Adjoint class number formula in the simplest case

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Lecture 2: Definite quaternions. For a Hecke eigenform f, we recall the adjoint L-value formula relative to each definite quaternion algebra D over $\mathbb{Q}$ with discriminant ∂ and reduced norm N. A key to prove the formula is the theta correspondence for the quadratic $\mathbb{Q}$-space (D, N). Under the $\mathcal{R} = \mathbb{T}$-theorem, p-part of the Bloch-Kato conjecture is known; so, the formula is an adjoint Selmer class number formula. The main reference is Chapter 5.

§**0. Setting of the simplest case.** For $E = \mathbb{Q} \times \mathbb{Q}$, $D_\sigma = \{(v, v^\iota) | v \in D\} \cong D$ by $(v, v^\iota) \mapsto v$ as quadratic spaces. Assume that D is definite. Let $G_D(\mathbb{Q}) = \{(h_l, h_r) \in D^\times \times D^\times | N(h_l) = N(h_r)\}$ which acts on D by $v \mapsto h_l^{-1} v h_r$; so, $\mathrm{SO}_D(\mathbb{Q}) = G_D(\mathbb{Q})/Z_D(\mathbb{Q})$ for the center $Z_D \subset G_D$. For a Bruhat function ϕ on $D_{\mathbb{A}}^{(\infty)}$ and $\tau \in \mathfrak{H}$ (the upper half complex plane), the theta kernel

$$\theta(\phi; \tau; h_l, h_r) = \sum_{\alpha \in D} \phi(h_l^{-1} \alpha h_r) \mathbf{e}(N(\alpha_\infty)\tau) \quad \text{on } \mathfrak{H} \times G_D(\mathbb{A})$$

can be extended to an automorphic form on $Y_\Gamma \times Sh \times Sh$ for $Sh := D^\times \backslash D_{\mathbb{A}}^\times / D_\infty^\times$ and $Y_\Gamma := \Gamma \backslash \mathfrak{H}$ for a suitable congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. An A-integral automorphic form $\mathcal{F}$ on Sh of level $\widehat{R}^\times$ is a function $\mathcal{F} : Sh/\widehat{R}^\times \to A$ with $\int_{Sh_R} \mathcal{F} d\mu = 0$ whose space is written as $S(A)$. For $f \in S_2(\Gamma)$ and $\mathcal{F}, \mathcal{G} \in S(\mathbb{C})$, we define

$$\theta^*(\phi)(f)(h_l, h_r) = \int_{Y_\Gamma} f(\tau)\theta(\phi)(\tau; h_l, h_r) y^{-2} dx dy \quad \text{(lift)},$$

$$\theta_*(\phi)(\mathcal{F} \otimes \mathcal{G})(\tau) = \int_{Sh \times Sh} \theta(\phi)(\tau; h_l, h_r) \mathcal{F}(h_l) \cdot \mathcal{G}(h_r) d\mu_l d\mu_r \quad \text{(descent)}.$$

§**1. Two good choices of ϕ.** Let R be an Eichler order of level M; so, $M = \partial N_0$ with $(N_0, \partial) = 1$.

Choice A: At N_0, we identify $R/N_0R = \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}) \in M_2(\mathbb{Z}/N_0\mathbb{Z})\}$. Let ϕ_R be the characteristic function of

$$\left\{x \in \widehat{R} | x \text{ mod } N_0\widehat{R} = (\begin{smallmatrix} * & * \\ 0 & d \end{smallmatrix}),\ d \in (\mathbb{Z}/N_0\mathbb{Z})^\times \right\}.$$

Then the first choice is $\Phi(v) = \phi_R(v^{(\infty)})\mathbf{e}(N(v_\infty)\tau)$ as a Schwartz-Bruhat function on $D_\mathbb{A}$. We have $\Gamma = \Gamma_0(M)$, as the level $M = N$ of a lattice for L is the smallest integer such that $M \cdot N(L^*) \subset \mathbb{Z}$ (see §3.1.4).

