Adjoint L-value and the Tate conjecture

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Lecture no.1 at NCTS, March 19, 2024

Lecture 0: Strategy. We describe the Tate conjecture for varieties over number fields and its back ground. Then we sketch the strategy to prove the conjecture for a good amount of quaternionic Shimura varieties. A key point is a twisted adjoint L-value formula relative to each quaternion algebra $D_{/F}$ for a totally real field F and its scalar extension $B = D \otimes_F E$ for a totally real quadratic extension $E_{/F}$. The theta base-change lift $\mathcal{F}$ of a Hilbert modular form f to $B^\times$ has period integral over the Shimura subvariety $Sh_D \subset Sh_B$ given by $L(1, Ad(f) \otimes \left(\frac{E/F}{}\right)) \neq 0$; so, Sh_D gives rise to a non-trivial Tate cycle in $H^{2r}(Sh_B, \mathbb{Q}_l(r))$ for $r = \dim Sh_D = \dim Sh_B/2$. All quoted section numbers without specification are from the book manuscript.

§**0. Algebraic Cycles.** Let S be a projective variety of dimension r defined over a number field $K \subset \mathbb{C}$. Let $T \subset S$ be a closed subvariety of codimension e; so, $T(\mathbb{C})$ is of real dimension $2r - 2e$. Regard $S(\mathbb{C})$ as a complex manifold of dimension r; so, it is a real manifold of dimension $2r$. For a closed smooth differential form ω of degree $2r-2e$, the integration $\omega \mapsto \int_T \omega$ induces a linear form $[T] : H^{2r-2e}_{DR}(S, \mathbb{C}) \to \mathbb{C}$ sending the de Rham cohomology class $[\omega]$ to its integral $\int_T \omega$. By Poincaré duality: $H^{2r-2e}_{DR}(S(\mathbb{C}), \mathbb{C}) \times H^{2e}_{DR}(S(\mathbb{C}), \mathbb{C}) \to \mathbb{C}$, we view $[T] = [T]_\infty \in H^{2e}_{DR}(S(\mathbb{C}), \mathbb{C})$. This class $[T]$ is called the algebraic cycle associated to T. Write $\mathrm{Alg}^e(S, A)$ for the A-span inside $H^{2e}_{DR}(S(\mathbb{C}), \mathbb{C})$ of $[T]$ for T running over all closed subvarieties of $S_{/\mathbb{C}}$ of codimension e. Here A is a subring of $\mathbb{C}$

§**1. Hodge Cycles.** An obvious question is:

Is there any good characterization of $\mathrm{Alg}^e(S,\mathbb{Q})$*?*

Define $H^{e,e}(S(\mathbb{C}),\mathbb{C})$ to be the degree (e,e) subspace of the Hodge decomposition $H^{2e}(S(\mathbb{C}),\mathbb{C}) = \bigoplus_{i+j=2e} H^{i,j}(S(\mathbb{C}),\mathbb{C})$. The Hodge conjecture [D06] tells us

$$\text{(H)} \qquad \mathrm{Alg}^e(S,\mathbb{Q}) \stackrel{?}{=} \mathrm{Hdg}^e(S) := H^{e,e}(S(\mathbb{C}),\mathbb{C}) \cap H^{2e}(S(\mathbb{C}),\mathbb{Q}).$$

The Hodge conjecture (H) is known for divisors by Lefschetz [PAG, 1.2] and some small number of cases. Of course, one can further ask to find an explicit set of subvarieties which span $\mathrm{Hdg}^e(S)$.

