL}_2^+(F_\infty)$ and $\{\mathrm{GL}_2(E_\mathfrak{p})\}_{\mathfrak{p}\in\mathbf{S}}$ (resp. $\{\mathrm{GL}_2(E_\mathfrak{p})\}_{\mathfrak{p}\in\mathbf{S}'}$). Chebotarev density 1 means that for each ray class modulo N of E for every integer $N > 0$, there is a prime $\mathfrak{p}|\mathfrak{p}_F \in \mathbf{S}$ in the class (cf. [CFN,V.6]).

For each $\mathfrak{p}_F$, we fix a choice of a prime ideal $\mathfrak{p}$ of E such that $\mathfrak{p}|\mathfrak{p}_F$ and identify $F_{\mathfrak{p}_F} = E_\mathfrak{p}$. We identify $\mathbf{S}$ with $\{\mathfrak{p}|\mathfrak{p}_F \in \mathbf{S}\}$. Since $\theta(\phi)(g,h) = \theta(g,\sigma\cdot h)$ by definition as $\sigma \in \mathrm{O}_{D_\sigma}(\mathbb{Q})$, the choice of $\mathfrak{p}$ does not matter. For $G = \mathbb{G}$ and $\mathbb{G}'$, we have the representation $\mathbf{w}$ extended to subgroup G of $\mathrm{GL}_2(F_\mathbb{A})$.

§1. $B^\times_{\mathbb{A}}$ action on $\mathcal{S}(D_{\sigma,\mathbb{A}})$. Define a representation L of $B^\times_{\mathbb{A}}$ on $\mathcal{S}(D_{\sigma,\mathbb{A}})$ by $(L(h)\phi)(v) = \phi(h^\iota v h^\sigma)$. Then *the action $L(h)$ of $B^\times_{\mathbb{A}}$ on the center factors through $N_{E/F}$.* This is because the algebraic group $B^\times$ is sent into GO_{D_σ} by the action $v \mapsto h^\iota v h^\sigma$ and the similitude map $N_{D_\sigma} : \mathrm{GO} \to \mathbb{G}_{m/F}$ is given by $N_{D_\sigma}(h) = N(hh^\sigma) = N_{E/F}(N(h))$. For the Weil representation $\mathbf{w}_b$ with respect to $x \mapsto \mathbf{e}_v(bx)$, Jacquet–Langlands [AFG, Chapter I] extended $\mathbf{w}$ to a representation $\mathbf{w}_{\mathfrak{p}}$ of $\mathrm{GL}_2(F_{\mathfrak{p}})$ by $\mathbf{w}_{\mathfrak{p}}(g\,\mathrm{diag}[b,1])\phi(v) = |b|_{\mathfrak{p}}\mathbf{w}_{b^{-1}}(g)$ for $g \in \mathrm{SL}_2(F_{\mathfrak{p}})$.

Consider $\theta(\phi)(g,h) : \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F)\mathrm{SL}_2(F_{\mathbb{A}}) \times B^\times\backslash B^\times_{\mathbb{A}} \to \mathbb{C}$ so that

$$\theta(\phi)(\gamma g, \delta h) = \theta(L(h)\mathbf{w}(g)\phi) = \theta(\phi)(g,h)$$

for $\gamma \in \mathrm{GL}_2(F)$, $g \in \mathrm{SL}_2(F_{\mathbb{A}})$, $\delta \in B^\times$ and $h \in B^\times_{\mathbb{A}}$ with $L(h)\phi(v) = \phi(h^\iota v h^\sigma)$.

