Siegel–Weil formula

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Lecture 5: We describe Tamagawa measure on a semi-simple group and the Tamagawa number for the group. Then we state the Siegel-Weil formula and describe an explicit description of the metaplectic group. At the end, we describe Fourier expansion of modular forms on Mp and SL(2) as a preparation for describing the Rankin product method explicit in Lecture 6.

§**0. Tamagawa measure.** Let $G_{/F}$ be an affine linear semi-simple algebraic group with $G(F_v) \neq \emptyset$ for all place v of F. Regard the group G as an affine F-scheme Spec($\mathcal{O}_G$). Write n for the dimension of the scheme G. We take a Haar measure dx on $F_{\mathbb{A}}$ so that $\int_{F_{\mathbb{A}}/F} dx = 1$, $\int_{O_{F_v}} dx_v = 1$ for almost all finite place v and dx_∞ is given by the Lebesgue measure. An algebraic differential form $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ defined everywhere on G, for each place v, we define a measure $|\omega|_v$ on $G(F_v)$ by

$$\int_{G(F_v)} \phi(x)d|\omega|_v = \int_{G(F_v)} \phi(x)|f(x)|_v dx_1 dx_2 \cdots dx_n$$

for the canonical measure dx_j on F_v induced by the above dx. Then define a measure $|\omega|_{\mathbb{A}}$ by $\bigotimes_v |\omega|_v$ on $G(F_{\mathbb{A}})$. The form ω is called a gauge form if $g^*\omega = \omega$ for the pull back of $x \mapsto gx$ for each $g \in G$, and the associated measure is unique and called the Tamagawa measure $d\omega$. The Tamagawa number $\tau(G)$ is defined by

$$\tau(G) = \int_{G(F_{\mathbb{A}})/G(F)} d\omega.$$

§**1. Gauge form on $\mathrm{O}_{V/\mathbb{Q}}$.** For simplicity, assume $F = \mathbb{Q}$ in this section. Writing $\mathrm{GL}(m) = \mathrm{Spec}(\mathbb{Z}[X_{ij}, \det(X_{ij})^{-1}])$, $\omega = \det(X_{i,j})^{-m} \bigwedge_{i,j} dX_{ij}$ induces a gauge form on $\mathrm{GL}(m)$. Since $\mathrm{GL}(m) = \mathbb{G}_m \ltimes \mathrm{SL}(m)$, for an $\mathrm{SL}(m)$-gauge form ω_{SL}, $\omega = \omega_{SL} \wedge dt/t$ writing $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$.

Choose a basis $v_1, \ldots, v_m$ of V over $\mathbb{Q}$ and put $S = (s(v_i, v_j))_{i,j} \in M_m(\mathbb{Q})$. Then we define $x^\iota = S^t x S^{-1}$, which is an involution of $M_m = \mathfrak{gl}_m$. Then $\mathrm{O}_V(A) = \{x \in \mathrm{GL}_m(A) | xx^\iota = 1\}$. We consider $\mathfrak{s}_\pm = \{x \in \mathfrak{gl}_m | x^\iota = \mp x\}$ ($\mathfrak{s}_+$ is the Lie algebra of O_V). We have $\mathfrak{gl}_m = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Since ω as above satisfies $\omega(axb) = \det(a)^m \det(b)^m \omega(x)$ for $a, b \in \mathrm{GL}(m)$, we can split $\omega = \omega_+ \wedge \omega_-$ according to the linear splitting $\mathfrak{gl}_m = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Then ω_+ restricted to $\mathrm{O}_S \subset \mathfrak{gl}_m$ gives a gauge form on the connected component of O_S.

It is known that $\tau(\mathrm{O}_V) = 2$ if $m \geq 2$. (See §1.2.2 and (4.46)).