Choice B: Let ϕ_L be the characteristic function of $\widehat{L}$. Choose $0 < c \in \mathbb{Z}$ and $L := \mathbb{Z} \oplus R_0$ for $R_0 = \{v \in R | \mathrm{Tr}(v) = 0\}$, and put $\phi'_{R_0} = (1 - c^3)^{-1}(\phi_{R_0} - \phi_{cR_0})$. Then we define, writing $v = z \oplus w$ with $z \in Z_\mathbb{A}$ and $w \in D_{0,\mathbb{A}}$

$$\phi'(v) = \phi'_c(v) = \phi_\mathbb{Z}(z^{(\infty)})\phi'_{R_0}(w^{(\infty)})\mathbf{e}(N(v_\infty)\tau).$$

We have $\Gamma = \Gamma_0(4c^2M)$.

§2. **Two theorems.** Let $Sh_R = Sh/\widehat{R}^\times$ and $\delta(Sh_R)$ is the diagonal image of Sh_R in $Sh_R \times Sh_R = Sh_B/(\widehat{R}^\times \times \widehat{R}^\times)$.

Theorem A: *Assume that $\int_{Sh_R} \mathcal{F} d\mu = \int_{Sh_R} \mathcal{G} d\mu = 0$. Take Φ as in Choice A. Then $\theta_*(\Phi)(\mathcal{F} \otimes \mathcal{G}) = \sum_{n=1}^\infty (\mathcal{F}, \mathcal{G}|T(n))q^n$ for $q = \exp(2\pi i\tau)$ with $(\mathcal{F}, \mathcal{G}) = \int_{\delta(Sh_R)} \mathcal{F}(h)\mathcal{G}(h) d\mu_h$. So $\theta_*(\Phi)$ and $\theta^*(\Phi)$ are Hecke equivariant.*

Theorem B: *If f is a Hecke eigen new form of $S_2(\Gamma_0(M))$, then for the canonical period $\Omega_\pm$ of f,*

$$\prod_{p|\partial}(1-p^{-2})^{-1}\mathfrak{m}_1 \frac{L(1, Ad(\rho_f))}{2\pi^3\Omega_+\Omega_-} = \int_{\delta(Sh_R)} \frac{\theta^*(\phi')(f)(h)}{\Omega_+\Omega_-} d\mu,$$

where $\mathfrak{m}_1$ is the mass factor of $L \cap D_0$: $\mathfrak{m}_1 \frac{\zeta(2)}{\pi^2} = \int_{Sh_R} d\mu \in \mathbb{Q}$ (Siegel's mass formula) *and if $\partial = p$ with $N_0 = 1$, $\mathfrak{m}_1 = (p-1)/2$.*

Theorem B is independent of c. Here $\int_{\widehat{R}^\times} d\mu = 1$.

§**3. Canonical periods.** If $D = M_2(\mathbb{Q})$, Sh_R is a Shimura curve $X_0(M)$. Let $\mathcal{W}$ be a DVR at a prime $\mathfrak{p}$ such that $\mathbb{Z}[\lambda] = \mathbb{Z}[\lambda(T(n))|n \in \mathbb{Z}] \subset \mathcal{W} \subset \mathbb{Q}[\lambda]$ for the Hecke field $\mathbb{Q}[\lambda]$ of f (i.e., $f|T(n) = \lambda(T(n))f$). Write $\mathbb{F} := \mathcal{W}/\mathfrak{m}_{\mathcal{W}}$. Define $f_\pm$ by

$$H_\lambda = H_\lambda^\pm := H^1(X_0(M), \mathcal{W})[\lambda, \pm] = \mathcal{W}[f_\pm],$$

where $\pm$ indicate the $\pm$-eigenspace of complex conjugation on Sh_R. Put $H := H^1(X_0(M), \mathcal{W})[\pm]$.

Define f by $H^0(X_0(M), \Omega_{X_0(M)/\mathcal{W}})[\lambda] = \mathcal{W}\omega_f$ ($f \in S_2(\Gamma_0(M); \mathcal{W})$) for $\omega_f = 2\pi i f(\tau)d\tau$. We project it to a unique element $\omega^\pm(f) = \omega_f \pm c^*\omega_f \in H^1(X_0(M), \mathbb{C})[\lambda, \pm]$ for complex conjugation $c : \tau \mapsto -\overline{\tau}$ and define the period $\Omega_\pm \in \mathbb{C}^\times$ as $\omega^\pm(f) = \Omega_\pm[f_\pm]$. Let $W = \varprojlim_n \mathcal{W}/\mathfrak{m}_{\mathcal{W}}^n$ and put

$$H(W) := W[T(n)|n = 1, 2, \ldots\} \subset \mathrm{End}_W(S(W))$$

as a W-algebra.