§2. **Legitimacy of extension in §1.** If $\gamma g = \gamma' g'$ for $g, g' \in \mathrm{SL}_2(F_{\mathbb{A}})$ and $\gamma, \gamma' \in \mathrm{GL}_2(F)$, then $g = \gamma^{-1}\gamma' g'$ and $\theta(\phi)(g,h) = \theta(\phi)(g',h)$ because of $\gamma^{-1}\gamma' \in \mathrm{SL}_2(F)$, and hence the value $\theta(\phi)(\gamma g, \delta h)$ is independent of the choice of $g, g' \in \mathrm{SL}_2(F_{\mathbb{A}})$ and $\gamma, \gamma' \in \mathrm{GL}_2(F)$ as long as $\gamma g = \gamma' g'$. For an element $\delta \in B^{\times}$, $\theta(\phi)(g, \delta h) = \theta(\phi)(g, \delta h)$ as the theta series is an average of the terms over D_σ and $\alpha \mapsto \delta^{\iota}\alpha\delta$ preserves D_σ. For a pair ψ_∞ and ϕ_∞ as in Lemma 4 in Lecture 7, we may define $\mathbf{w}(zg)\phi(v) = \psi_\infty(z)\mathbf{w}(g)\phi(v)$ for $g \in \mathrm{SL}_2(F_\infty)$.

For $G = \mathbb{G}$ or $\mathbb{G}'$, we extend further θ to $\mathrm{GL}_2(F)\mathrm{SL}_2(F_{\mathbb{A}}) \cdot G \times B^{\times}_{\mathbb{A}}$ by

$$\text{(E)} \qquad \theta(\phi)(\gamma g\mathbf{g}, h) := \theta(\phi)(g\mathbf{g}, h)$$

for $\gamma \in \mathrm{GL}_2(F)$, $g \in \mathrm{SL}_2(F_{\mathbb{A}})$, $\mathbf{g} \in G$.

§3. Legitimacy of extension adding G.

Lemma 1. *The extension* (E) *of* $\theta(\phi)$ *to* $\mathrm{GL}_2(F)\mathrm{SL}_2(F_{\mathbb{A}})\cdot G \times B_{\mathbb{A}}^{\times}$ *is well defined and is left invariant under* $GL_2(F) \times B^{\times}$*.*

Proof. If $g\mathbf{g} = g'\mathbf{g}'$ with $g, g' \in \mathrm{SL}_2(F_{\mathbb{A}})$ and $\mathbf{g}, \mathbf{g}' \in G$, then

$$\mathbf{w}(g'^{-1})\mathbf{w}(g) = \mathbf{w}(\mathbf{g}'\mathbf{g}^{-1}) = \mathbf{w}(\mathbf{g}')\mathbf{w}(\mathbf{g})^{-1}$$

as the extension of $\mathbf{w}$ to $\mathrm{SL}_2(F_{\mathbb{A}})G$ coincides with the original Weil representation $\mathbf{w}$ of $\mathrm{SL}_2(F_{\mathbb{A}})$. Thus $\mathbf{w}(g\mathbf{g}) = \mathbf{w}(g'\mathbf{g}')$ and hence $\theta(\phi)(g\mathbf{g}, h) = \theta(\phi)(g'\mathbf{g}', h)$.

For $\gamma, \gamma' \in \mathrm{GL}_2(F)$, if $\gamma g\mathbf{g} = \gamma' g'\mathbf{g}'$, $\gamma'^{-1}\gamma = g'\mathbf{g}'\mathbf{g}^{-1}g^{-1}$. Taking a prime $\mathfrak{l}$ of F non-split in E outside $\mathbf{S}'$, we find $\gamma'^{-1}\gamma = g'_{\mathfrak{l}}g_{\mathfrak{l}}^{-1}$, and hence $\det(\gamma_{\mathfrak{l}}'^{-1}\gamma_{\mathfrak{l}}) = 1 \Rightarrow \det(\gamma'^{-1}\gamma) = 1$. This implies $\theta(g\mathbf{g}, h) = \theta(g'\mathbf{g}', h)$, and therefore, $\theta(g\mathbf{g}, h)$ factors through $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F)\mathrm{SL}_2(F_{\mathbb{A}})G \times B_{\mathbb{A}}^{\times}$. Since $L(\xi)$ for $\xi \in B^{\times}$ permutes elements of D_{σ}, plainly as a function of $h \in B_{\mathbb{A}}^{\times}$, $\theta(\phi)(g, h)$ factors through $B^{\times}\backslash B_{\mathbb{A}}^{\times}$. □

§4. Norm subgroup.

