ADJOINT L-VALUE AS A PERIOD INTEGRAL AND THE MASS FORMULA OF SIEGEL–SHIMURA

HARUZO HIDA

Abstract. Let F be an elliptic Hecke eigen cusp form. For a quaternion algebra $D_{/\mathbb{Q}}$ and a semi-simple quadratic extension $E=\mathbb{Q}[\sqrt{\Delta}]_{/\mathbb{Q}}$ (including $E=\mathbb{Q}\times\mathbb{Q}$), we determine the period of the Doi-Naganuma theta lift $\theta^*(F)$ of F to the algebraic group $(D\otimes_{\mathbb{Q}}E)^\times$ over Shimura subvarieties associated to D, as the product of $L(1,Ad(\rho_F)\otimes\left(\frac{\Delta}{}\right))$ and the mass factor $\mathfrak{m}_1$ of Siegel–Shimura for a ternary quadratic form. Here ρ_F is the compatible system of Galois representations associated to F, and $Ad(\rho_F)$ acts on the Lie algebra $\mathfrak{sl}(2)$ via conjugation by ρ_F.

To state a simplest example of the adjoint L-value formula for a definite quaternion algebra $D_{/\mathbb{Q}}$ ramified at one prime p, set $E=\mathbb{Q}\times\mathbb{Q}$ and let $F\in S_2(\Gamma_0(p))$ be a Hecke eigenform normalized so that its Fourier coefficients in $\exp(2\pi\sqrt{-1}\tau)$ ($\tau\in\mathfrak{H}:=\{z\in\mathbb{C}|\operatorname{Im}(z)>0\}$) is equal to 1. Defining λ by $F|T(n)=\lambda(T(n))F$ for Hecke operators $T(n)$, λ gives rise to an algebra homomorphism of the Hecke algebra for $S_2(\Gamma_0(p))$ to the integer ring A of the Hecke field of F. Inverting finite number of primes, we may assume that A is a principal ideal domain; so, we have the canonical period $\Omega_\pm=\Omega_\pm(F,\lambda;A)\in\mathbb{C}^\times$ (up to units in A) as in [EMI, §9.2.4]. Of course, if F is associated with a rational elliptic curve, we take $A:=\mathbb{Z}$ and $\Omega_\pm$ is determined up to a sign. Take a definite quaternion algebra D ramified only at a prime p with a maximal order $R\subset D$. Choose a finite subset $Sh=Sh_1\subset D_{\mathbb{A}}^\times$ such that $Sh\cong D^\times\backslash D_{\mathbb{A}}^\times/\widehat{R}^\times D_\infty^\times$ for the profinite completion $\widehat{R}$ of R, and set $e_h=h\widehat{R}^\times h^{-1}\cap D^\times$ for $h\in Sh$. The following formula is a special case of Corollary 5.3 (with an explicit value of $\mathfrak{m}_1$ computed in Remark 4.3):

Theorem 0.1. *Let ρ_F be the compatible system of Galois representations associated to F and $Ad(\rho_F)$ act on the Lie algebra $\mathfrak{sl}(2)$ via conjugation by ρ_F. For the canonical period $\Omega_\pm$ of F and the mass factor $\mathfrak{m}_1=\frac{p-1}{2}$ of Siegel,*

$$(1-p^{-2})^{-1}\mathfrak{m}_1\frac{L(1,Ad(\rho_F))}{2\pi^3\Omega_+\Omega_-}=\sum_{h\in Sh}e_h^{-1}\frac{\theta^*(F)(h,h)}{\Omega_+\Omega_-}.$$