§**4.** $\mathcal{R} = \mathbb{T}$. For simplicity, assume $N_0 = 1$. Let

$$S(\mathcal{W})_\lambda = S(\mathcal{W})[\lambda], \quad S(\mathcal{W})^\lambda = \{\mathcal{G} \in S(\mathbb{Q}(\lambda)) | (\mathcal{G}, S(\mathcal{W})_\lambda) \subset \mathcal{W}\}.$$

Define the D-congruence module by $C^D(\lambda; \mathcal{W}) := S(\mathcal{W})^\lambda / S(\mathcal{W})_\lambda$. Write p for the residual characteristic of $\mathcal{W}$ and $\mathbb{Q}^{(p\partial)}$ for the maximal extension of $\mathbb{Q}$ unramified outside $p\partial$. Let $\rho_\lambda : \mathfrak{g} \to \mathrm{GL}_2(W)$ for $\mathfrak{g} := \mathrm{Gal}(\mathbb{Q}^{(p\partial)}/\mathbb{Q})$ be the Galois representation of λ. Put $\overline{\rho} := \rho_\lambda \bmod \mathfrak{m}_{\mathcal{W}} : \mathfrak{g} \to \mathrm{GL}_2(\mathbb{F})$. Suppose $p > 3$ and that $\lambda(T(p)) \in \mathcal{W}^\times$ and that $\overline{\rho}$ ramifies at $l | p\partial$. It is known that $\rho_\lambda|_{D_l} \cong \left(\begin{smallmatrix} \varepsilon_{\lambda,l} & u_{\lambda,l} \\ 0 & \delta_{\lambda,l} \end{smallmatrix}\right)$ with unramified δ_l such that $\delta_{\lambda,l}(\mathrm{Frob}_l) = \lambda(U(l))$ for each prime $l | p\partial$. Let $\mathcal{R}$ be the universal minimal deformation ring unramified outside $p\partial$ and $\mathbb{T}$ be the local component of $H(W)$ through which λ factors. See [EMI, Section 9.3] for

$\mathcal{R} = \mathbb{T}$ **Theorem** (Taylor–Wiles, Mazur). *Suppose absolute irreducibility of $\overline{\rho}$, $\delta_{\lambda,p} \not\equiv \varepsilon_{\lambda,p} \bmod \mathfrak{m}_W$ and $u_{\lambda,l} \bmod \mathfrak{m}_W \neq 0$ for $l | p\partial$. Then $\mathcal{R} \cong \mathbb{T}$, $S(W) \otimes_{H(W)} \mathbb{T} \cong \mathbb{T}$, and*

$$|C^D(\lambda; \mathcal{W})| \overset{\text{T-W}}{=} |\Omega_{\mathbb{T}/W} \otimes_{\mathbb{T},\lambda} W| \overset{\text{M}}{=} |\mathrm{Sel}(Ad(\rho_\lambda))|$$

for the minimal ordinary Selmer group $\mathrm{Sel}(Ad(\rho_\lambda))$.

§**5. Universal ring $\mathcal{R}$.** A Galois representation $\rho : \mathfrak{g} \to \mathrm{GL}_2(A)$ for a local profinite W-algebra with $A/\mathfrak{m}_A = \mathbb{F}$ is called a minimal ordinary deformation of $\overline{\rho}$ if $\rho \mod \mathfrak{m}_A \cong \overline{\rho}$, ρ is unramified outside $p\partial$, $\det(\rho)$ is the cyclotomic character $\nu : \mathfrak{g} \to W^\times$ composed with the structure morphism: $W \to A$ and $\rho|_{D_l} \cong \left(\begin{smallmatrix} \varepsilon_l & u_l \\ 0 & \delta_l \end{smallmatrix}\right)$ keeping the upper triangular form over D_l for all $l|p\partial$. A profinite local W-algebra $\mathcal{R}$ with $\mathcal{R}/\mathfrak{m}_{\mathcal{R}} = \mathbb{F}$ is called minimal (ordinary) universal ring if