Lemma 2. *For a quadratic field extension $E_{/F}$, inside the idele group $F_{\mathbb{A}}^{\times}$, the closed subgroup H topologically generated by $\{F_{\mathfrak{l}}^{\times}\}_{\mathfrak{l}\in\mathbf{S}}$, $F^{\times}$, $F_{\infty+}^{\times}$, and $N_{E/F}(E_{\mathbb{A}}^{\times})$ has index 2 and is open, and Artin symbol induces $F_{\mathbb{A}}^{\times}/H \cong \mathrm{Gal}(E/F)$. Here $F_{\infty+}^{\times}$ is the identity connected component of $F_{\mathbb{R}}^{\times}$.*

Proof: We show that $E_{\mathbb{A}}^{\times}$ is generated topologically by

$$E^{\times},\ \{E_{\mathfrak{p}}^{\times}\}_{\mathfrak{p}|\mathfrak{p}_F\in\mathbf{S}} \ \text{ and } \ E_{\infty+}^{\times},$$

where $\mathbf{S}$ is identified with subset of prime ideals of E. Let $U(N) := \{x \in \widehat{O}_E^{\times} | x \equiv 1 \mod N\}$ for a positive integer N. Then the finite group $Cl_N := E_{\mathbb{A}}^{\times}/E^{\times}U(N)E_{\infty+}^{\times}$ is the strict ray class group modulo N of E. Thus it is canonically isomorphic to the Galois group H_N of ray class field modulo N over E by class field theory.

§5. Proof continues.

Since $E_{\mathbb{A}}^{\times}/\overline{E^{\times}E_{\infty+}^{\times}} = \varprojlim_N Cl_N$, we need to prove that Cl_N is generated by totally split primes in E. Since totally split primes of $E_{/\mathbb{Q}}$ has density one among primes of E, by Chebotarev density theorem, $E_{\mathbb{A}}^{\times}$ is generated topologically by $E^{\times}$, $\{E_{\mathfrak{p}}^{\times}\}_{\mathfrak{p}|\mathfrak{p}_F \in \mathbf{S}}$ and $E_{\infty+}^{\times}$.

Now by global class field theory,

$$\mathrm{Coker}(N_{E/F} : E_{\mathbb{A}}^{\times}/\overline{E^{\times}E_{\infty+}^{\times}} \to F_{\mathbb{A}}^{\times}/\overline{F^{\times}F_{\infty+}^{\times}}) \cong \mathrm{Gal}(E/F).$$

Since $N_{E/F} : E_{\mathbb{A}}^{\times} \to F_{\mathbb{A}}^{\times}$ is an open map, the assertion follows. $\square$

§6. GL(2)-version.

Corollary 1. *Write* χ_E^G *for the composite of* $\det : \mathrm{GL}_2(F_{\mathbb{A}}) \to F_{\mathbb{A}}^\times$ *with* $\chi_E = \left(\frac{E/F}{}\right) : F_{\mathbb{A}}^\times \to \{\pm 1\}$, *and define* $G_{E/F}(\mathbb{A}) := \mathrm{Ker}(\chi_E^G)$. *Then* $G_{E/F}(\mathbb{A})$ *is an open-closed subgroup of* $\mathrm{GL}_2(F_{\mathbb{A}})$ *of index 2 containing* $\mathrm{GL}_2(F)$, $\mathrm{GL}_2(F_\infty)$ *and the center* $F_{\mathbb{A}}^\times$.