§2. **Étale Poincaré duality.** Fix an algebraic closure $\overline{K} \subset \mathbb{C}$ and put $\overline{S} = S \otimes_K \overline{K}$. Let μ_N (resp. $\mathbb{Z}/N\mathbb{Z}$) be the sheaf of N-th roots of unity (resp. the constant sheaf of $\mathbb{Z}/N\mathbb{Z}$) on S; so, $\mu_N = \mathrm{Spec}_S(\mathcal{O}_S[t]/(t^N - 1))$. Then the pairing $\mu_N \times \mathbb{Z}/N\mathbb{Z} \ni (\zeta, a) \mapsto \zeta^a \in \mu_N$ induces $\mathrm{End}_{S\text{-gp}}(\mu_N) \cong \mathbb{Z}/N\mathbb{Z}$ canonically; i.e., the pairing is perfect. Let $\mathbb{Z}/N\mathbb{Z}(e) = \mathrm{Hom}(\overbrace{\mu_N \otimes \cdots \otimes \mu_N}^{e}, \mathbb{Z}/N\mathbb{Z})$ (the Pontryagin dual). By cup product, writing $H^{2e}(?) := H^{2e}_{et}(\overline{S}, ?)$, we have a perfect pairing

$$H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r)) \times H^0_{et}(\mathbb{Z}/N\mathbb{Z}) \to H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r)).$$

By the comparison isomorphism between étale cohomology and Betti cohomology, $H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r))$ is free of rank 1 over $\mathbb{Z}/N\mathbb{Z}$, and the paring is perfect, inducing a canonical isomorphism $\mathrm{Tr} : H^{2r}_{et}(\mathbb{Z}/N\mathbb{Z}(r)) \cong \mathbb{Z}/N\mathbb{Z}$. By an abstract non-sence, this perfect duality extends to

$$H^{2e}_{et}(\mathbb{Z}/N\mathbb{Z}(e)) \times H^{2r-2e}_{et}(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}/N\mathbb{Z} \quad (0 \le e \le r).$$

§3. **Tate conjecture.** The immersion $i : T \hookrightarrow S$ induces $i^* : H^{2r-2e}(\mathbb{Z}/N\mathbb{Z}(r-e)) \to H^{2r-2e}(\overline{T}, \mathbb{Z}/N\mathbb{Z}(r-e)) \cong \mathbb{Z}/N\mathbb{Z}$. By Poincaré duality, we have an inclusion $i_* : \mathbb{Z}/N\mathbb{Z} \hookrightarrow H^{2e}(\mathbb{Z}/N\mathbb{Z}(e))$. So we have the fundamental class $[T] = [T]_N = i_*(1)$. Taking $N = l^n$ for a prime l and passing to the limit, $[T]_{et} = \varprojlim_n [T]_{l^n} \in H^{2e}_{et}(\mathbb{Z}_l(e)) = \varprojlim_n H^{2e}(\mathbb{Z}/l^n\mathbb{Z}(e))$. Extending scalars to $\overline{\mathbb{Q}}_l$ and identifying $\overline{\mathbb{Q}}_l$ with $\mathbb{C}$, by the comparison isomorphism, $H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) \ni [T]_{et} \leftrightarrow [T]_\infty \in H^{2e}(\mathbb{C})$. Thus we may regard $\mathrm{Alg}^e_K(S;\mathbb{Q}) \subset H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))$. We define $\mathrm{Tate}^e_K(S) = \overline{\mathbb{Q}}_l \cdot \mathrm{Alg}^e_K(S;\mathbb{Q})$ (the $\overline{\mathbb{Q}}_l$-span of $\mathrm{Alg}^e_K(S;\mathbb{Q})$ inside $H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))$. The Tate conjecture states,

$$\text{(T)} \qquad \mathrm{Tate}^e(S) \overset{?}{=} H^0(\overline{K}/K, H^{2e}_{et}(\overline{\mathbb{Q}}_l(e))).$$

Here $H^0(\overline{K}/K, ?) = H^0(\mathrm{Gal}(\overline{K}/K), ?)$ for a Galois module ?. Replacing $\mathrm{Tate}^e_K(S)$ by $\mathbb{Q}_l \cdot \mathrm{Alg}^e_K(S)$, we ask the same question for $\mathbb{Q}_l$ in place of $\overline{\mathbb{Q}}_l$.

§4. Known cases.