(i) we have a minimal ordinary deformation $\boldsymbol{\rho} : \mathfrak{g} \to \mathrm{GL}_2(\mathcal{R})$ unramified outside $p\partial$ with $\boldsymbol{\rho}|_{D_l} \cong \left(\begin{smallmatrix} \boldsymbol{\varepsilon}_l & \mathbf{u}_l \\ 0 & \boldsymbol{\delta}_l \end{smallmatrix}\right)$,

(ii) for any minimal ordinary deformation $\rho : \mathfrak{g} \to \mathrm{GL}_2(A)$ for a local profinite W-algebra with $A/\mathfrak{m}_A = \mathbb{F}$ such that

$$\rho \mod \mathfrak{m}_A \cong \overline{\rho}$$

in $\mathrm{GL}_2(\mathbb{F})$, there is a unique W-algebra homomorphism $\varphi : \mathcal{R} \to A$ such that $\varphi \circ \boldsymbol{\rho} \cong \rho$ with $\varphi \circ \boldsymbol{\delta}_l = \delta_l$.

Since $\mathbb{T}$ carries a minimal ordinary deformation $\rho_{\mathbb{T}}$ satisfying $\mathrm{Tr}(\rho_{\mathbb{T}}(\mathrm{Frob}_l)) = T(l) \in \mathbb{T}$ for $l \nmid p\partial$, the isomorphism ι in $\mathcal{R} = \mathbb{T}$ theorem is canonical with $\iota \circ \boldsymbol{\rho} \cong \rho_{\mathbb{T}}$.

§**6. Selmer group.** Define a representation $Ad(\rho_\lambda)$ acting on $\mathfrak{sl}_2(W)$ by $x \mapsto \rho_\lambda(\sigma)x\rho_\lambda^{-1}(\sigma)$. Let $T := \mathfrak{sl}_2(W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. At primes $l|p\partial$, fixing decomposition group and inertia group $D_l \supset I_l$, we have $T_{l,-}$ such that D_l acts by $\varepsilon_l\delta_l^{-1}$ and put $T_{l,+} = T_l/T_{l,-}$. Define

$$\mathrm{Sel}(Ad(\rho_\lambda)) := \mathrm{Ker}(H^1(\mathfrak{g}, T) \to \prod_{l|p\partial} H^1(I_l, T_{l,+})).$$

We need a more general definition of the Selmer group to relate $\mathrm{Sel}(Ad(\rho_\lambda))$ and $\Omega_{\mathbb{T}/W} \otimes_{\mathbb{T},\lambda} W$. Let X be a finite A-module and consider $A[X] := A \oplus X$ as an A-algebra so that $X^2 = 0$. The ring $A[X]$ is still local profinite. Write $\mathcal{D}(A)$ for the set of all deformations with values in $\mathrm{GL}_2(A)$ modulo isomorphisms and taking $A = \mathcal{R}$, put

$$\Phi(X) = \frac{\{\rho : \mathfrak{g} \to \mathrm{GL}_2(\mathcal{R}[X]) | (\rho \bmod X) = \boldsymbol{\rho}, [\rho] \in \mathcal{D}(\mathcal{R}[X])\}}{1 + M_2(X)}.$$

§**7. Local conditions.** Write $\rho \in \Phi(X)$ as $\rho = \boldsymbol{\rho} \oplus u'$ for $u' : G \to M_2(X)$ as $\mathrm{GL}_2(\mathcal{R}[X]) = \mathrm{GL}_2(\mathcal{R}) \oplus M_2(X)$. By computation, writing $u(g) = u'(g)\boldsymbol{\rho}(g)^{-1}$, $\rho(gh) = \rho(g)\rho(h)$ produces the relation $u'(gh) = \boldsymbol{\rho}(g)u'(h) + u'(g)\boldsymbol{\rho}(h) \Leftrightarrow Ad(g)u(h) + u(g)$; so, u is a 1-cocycle with values in $M_2(X) = M_2(\mathcal{R}) \otimes_{\mathcal{R}} X$. By $\det(\boldsymbol{\rho}) = \det(\rho) = \nu$, we find $1 = \det(\rho\boldsymbol{\rho}^{-1}) = \det(1 \oplus u) = 1 + \mathrm{Tr}(u)$ as $X^2 = 0$. Thus u has values in $\mathfrak{sl}_2(X) := \mathfrak{sl}_2(\mathcal{R}) \otimes_{\mathcal{R}} X$. Writing $u|_{I_l} = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, by unramifiedness of δ_l and $\mathrm{Tr}(u) = 0$, we find $u|_{I_l}$ is upper nilpotent; so, verifying the equivalence classes in $\Phi(X)$ corresponds cohomology classes, we find $\Phi(X) \hookrightarrow H^1(\mathfrak{g}, Ad(X))$ and