Proof. Since χ_E^G is a continuous open character of order 2, its kernel $G_{E/F}(\mathbb{A})$ is an open subgroup of index 2 containing $(F_{\mathbb{A}}^\times)^2$ which is the image of the center under det. Since χ_E factors through the idele class group $F_{\mathbb{A}}^\times / F^\times N_{E/F}(E_{\mathbb{A}}^\times)$, $G_{E/F}(\mathbb{A})$ contains $\mathrm{GL}_2(F)$ and $\mathrm{GL}_2(F_\infty)$. □

Since $G_{E/F}(\mathbb{A})$ contains $\mathrm{GL}_2(F_\infty)$ and projects surjectively to $\mathrm{GL}_2(F_\infty)$, writing $G_{E/F}(\mathbb{A}^{(\infty)})$ for $\mathrm{Ker}(G_{E/F}(\mathbb{A}) \to \mathrm{GL}_2(F_\infty))$,

$$G_{E/F}(\mathbb{A}) = G_{E/F}(\mathbb{A}^{(\infty)}) \times \mathrm{GL}_2(F_\infty).$$

§7. Extension Theorem.

Theorem 1. *The theta function*

$$\theta(\phi)(g,h) : \mathsf{GL}_2(F)\backslash \mathsf{GL}_2(F)\mathsf{SL}_2(F_{\mathbb{A}}) \cdot \mathbb{G} \times B^\times \backslash B^\times_{\mathbb{A}} \to \mathbb{C}$$

extends to an automorphic form on $\mathsf{GL}_2(F)\backslash G_{E/F}(\mathbb{A}) \times B^\times \backslash B^\times_{\mathbb{A}}$ *independent of the choice of* $\mathbf{S}$*.*

Here "automorphic form" means analytic on the infinite component and right invariant under an open subgroup of $G_{E/F}(\mathbb{A}^{(\infty)}) \times B^\times_{\mathbb{A}^{(\infty)}}$. Since $\theta'(\phi)(g,h) = \theta(\phi)(g,h^{-\iota})$, the twisted one $\theta'(\phi)$ is also well defined over $\mathsf{GL}_2(F)\backslash G_{E/F}(\mathbb{A}) \times B^\times \backslash B^\times_{\mathbb{A}}$. Since $G_{E/F}(\mathbb{A})$ is independent of the choice of $\mathbf{S}$, the theta function is also independent of the choice of $\mathbf{S}$.

§8. **Proof.** We need to show that $\theta(g,h)$ extends to $G_{E/F}(\mathbb{A})$ from the subset $\mathrm{GL}_2(F)\mathbb{G}\mathrm{GL}_2^+(F_\infty)$. For $\phi = \phi^{(\infty)}\phi_\infty \in \mathcal{S}(D_{\sigma,F_{\mathbb{A}}})$, $\phi^{(\infty)}$ is a finite linear combination of factorizable Bruhat functions in $\mathcal{S}(D_{\sigma,F_{\mathbb{A}}^{(\infty)}})$. Thus we may assume that $\phi^{(\infty)}$ is factorizable to prove the extension property of $\theta(g,h) = \theta(\phi)(g,h)$. By Chebotarev density Lemma 2, for $F_{\mathbf{S}}^\times := \prod_{\mathfrak{p}_F \in \mathbf{S}} F_{\mathfrak{p}}^\times \cap F_{\mathbb{A}^{(\infty)}}^\times$, $F^\times F_{\mathbf{S}}^\times F_{\infty+}^\times$ is dense in $F^\times N_{E/F}(E_{\mathbb{A}}^\times)$, and hence its pull-back $\mathrm{GL}_2(F)\mathbb{G}\mathrm{GL}_2^+(F_\infty)$ in $\mathrm{GL}_2(F_{\mathbb{A}})$ by the determinant map is dense in $G_{E/F}(\mathbb{A})$. Thus for any open subgroup U of $G_{E/F}(\mathbb{A})$, we have $\mathrm{GL}_2(F)\mathbb{G}U\mathrm{GL}_2^+(F_\infty) = G_{E/F}(\mathbb{A})$. As we saw, $U^\phi = \prod_{\mathfrak{p}_F} U_{\mathfrak{p}}^\phi$ is an open subgroup of the finite part $G_{E/F}(\mathbb{A}^{(\infty)})$ and $\theta(\phi)(g,h)$ extends to $\mathrm{GL}_2(F)\mathbb{G}U^\phi\mathrm{GL}_2^+(F_\infty) = G_{E/F}(\mathbb{A})$ left invariant under $\mathrm{GL}_2(F)$ and right invariant under U^ϕ. □

By the same proof using $\mathbb{G}'$, we have an extension of $\theta(g,h)$ to $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F_{\mathbb{A}}) \times B^\times\backslash B_{\mathbb{A}}^\times$.