§2. **Siegel–Weil Eisenstein series.** Consider the function Φ : Mp$(F_{\mathbb{A}}) \ni g \to (\mathbf{w}(g)\phi)(0) \in \mathbb{C}$ for $\phi \in \mathcal{S}(V_{\mathbb{A}})$. This means that we first apply $\mathbf{w}(g)$ to ϕ and then evaluate at $0 \in V$. We have the splitting $B(F_{\mathbb{A}}) \hookrightarrow$ Mp$(F_{\mathbb{A}})$ and SL$_2(F) \hookrightarrow$ Mp$(F_{\mathbb{A}})$ which coincide with $\mathbf{r}$ on $B(F)$ up to constants, and by the definition of $\mathbf{r}$ on $b \in B(F)$, writing $\mathbf{r}(b) = \mathbf{r}(\upsilon(u))\mathbf{r}(\text{diag}[a, a^{-1}])$,

$$\Phi(bg) = \mathbf{e}_F(uQ(v))|a|_{F_{\mathbb{A}}}^{m/2}(\mathbf{w}(g)\phi)(av)|_{v=0} = \Phi(b).$$

Therefore $\Phi(g)$ is a left $B(F)$-invariant function on Mp$(F_{\mathbb{A}})$. For $g \in$ Mp$(F_{\mathbb{A}})$, by Iwasawa decomposition applied to SL$_2(F_{\mathbb{A}})$, write $g = \text{diag}[a, a^{-1}]\upsilon(u)k$ with $k \in \text{SL}_2(\widehat{O}_F)C_\infty$ for the standard maximal compact subgroup C_∞ of SL$_2(F_\infty)$, we define $a(g) := |a|_{F_{\mathbb{A}}}$ and $\Phi_s(g) := \Phi(g)|a(g)|_{F_{\mathbb{A}}}^s$. Define Siegel–Weil Eisenstein series by

$$E(\Phi; s) := \sum_{\gamma \in B(F)\backslash \text{SL}_2(F)} \Phi_s(\gamma g),$$

which is absolutely and locally uniformly convergent if Re$(s) \gg 0$.

§**3. Siegel–Weil formula.** When $n > 4$, $E(\Phi; s)$ converges absolutely if $s = 0$, and $E(\Phi; s)$ has a meromorphic continuation to the whole $s \in \mathbb{C}$. If V is anisotropic and $n \geq 2$, $E(\Phi, s)$ is finite at $s = 0$. When well defined, we write $E(\Phi)$ for $E(\Phi; 0)$.

Let K be a maximal compact subgroup of $\mathrm{O}_S(\mathbb{A})$. Then we have, if either $n > 4$ or S is anisotropic with $n > 1$,

$$\int_{\mathrm{O}_V(\mathbb{Q})\backslash \mathrm{O}_V(\mathbb{A})} \theta(\Phi)(g,h)d\omega(g) = \tau(\mathrm{O}_V)E(\Phi)(g) = 2 \cdot E(\Phi)(g)$$

for $g \in \mathrm{Mp}(\mathbb{A})$, $h \in \mathrm{O}_S(\mathbb{A})$ all K-finite $\Phi \in \mathcal{S}(V_{\mathbb{A}})^{\infty}$.

See §4.4.3 for a proof.

§4. **Standard automorphic factor:** §4.5.2. Let F be a totally real field. We consider $\phi(\tau;\mathfrak{z}_\infty) : \mathfrak{z}_\infty \mapsto \mathbf{e}_\infty(\mathfrak{z}_\infty^2\tau)$ as a Schwartz function of $\mathfrak{z}_\infty \in F_\infty$ with $\tau \in \mathfrak{Z}_F := \mathfrak{H}^{I_F}$. Define a function $h(g,\tau)$ of $g \in \mathrm{Mp}(F_\mathbb{A})$, $\tau \in \mathfrak{Z}_F$ by

$$\Phi_\infty(g) = (\mathbf{w}(g)\phi)(\tau;0) = |a(g)|_{F_\mathbb{A}}^{-1/2} h(g,\tau)^{-1}.$$