1. Divisors on abelian varieties, $K3$ surfaces and products of two curves (Faltings) [Ta94b, §5],
2. Hilbert modular surfaces by Harder–Langlands–Rapoport [HLR], Murty–Ramakrishnan [MR87] and Klingenberg [K87] (increasingly general cases),
3. non-CM motives on Hilbert modular fourfolds [R04] when the base totally real field is a Galois extension of $\mathbb{Q}$,
4. Picard surfaces by Blasius and Rogawski [BR92],
5. General Hilbert modular motives under some restrictive conditions by Getz and Hahn [GH14],
6. Good motives occurring on a product of a Hilbert-Siegel variety and a Hilbert modular variety by Sweeting [S22].

The results of [HLR], [GH14] and [S22] are explicit in the sense that Shimura subvarieties span $\mathrm{Tate}^1_K(S)$, and in all the other cases, we do not know the generators explicitly.

§**5. Explicit analog for Shimura varieties.** If S is a Shimura variety associated to a reductive group G defined over the reflex field $\mathcal{E}$, the finite adele $G(\mathbb{A}^{(\infty)})$ acts on S which induces an action of $G(\mathbb{A}^{(\infty)})$ on $H^{2e}_{et}(\overline{\mathbb{Q}}_l)$ commuting with the Galois action. We can then decompose

$$H^{2e}_{et}(\overline{\mathbb{Q}}_l) = \bigoplus_{\pi} \Pi_\pi \otimes \pi^{(\infty)}$$

for automorphic representations π of $G(\mathbb{A})$ with its finite part $\pi^{(\infty)}$ and associated l-adic Galois representation Π_π. Then

$$H^0(X, H^{2e}_{et}(\overline{\mathbb{Q}}_l(e)) = \bigoplus_{\pi} H^0(X, \Pi_\pi(e)) \otimes \pi^{(\infty)}$$

for $\Pi_\pi(e) = \Pi_\pi \otimes_{\mathbb{Q}_l} \mathbb{Q}_l(e)$, where $H^0(X, ?) = H^0(\overline{X}/X, ?)$ for a finite extension $X_{/\mathcal{E}}$ with algebraic closure $\overline{X}$. As Langlands suggested, we may describe well Π_π from the automorphic data; so, we might be able to compute the space $H^0(X, \Pi_\pi(e))$ of Tate cycles, or at least, we ask how much of it is spanned by Shimura subvarieties $[S_H]$ for reductive subgroups $H \subset G$.

§**6. Quaternionic Shimura varieties.** In this series of lectures, we try to answer the questions in §5 for G given by G^B for a quaternion algebra B over the totally real field E, as we have an explicit description of Π_π as a tensor induction of 2-dimensional compatible system ρ_π. Our method is the use of the theta correspondence exploiting the following facts:

1. An appropriate theta lift $\theta^*(\phi)(f)$ of a Hilbert modular form f gives the Doi–Naganuma lift of f to a quaternionic automorphic form over a quadratic extension $E_{/F}$ (see Section 4.7);
2. The integral over the Shimura subvariety Sh_D of appropriate dimension in the quaternionic Shimura variety Sh_B is equal to a nonzero twisted adjoint L-value of f (see Chapter 5);
3. The theta descent $\theta_*(\phi)(\mathcal{F})$ of each quaternionic automorphic form $\mathcal{F}$ has Hilbert modular Fourier expansion whose coefficients are the integral of $\mathcal{F}$ over appropriate Shimura subvarieties Sh_α.

§7. **Twisted 4-dimensional quadratic spaces.** In the earlier lectures, we assume $F = \mathbb{Q}$. Let $D_{/\mathbb{Q}}$ be a quaternion algebra with discriminant ∂. Choose a semi-simple quadratic extension $E = \mathbb{Q}[\sqrt{\Delta}]_{/\mathbb{Q}}$ including $E = \mathbb{Q} \times \mathbb{Q}$ with $\sqrt{\Delta} = (1, -1)$, and let $B := D \otimes_{\mathbb{Q}} E$. Write simply $\delta = \sqrt{\Delta}$ with square-free $\Delta \in \mathbb{Z}$. Let $\langle \sigma \rangle = \mathrm{Gal}(E/\mathbb{Q})$ act on B through the factor E. Then $D_\sigma := \{v \in B | v^\sigma = v^\iota\}$ for $v + v^\iota = \mathrm{Tr}(v)$ is a 4-dimensional quadratic space with a quadratic $\mathbb{Q}$-form induced by the reduced norm $N : B \to E$. We have an orthogonal decomposition $D_\sigma = Z \oplus D_0$ where $Z = E \cap D_\sigma = (\mathbb{Q}, z^2)$ and