§**9. Induction from $G_{E/F}(\mathbb{A})$ to $\mathrm{GL}_2(F_\mathbb{A})$.** Choose a pair $(\psi_\infty, \phi_\infty)$ with $\psi_\infty|_{F^\times_{\infty+}} = 1$ as in §10 of the last lecture. Then by all the extension lemmas combined, this newly extended $\theta(\phi)(g,h)$ is an automorphic form on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F_\mathbb{A}) \times B^\times\backslash B^\times_\mathbb{A}$. Since $\psi_\infty|_{F^\times_{\infty+}} = 1$, we have

$$\theta(\phi)(zg,h) = \theta(\phi)(g,h) \text{ for } z \in F^\times_{\infty+}.$$

Since $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(F_\mathbb{A})/F^\times_\infty$ has finite volume, we can project $\theta(\phi)$ to the cuspidal subspace, which is written as $\theta_{cusp}(\phi)$. Here is an obvious corollary

Corollary 2. *Writing* Θ *(resp.* Θ_0*) for the representation of* $\mathrm{GL}_2(F_\mathbb{A}) \times B^\times_\mathbb{A}$ *(resp.* $G_{E/F}(\mathbb{A}) \times B^\times_\mathbb{A}$*) generated by* $\theta_{cusp}(\phi)$ *(i.e.,* $\Theta(g,h)(\theta_{cusp}(\phi)(g',h')) = \theta_{cusp}(\phi)(g'g,h'h)$*),* Θ *is isomorphic to the induction* $\mathrm{Ind}_{G_{E/F}(\mathbb{A})}^{\mathrm{GL}_2(F_\mathbb{A})}\Theta_0$*.*

§**10. Quaternion subalgebras of** B**.** For each $\alpha \in D_\sigma \cap B^\times$, define the α-twist σ_α of σ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma\alpha}$. Then σ_α is another action of Gal$(E/\mathbb{Q})$ on B, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of B.

- All quaternion $\mathbb{Q}$-subalgebras of B are realized as D_α for some $\alpha \in D_\sigma$, and $D_z = D \Leftrightarrow z \in Z$;
- $\alpha = \xi^{-1}\beta\xi^{-\iota\sigma}$ for $\xi \in B^\times \Leftrightarrow D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of B if $N(\alpha) = N(\beta)$ and $B_\infty \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group G_α of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \mathrm{Tr}(v) = 0\}$ is $D_\alpha^\times$ and $B^\times$ is a covering of the similitude group GO_{D_σ} of D_σ.

Let $\widehat{\Gamma}_\phi = \{h \in B^\times_{\mathbb{A}^{(\infty)}} | \phi^{(\infty)}(h^{-1}vh^{-\iota\sigma}) = \phi^{(\infty)}(v), \forall v \in D_{\sigma,\mathbb{A}^{(\infty)}}\}$ for each Schwartz-Bruhat function ϕ on $D_{\sigma,\mathbb{A}^{(\infty)}}$.
Let $Sh_B = B^\times \backslash B^\times_{\mathbb{A}} / E^\times_{\mathbb{A}} \widehat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\widehat{\Gamma}_\phi$, and Sh_α be the image of $D^\times_{\alpha,\mathbb{A}}$ in Sh_B for $\alpha \in D_\sigma$. Regard $Sh_\alpha \in H_2(Sh_B, \mathbb{Z})$ and write $(\cdot,\cdot) : H^2 \times H_2 \to \mathbb{C}$ for the Poincaré duality.