Then $h : \mathrm{Mp}(F_\mathbb{A}) \times \mathfrak{Z}_F \to \mathbb{C}^\times$ is a holomorphic function in τ as long as $\pi(g) \in B(F_\mathbb{A})\widehat{\Gamma}_0(4)\mathrm{SO}_2(F_\infty)$. Set

$$j(\gamma,\tau) = (j(\gamma^\nu,\tau_\nu))_{\nu\in I_F} = (c^\nu\tau_\nu + d^\nu)_{\nu\in I_F},\ j(\gamma,\tau)^{I_F} = \prod_\nu (c^\nu\tau_\nu + d^\nu)$$

and $j(\gamma,\tau)^k = \prod_\nu (c^\nu\tau_\nu + d^\nu)^{k_\nu}$ for $k = \sum_\nu k_\nu\nu \in \mathbb{Z}[I_F]$. We denote also by I_F the element $\sum_\nu \nu \in \mathbb{Z}[I_F]$. Then we have
(h1) $h(g,\tau)^2 = t\cdot j(\pi(g),\tau)^{I_F}$ for $t \in S^1$;
(h2) h is an automorphic factor of $g \in \pi^{-1}(B(F_\mathbb{A})\widehat{\Gamma}_0(4)\mathrm{SO}_2(F_\infty))$;
(h3) if $\gamma \in \mathrm{SL}_2(F) \cap B(F_\mathbb{A})\widehat{\Gamma}_0(4)$, $h(\gamma,\tau)^4 = j(\gamma,\tau)^{2I_F}$;
(h4) if $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F) \cap B(F_\mathbb{A})\widehat{\Gamma}_0(4)$ (see §4.5.2),

$$h(\gamma,\tau)^2 = \frac{N(d)}{|N(d)|}\left(\frac{F[\sqrt{-1}]/F}{adO_F}\right) j(\gamma,\tau)^{I_F}.$$

§**5. The case** $F = \mathbb{Q}$. Assume that $n = 1$; so, $\mathfrak{H}_n = \mathfrak{H}$. For integers $a, b \neq 0$, we define Shimura's symbol $\left(\frac{a}{b}\right)$ by

1. $\left(\frac{a}{b}\right) = 0$ if $(a, b) \neq 1$ (where (a, b) is the GCD of a and b),
2. If b is an odd prime, $\left(\frac{a}{b}\right)$ is the Legendre symbol (i.e., it is less one than the number of solutions of $x^2 \equiv a \mod b$),
3. If $b > 0$, $a \mapsto \left(\frac{a}{b}\right)$ is a character modulo b,
4. If $a \neq 0$, $b \mapsto \left(\frac{a}{b}\right)$ is a character modulo $4a$ whose conductor is the conductor of $\mathbb{Q}[\sqrt{a}]_{/\mathbb{Q}}$,
5. $\left(\frac{a}{-1}\right) = 1$ or -1 according as $a > 0$ or $a < 0$,
6. $\left(\frac{0}{\pm 1}\right) = 1$.

Recall $\theta : \mathfrak{H} \to \mathbb{C}$ given by $\theta(\tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau)$. For $\gamma \in \Gamma_0(4)$, we have $h(\gamma, \tau) := \theta(\gamma(\tau))/\theta(\tau)$ and

$$h(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau)^{1/2},$$

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$, $\tau^{1/2} = \sqrt{|\tau|}\exp(\pi i\theta)$ if $\tau = |\tau|\exp(2\pi i\theta)$ with $-1 < \theta \leq 1$ and $\varepsilon_d = \sqrt{-1}$ or 1 according as $d \equiv 3$ or $1 \mod 4$. See §4.3.1.

§6. **Quadratic space over a totally real field F.** The extension $S^1 \hookrightarrow \mathrm{Mp}(F_{\mathbb{A}}) \twoheadrightarrow \mathrm{SL}_2(F_{\mathbb{A}})$ actually descends down to $\mu_2 \hookrightarrow \widetilde{\mathrm{SL}}_2(F_{\mathbb{A}}) \twoheadrightarrow \mathrm{SL}_2(F_{\mathbb{A}})$. The 2-cocycle: $\mathrm{SL}_2(F_{\mathbb{A}}) \to S^1$ giving rise to the extension $\mathrm{Mp}(F_{\mathbb{A}})$ is cohomologous to another one $\kappa : \mathrm{SL}_2(F_{\mathbb{A}}) \to \mu_2$ with values in μ_2 (found by T. Kubota; see §4.3.3), and we have the following commutative diagram:

$$\begin{array}{ccccc} \mu_2 & \xrightarrow{\hookrightarrow} & \widetilde{\mathrm{SL}}_2(F_{\mathbb{A}}) & \xrightarrow{\twoheadrightarrow} & \mathrm{SL}_2(F_{\mathbb{A}}) \\ \cap\downarrow & & \cap\downarrow & & \downarrow\cap \\ S^1 & \underset{\hookrightarrow}{\longrightarrow} & \mathrm{Mp}(F_{\mathbb{A}}) & \underset{\twoheadrightarrow}{\longrightarrow} & \mathrm{SL}_2(F_{\mathbb{A}}). \end{array}$$

For F, we put $j(g_\infty, \tau) = \prod_{\nu \in I}(c_\nu \tau_\nu + d_\nu)$ for $\tau = (\tau_\nu)_{\nu \in I} \in \mathfrak{H}^I$ and $g_\infty = (g_\nu)_{\nu \in I} \in \mathrm{SL}_2(\mathbb{R})^I = \mathrm{SL}_2(F_\infty)$. We can realize

$$\widetilde{\mathrm{SL}}_2(F_\infty) = \{(g, J(g,\tau)) | g \in \mathrm{SL}_2(F_\infty), J(g,\tau)^2 = j(g,\tau)\}$$

with product given by $(g, J(g,\tau))(h, J(h,\tau)) = (gh, J(g,h(\tau))J(h,\tau))$. Thus we have the central extension $\mu_2 \overset{i}{\hookrightarrow} \widetilde{\mathrm{SL}}_2(F_\infty) \overset{\pi}{\twoheadrightarrow} \mathrm{SL}_2(F_\infty)$ with $i(-1) = (1_2, -1)$ and $\pi(g, J) = g$. The center of $\widetilde{\mathrm{SL}}_2$ is given by $\mu_2 \times \mu_2(F_\infty)$. See §4.3.3.

§**7. Half integral weight for $F = \mathbb{Q}$.** Let $\widehat{\Gamma}$ is an open subgroup of $\widehat{\Gamma}_0(4)$ and $\Gamma = \widehat{\Gamma} \cap \mathrm{SL}_2(\mathbb{Z})$. A modular form $f \in M^{\pm}_{\ell/2}(\Gamma, \psi)$ (which is a holomorphic or anti-holomorphic function on $\mathfrak{H}$ depending on the sign) is called a modular form of weight $\frac{\ell}{2}$ for odd ℓ if it satisfies $f(\gamma(\tau)) = \psi(\gamma) f(\tau) h(\gamma, \tau^{\pm})^{\ell}$ for $\gamma \in \Gamma$, $\tau^{+} = \tau$ and $\tau^{-} = \overline{\tau}$. Here $\psi : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}^{\times}$ is a character and $\psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(d)$. The modular form f has its Fourier expansion: $f(\tau) = \sum_{0 \leq n \in L}^{\infty} a_n(f) \mathbf{e}(\pm n \tau^{\pm})$ for a lattice $L \subset \mathbb{Q}$. We extend ψ to a character of $\widetilde{\psi} : \widehat{\Gamma}_0(M) \to \mathbb{C}^{\times}$ so that $\widetilde{\psi}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \psi(d_N)$. We lift f to $\mathbf{f} : \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ by putting

$$\mathbf{f}(\alpha(u, \zeta J(u_\infty, \tau))) = \widetilde{\psi}(u) f(u_\infty(\sqrt{-1})) \zeta^{\ell} J(u_\infty, \pm i)^{-\ell}$$

for $\alpha \in \mathrm{SL}_2(\mathbb{Q}) \subset \mathrm{Mp}(\mathbb{A})$ and $(u, J(u_\infty, \tau)) \in \widehat{\Gamma} \cdot \mathrm{Mp}(\mathbb{R})$ ($\zeta \in S^1$ and $u_\infty = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$) regarding $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A}) \subset \mathrm{Mp}(\mathbb{A})$.