$$D_0 := \{v \in D_\sigma | \mathrm{Tr}(v) = 0\} = \{\sqrt{\Delta} w | w \in D, \mathrm{Tr}(w) = 0\}.$$

When $E = \mathbb{Q} \times \mathbb{Q}$, $\sigma(x, y) = (y, x)$ and $D_\sigma = \{(x, x^\iota) | x \in D\} \cong D$ by $(x, x^\iota) \leftrightarrow x$.

Let R be a Eichler order of level N of B; so, $N = \partial N_0$ with N_0 prime to ∂, maximal outside ∂ and $R/N_0R \cong \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$. Let $\widehat{R} = \varprojlim_N R/NR$ (the profinite completion).

$\S$**8. Quaternion subalgebras of B.** For each $\alpha \in D_\sigma \cap B^\times$, define the α-twist σ_α of σ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma_\alpha}$. Then σ_α is another action of $\mathrm{Gal}(E/\mathbb{Q})$ on B, and $D_\alpha = H^0(E/F, B)$ under this twisted action is a quaternion subalgebra of B.

- All quaternion $\mathbb{Q}$-subalgebras of B are realized as D_α for some $\alpha \in D_\sigma$, and $D_z = D \Leftrightarrow z \in Z$;
- $\alpha = \xi^{-1}\beta\xi^{-\iota\sigma}$ for $\xi \in B^\times \Leftrightarrow D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\alpha \cong D_\beta$ by an inner automorphism of B if $N(\alpha) = N(\beta)$ and $D_{E_\infty} \cong M_2(E_\infty)$ (strong approximation);
- The even Clifford group G_α of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \mathrm{Tr}(v) = 0\}$ is $D_\alpha^\times$ and $B^\times$ is a covering of the similitude group GO_{D_σ} of D_σ.

Let $\widehat{\Gamma}_\phi = \{h \in D^\times_{E_{\mathbb{A}}^{(\infty)}} | \phi^{(\infty)}(h^{-1}vh^{-\iota\sigma}) = \phi^{(\infty)}(v), \forall v \in D_{\sigma,\mathbb{A}^{(\infty)}}\}$ for each Schwartz-Bruhat function ϕ on $D_{\sigma,\mathbb{A}^{(\infty)}}$.

Let $Sh_B = B^\times \backslash D^\times_{E_{\mathbb{A}}} / E^\times_{\mathbb{A}} \widehat{\Gamma}_\phi C_\infty$ be the Shimura variety for $B^\times$ of level $\widehat{\Gamma}_\phi$, and Sh_α be the image of $D^\times_{\alpha,\mathbb{A}}$ in Sh_B for $\alpha \in D_\sigma$. Let $d = \dim_{\mathbb{R}} Sh_\alpha \in \{0, 2\}$. Regard $Sh_\alpha \in H_d(Sh_B, \mathbb{Z})$ and write $(\cdot,\cdot) : H^d \times H_d \to \mathbb{C}$ for the Poincaré duality.

§**9. Two formulas.** Write $f \mapsto \theta^*(\phi)(f) = \int_{Y_0(N)} \theta(\phi)(\tau, g) f(\tau) d\mu_\tau$ for the theta lift of $f \in S_2^{new}(\Gamma_0(N))$ to G and $\theta_*(\phi)(\mathcal{F}) \in S_2(\Gamma_0(N))$ for the theta descent of a cuspidal harmonic differential form $\mathcal{F}$ on Sh_B of degree d. The lift $\theta^*(\phi)(f)$ is such a differential form of matching degree d' or d with $d' + d = \dim_{\mathbb{R}} Sh_B$.