$$\Phi(X) = \mathrm{Sel}(Ad(X)) := \mathrm{Ker}(H^1(\mathfrak{g}, Ad(X)) \to \prod_{l|p\partial} H^1(I_l, T^X_{l,+})),$$

where $T^X_{l,-} \subset Ad(X)$ is the maximal subspace on which D_l acts by $\boldsymbol{\varepsilon}_l\boldsymbol{\delta}_l^{-1}$ and $T^X_{l,+} = Ad(X)/T^X_{l,-}$. Now writing $T = \varinjlim_n \mathfrak{sl}_2(W_n)$ and applying the above result to each $X = W_n := p^{-n}W/W$,

$$\mathrm{Sel}(Ad(\rho_\lambda)) = \varinjlim_n \Phi(W_n).$$

§**8. $\Phi(X)$ and differentials.** Let $\varphi = \mathrm{id}_{\mathcal{R}} \oplus \delta_\varphi : \mathcal{R} \to \mathcal{R}[X]$ be an $\mathcal{R}$-algebra homomorphism. Since $(r \oplus x)(r' \oplus x') = rr' \oplus (rx' + r'x)$ for $r, r' \in \mathcal{R}$ and $x, x' \in X$, we find φ projected to X written as δ_φ satisfies $\delta_\varphi(rr') = r\delta_\varphi(r') + r'\delta_\varphi(r)$; so, it is an $\mathcal{R}$-derivation with values in X. Thus byn definition,

$$\mathrm{Sel}(Ad(W_n)) \cong \Phi(W_n) \cong \mathrm{Hom}_W(\Omega_{\mathcal{R}/W} \otimes_{\mathcal{R},\lambda} W_n, W_n),$$

and passing to the injectve limit

$$\mathrm{Sel}(Ad(\rho_\lambda)) = \mathrm{Hom}_W(\Omega_{\mathcal{R}/W} \otimes_{\mathcal{R},\lambda} W, W \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) = (\Omega_{\mathcal{R}/W} \otimes_{\mathcal{R},\lambda} W)^*$$

with "$*$" indicating Pontryagin dual. Since $\mathbb{T}$ is reduced finite flat over W, $\Omega_{\mathcal{R}/W} \otimes_{\mathcal{R},\lambda} W$ is finite, and we conclude Mazur's result

$$|\mathrm{Sel}(Ad(\rho_\lambda))| = |\Omega_{\mathcal{R}/W} \otimes_{\mathcal{R},\lambda} W|.$$

§9. **Congruence module.** Replacing $S(\mathcal{W})$ by $S_2^{new}(\Gamma_0(M);\mathcal{W}) = S_2^{new}(\Gamma_0(M)) \cap \mathcal{W}[[q]]$ in the definition of $C^D(\lambda;\mathcal{W})$ for $M = [p,\partial]$, we define $C(\lambda;\mathcal{W})$ (the congruence module for $M_2(\mathbb{Q})$). Let $H_2(W) := W[T(n)|n = 1,2,\dots] \subset \mathrm{End}(S_2^{new}(\Gamma_0(M);W))$ for $S_2^{new}(\Gamma_0(M);W) = S_2^{new}(\Gamma_0(M);\mathcal{W}) \otimes_{\mathcal{W}} W \subset W[[q]]$. By Jacquet–Langlands correspondence, $H_2(W) \cong H(W)$ by $T(n) \mapsto T(n)$; so, $\mathbb{T}$ is a factor of $H_2(W)$. Again by Taylor–Wiles argument, we get

$\mathcal{R} = \mathbb{T}$ **Theorem 2.** *Let the assumption be as in* $\mathcal{R} = \mathbb{T}$ Theorem. *Then* $\mathcal{R} \cong \mathbb{T}$, $S_2^{new}(\Gamma_0(p\partial);W) \otimes_{H_2(W)} \mathbb{T} \cong \mathbb{T} \cong H_\lambda^{\pm} \otimes_{\mathcal{W}} W$, *and*

$$|C(\lambda;\mathcal{W})| = |C^D(\lambda;\mathcal{W})| = |\Omega_{\mathbb{T}/W} \otimes_{\mathbb{T},\lambda} W| = |\mathrm{Sel}(Ad(\rho_\lambda))|$$

for the minimal ordinary Selmer group $\mathrm{Sel}(Ad(\rho_\lambda))$.