§8. **Adelic half integral weight forms;** §**4.3.4.** We define the space of adelic modular forms $M^{\pm}_{\ell/2}(\widehat{\Gamma},\psi)$ on $\widehat{\Gamma}$ of weight $\ell/2$ as a function $\mathbf{f} : \mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:
(hi1) $\mathbf{f}(\xi g(u, \zeta J(u_\infty,\tau))) = \widetilde{\psi}(u)\mathbf{f}(g)\zeta^{\ell} J(u_\infty, \pm i)^{-\ell}$ for all $\xi \in \mathrm{SL}_2(\mathbb{Q})$, $\zeta \in S^1$ and $u \in \widehat{\Gamma} \cdot \mathrm{Mp}(\mathbb{R})$;
(hi2) $f(\tau) := \mathbf{f}(g_\tau, \eta^{-1/4})\eta^{-\ell/4}$ for $g_\tau = \eta^{-1/2}\left(\begin{smallmatrix}\eta & \xi\\ 0 & 1\end{smallmatrix}\right)$ is holomorphic or anti-holomorphic according to the sign;
(hi3) $f(\tau)$ is finite at cusps.

We define similarly the space $S^{\pm}_{\ell/2}(\widehat{\Gamma},\psi)$ of cusp forms, requiring $a_0(f|_{\ell/2}\alpha) = 0$ for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, where $f|_{\ell/2}\alpha(\tau) = f(\alpha(\tau))h(\alpha,\tau^{\pm})^{-\ell}$ taking a square root holomorphic function $\tau \mapsto h(\alpha,\tau)$ of $j(\alpha,\tau)$ suitably.

§**9. Fourier expansion; Section 4.6.** We extend ψ to a character of $\psi^* : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ so that $\psi^*(\varpi_l) = \psi(l)$ for each prime l prime to M and then ψ^* to $\widehat{\Gamma}_0(M)$ so that $\widetilde{\psi}(u) = \psi(u)^{-1}$. Define an idele character $\boldsymbol{\psi} : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ by $\boldsymbol{\psi}(a) = \psi^*(a)|a|_{\mathbb{A}}^{-\ell/2}$. Thus for $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B(\widehat{\mathbb{Z}})B(\mathbb{R}) \subset \widetilde{\mathrm{SL}_2}(\mathbb{A})$, we find for $\tau = a_\infty(a_\infty i + b_\infty)$ and a lattice $L \subset \mathbb{Q}$

$$\mathbf{f}(g) = \boldsymbol{\psi}^{-1}(a) \sum_{0 \leq n \in L} a_n(f) \exp(-2\pi n a_\infty^2) \mathbf{e}(\pm n a_\infty b_\infty).$$

Let $v(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U(\mathbb{A})$. Then we consider for a general $b = v(u)\,\mathrm{diag}[a, a^{-1}] \in B(\mathbb{A})$. Write $\mathbf{f}(a,u) := \mathbf{f}(b)$. Then $\mathbf{f}(a, u+\alpha) = \mathbf{f}(v(\alpha)b) = \mathbf{f}(b)$ if $\alpha \in \mathbb{Q}$. Thus $\mathbf{f}(a,u)$ has a Fourier expansion over $u \in \mathbb{A}$ of the form

$$\mathbf{f}(a,u) = \sum_{\alpha \in \mathbb{Q}} a_{\mathbf{f}}(\alpha; a)\mathbf{e}(\alpha u) \quad \text{with } a_{\mathbf{f}}(\alpha; a) = \int_{\mathbb{A}/\mathbb{Q}} \mathbf{f}(a,u)\mathbf{e}_{\mathbb{A}}(-\alpha u)du$$

for the volume one Haar measure du of $\mathbb{A}/\mathbb{Q}$.