Theorem A: $\theta_*(\Phi)(\mathcal{F}) = * \sum_{0<n\in\mathbb{Q}}^{\infty} \sum_{\alpha, N(\alpha)=n} \Phi(\alpha)(\mathcal{F}, Sh_\alpha) q^n$ *for* $q = \exp(2\pi i\tau)$, *where* $* = (8\sqrt{-1})^{-1}$ *if* $E_{\mathbb{R}} := E \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R} \times \mathbb{R}$ *and* $* = 4$ *if* $E_{\mathbb{R}} = \mathbb{C}$. *If* $E_{\mathbb{R}} = \mathbb{C}$, *only* α *with* $\dim Sh_\alpha = 1$ *appears.*

Note $(\mathcal{F}, Sh_\alpha) = \int_{Sh_\alpha} \mathcal{F}$. The Tate conjectures only concern D and E with $D_{\mathbb{R}} = M_2(\mathbb{R})$ and $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$.

Theorem B: $i^{-3} E \mathfrak{m}_1 \frac{L(1, Ad(\rho_f) \otimes \chi_E)}{2^2 \pi^3} = (\theta^*(\phi')(f), Sh_1)$ *for an explicit constant* $E \neq 0$. *Here choosing a Haar measure* $d\mu_1$ *on* $D_{\mathbb{A}}^\times$ *so that it is Dirac on* $D^\times$ *and having volume* 1 *on* $\widehat{\Gamma}_{\phi'} \cap D_{\mathbb{A}}^\times C_\infty$, $\mathfrak{m} = \mathfrak{m}_1 \zeta(2)/\pi$ *is defined by* $d\mu_1 = (\mathfrak{m}/2) d\omega$ *for the Tamagawa measure.* Note $0 < \mathfrak{m}_1 \in \mathbb{Q}^\times$ by Siegel.

§10. Strategy.

1. We prove that $f \mapsto \theta^*(\phi)(f)$ is Hecke equivariant for $T(p)$ with p splitting in $E/\mathbb{Q}$ if we choose a Schwartz–Bruhat function $\phi : D_{\sigma,\mathbb{A}} \to \mathbb{C}$ well.

2. By Chebotarev density, Item 1 is sufficient to see $\mathcal{F} = \theta^*(\phi)(f)$ is a Hecke eigenform for $T(\mathfrak{p})$ for all prime $\mathfrak{p}$ of E giving the Doi–Naganuma lift. Non-vanishing $\mathcal{F} \neq 0$ comes out from Theorem A and Rallis' work (a bit over-simplified).

3. Analyze the Galois representation Π_π for the automorphic representation π generated by a general Hecke eigenform $\mathcal{F}$ on Sh_B to see that $\dim \mathrm{Tate}^1_K[\pi^{(\infty)}]$ has dimension 1 only when $\mathcal{F} = \theta^*(\phi)(f)$ for some f.

4. By Theorem B, if $(Sh_\alpha, \mathcal{F}) = 0$ for all $\alpha \in D_\sigma \cap B^\times$, $\theta_*(\phi)(\mathcal{F}) = 0$, contradicting to $\theta^*(\phi)(f) \neq 0$. This proves $\sum_\alpha \overline{\mathbb{Q}}_l[Sh_\alpha] \supset \dim \mathrm{Tate}^1_K[\pi^{(\infty)}]$, proving the conjecture.

§**11. Further generalizations.** As we already mentioned, Sweeting [S22] applied theta correspondence to show the Tate conjecture to some extent for a product of a Hilbert-Siegel modular variety of genus 2 and a Hilbert modular variety.

The use of the theta correspondence for this type of problems seems quite effective. The case of orthogonal Shimura variety associated to quadratic forms of signature $(2n, 2)$ or $(2n, 0)$ at different infinite places of a totally real field F can be treated to some extent; at least, an analog of the adjoint L-value formula is expected in these cases.