Since the congruence modules only depends on $\mathbb{T}$-module structure of $S_2^{new}(\Gamma_0(p\partial);W) \otimes_{H_2(W)} \mathbb{T}$ and $S(W) \otimes_{H(W)} \mathbb{T}$, the identity $|C(\lambda;\mathcal{W})| = |C^D(\lambda;\mathcal{W})|$ follows from $S_2^{new}(\Gamma_0(p\partial);W) \otimes_{H_2(W)} \mathbb{T} \cong \mathbb{T} \cong S(W) \otimes_{H(W)} \mathbb{T}$.

§10. $|C_0^D(\lambda; W)|$. If $S(\mathcal{W})_\lambda = \mathcal{W}\mathcal{F}$, then $|C^D(\lambda; W)| = |(\mathcal{F},\mathcal{F})|_p^{-1}$, where choosing a representative set $S \subset D^\times_{\mathbb{A}^{(\infty)}}$ for Sh_R and writing $R_h := h\widehat{R}h^{-1} \cap D$ (another Eichler order) and $e_h = |R_h^\times|$,

$$(\mathcal{F},\mathcal{G}) = \sum_{h \in S} \mathcal{F}(h)\mathcal{G}(h)/e_h = \int_{Sh_R} \mathcal{F}\mathcal{G}d\mu.$$

Similarly, ignoring powers of π for simplicity,

$$|C(\lambda; W)| = |\langle [f_+],[f_-]\rangle|_p^{-1} = \left|\frac{(f,f)}{\Omega_+\Omega_-}\right|_p^{-1} \overset{\text{H, 1981}}{=} \left|\frac{L(1,Ad(\rho_\lambda))}{\Omega_+\Omega_-}\right|_p^{-1}.$$

By Hecke equivariance, $\theta^*(\Phi)(f) = \Omega^D(\mathcal{F}\otimes\mathcal{F})$ ($\Omega^D \in \mathbb{C}$); so,

$$|C(\lambda; \mathcal{W})| = \frac{L(1,Ad(\rho_\lambda))}{\Omega_+\Omega_-}$$

$$= \int_{Sh_R} \frac{\theta^*(\Phi)(f)}{\Omega_+\Omega_-} d\mu = \frac{\Omega^D}{\Omega_+\Omega_-}(\mathcal{F},\mathcal{F}) = \frac{\Omega^D}{\Omega_+\Omega_-}|C^D(\lambda; \mathcal{W})|$$

up to $\mathcal{W}$ units. We conclude from p-adic limit $\Phi \doteqdot \lim_{n\to\infty} \phi'_{p^n}$

Period Theorem. $\Omega^D = \Omega_+\Omega_-$ *up to $\mathcal{W}$-units.*

§**11. Adjoint Selmer class number formula.** We have

$$|\mathrm{Sel}(Ad(\rho_\lambda))| \doteqdot \mathfrak{m}_1 \frac{L(1, Ad(\rho))}{2\pi^3\Omega_+\Omega_-(1-p^{-2})} = \sum_{h\in S} e_h^{-1} \frac{\theta^*(f)(h,h)}{\Omega_+\Omega_-}.$$

The above formula is an adjoint generalization of the mass formula of Siegel:

$$\mathfrak{m}_1 \frac{\zeta(2)}{\pi^2} = \sum_{h\in S} e_h^{-1},$$

and also an obvious generalization of the Dirichlet class number formula for an imaginary quadratic field $K := \mathbb{Q}[\sqrt{-d}]$ with discriminant $-d < 0$:

$$\frac{\sqrt{d}\cdot L(1, \left(\frac{-d}{}\right))}{2\pi} = \frac{h(-d)}{e} = \sum_{\mathfrak{a}\in Cl_K} e^{-1},$$

where Cl_K is the class group of K, $h(-d) := |Cl_K|$ and e is the number of roots of unity in K.