§**10. Properties of expansion coefficients.** By

$$\mathrm{diag}[\beta,\beta^{-1}]\upsilon(u)\,\mathrm{diag}[a,a^{-1}]=\upsilon(\beta^2u)\,\mathrm{diag}[\beta a,(\beta a)^{-1}]$$

for $\beta\in\mathbb{Q}$, we have

$$\sum_{\alpha\in\mathbb{Q}}a_{\mathbf{f}}(\alpha;a)\mathbf{e}(\alpha u)=\mathbf{f}(a,u)=\mathbf{f}(\beta a,\beta^2u)=\sum_{\alpha\in\mathbb{Q}}a_{\mathbf{f}}(\alpha;\beta a)\mathbf{e}(\alpha\beta^2u).$$

By the uniqueness of the expansion,

(*) $$a_{\mathbf{f}}(\alpha;a)=a_{\mathbf{f}}(\alpha\beta^{-2},\beta a)$$

$$a_{\mathbf{f}}(\alpha;a)=\begin{cases}\psi^{-1}(a)a_\alpha(f)\exp(-2\pi\alpha a_\infty^2) & \text{if } 0\le\alpha\in L,\\ 0 & \text{otherwise.}\end{cases}$$

For $t\in\widehat{\mathbb{Z}}^\times$, we get

$$a_{\mathbf{f}}(\alpha;at)=\psi^{-1}(t)a_{\mathbf{f}}(\alpha,a),$$

since $\upsilon(u)\,\mathrm{diag}[a,a^{-1}]\,\mathrm{diag}[t,t^{-1}]=\upsilon(u)\,\mathrm{diag}[ta,(ta)^{-1}]$.

§11. **Normalization.** Define for $a \in \mathbb{A}^\times$, if $\alpha a^2 \in L(U_{\widehat{\Gamma}}\mathbb{R}_+)^2 \cap (\mathbb{A}^\times \sqcup \{0\})$,

(∗∗) $\quad \mathbf{a_f}(\alpha a^2) := \psi(a) a_{\mathbf{f}}(\alpha, a) \exp(2\pi\alpha_\infty a_\infty^2) = a_\alpha(f),$

and otherwise, put $\mathbf{a_f}(\alpha a^2) = 0$. If $\alpha a^2 (U_{\widehat{\Gamma}})^2 = \xi b^2 (U_{\widehat{\Gamma}})^2$ ($\xi \in \mathbb{Q}$), then writing $\alpha a^2 a^2 = \xi b^2 t^2$ for $t \in U_{\widehat{\Gamma}}$, $\xi^{-1}\alpha$ is locally square; so, $\xi = \alpha\beta^{-2}$ with $b = \beta a$, and we have

$$\mathbf{a_f}(\alpha a^2) := \psi(a) a_{\mathbf{f}}(\alpha, a) \exp(2\pi\alpha_\infty a_\infty^2)$$
$$\overset{(*)}{=} \psi(\beta a) a_{\mathbf{f}}(\alpha\beta^{-2}, \beta a) \exp(2\pi(\alpha_\infty \beta_\infty^{-2})(\beta a)_\infty^2) = \mathbf{a_f}(\xi b^2),$$

since $\psi(\beta a) = \psi(a)$. This shows that $\mathbf{a_f}(x)$ is well defined independent of the choice of the expression $x = \alpha a^2$ with $a \in \mathbb{A}^\times$ and $\alpha \in \mathbb{Q}^\times$.

By (∗∗), we have

$$\mathbf{a_f}(x) = a_\alpha(f) = \mathbf{a_f}(xt^2) \text{ for } t \in \widehat{\mathbb{Z}}^\times \mathbb{R}^\times \text{ and } x = \alpha a^2.$$

Since $\mathbf{a_f}$ is supported over $\mathbb{A}_+^\times = \mathbb{A}^{(\infty)}\mathbb{R}_+^\times$, $\mathbf{a_f}$ only depends on the finite part of the idele. Thus we can recover

$$\mathbf{f}(a, u) = \psi(a)^{-1} \sum_{0<\alpha\in\mathbb{Q}} \mathbf{a_f}(\alpha a^2) \exp(-2\pi\alpha_\infty a_\infty^2) \mathbf{e}(\pm\alpha u).$$