Writing D_0 for the quadratic space $\{v\in D|\operatorname{Tr}(v)=0\}$, the mass factor $\mathfrak{m}_1$ above for the lattice $R\cap D_0$ is computed by Siegel. The automorphic form $\theta^*(F)$ on $D^\times\backslash D_{\mathbb{A}}^\times\times D^\times\backslash D_{\mathbb{A}}^\times$ is the base change theta lift of F to $D^\times\times D^\times$ which is a function of $(h,g)\in D_{\mathbb{A}}^\times\times D_{\mathbb{A}}^\times$. The above formula is an adjoint generalization of the mass formula of Siegel:

$$\mathfrak{m}=\mathfrak{m}_1\frac{\zeta(2)}{\pi^2}=\sum_{h\in Sh}e_h^{-1},$$

and also an obvious generalization of the Dirichlet class number formula. We will prove this type of an explicit formula of $L(1,Ad(\rho_F)\otimes\chi_E)$ for any Hecke eigen new form F of appropriate level with quadratic twist by the character χ_E of an arbitrary semi-simple quadratic extension $E/\mathbb{Q}$ and an arbitrary division quaternion algebra $D_{/\mathbb{Q}}$ (the case where $D=M_2(\mathbb{Q})$ has been treated in [H99] in a different manner). The exact formulas are in Theorem 4.7 when D is indefinite and E is real, Theorem 5.2 when D is definite and E is real, and Theorem 7.1 when D is definite and E is imaginary and in Theorem 8.3 when D is indefinite and E is imaginary.

Date: April 18, 2023.
2020 *Mathematics Subject Classification.* primary 11F67, 11G18, 11F27; secondary 11F75, 11E20.
Key words and phrases. Adjoint L-value, mass formula, congruence number.

We apply the seesaw identity [K84, (1.9)] to the pairs $(\mathrm{SO}_V, \mathrm{SL}(2))$ and $(\mathrm{SO}_{W_0} \times \mathrm{SO}_Z, \mathrm{Mp}(2) \times \mathrm{Mp}(2))$ for an orthogonal decomposition $V = Z \oplus W_0$ of a quadratic space V. The seesaw identity (R) below itself is tautological, but an important point is an idea of Waldspurger to convert it into a Rankin–Selberg convolution via the Siegel–Weil formula taking a constant as an automorphic form on SO_{W_0} and a character on SO_Z. If $\dim Z = 1$, the Siegel-Weil formula does not have much effect and produces the half-integral theta series of the character for Z, and thus the adjoint L-values appears as a period integral by Shimura's calculation of the Rankin product [Sh75].

More precisely, for an elliptic cusp form F, an idea of Waldspurger [W85] of computing the period of a theta lift of F to SO_V over an orthogonal Shimura subvariety $S_Z \times S_{W_0} \subset S_V$ is two-folds:

(S) Split $\theta(\phi)(\tau; h_Z, h_0) = \theta(\psi)(h_Z) \cdot \theta(\phi_0)(h_0)$ (the variable τ in the Poincaré half plane $\mathfrak{H}$, $h = (h_Z, h_0) \in \mathrm{O}_{W_0}(\mathbb{A}) \times \mathrm{O}_Z(\mathbb{A})$) for a decomposition $\phi = \psi \otimes \phi_0$ (here ψ and ϕ_0 are Schwartz–Bruhat functions on $Z_{\mathbb{A}}$ and $W_{0,\mathbb{A}}$, respectively);

(R) For the theta lift $\Theta(F)(h) = \int_X F(\tau)\theta(\phi)(\tau; h)d\mu_\tau$ with a modular curve X, the period P over the Shimura subvariety $S_Z \times S_0$ (S_Z for SO_Z and $S_0 = S_{W_0}$ for SO_{W_0}) is given by a trivial seesaw identity:

$$\int_{S_Z \times S_0} \int_X F(\tau)\theta(\phi)(\tau; h)d\mu_\tau d\mu_h = \int_X F(\tau) \left(\int_{S_Z} \theta(\psi)(\tau; h_Z)d\mu_h\right) \cdot \left(\int_{S_0} \theta(\phi_0)(\tau; h_0)d\mu_{h_0}\right) d\mu_\tau,$$

where $\tau = \xi + \eta\sqrt{-1}$ is the variable of the metaplectic side and $d\mu_\tau = \eta^{-2}d\xi d\eta$. Then invoke the Siegel–Weil formula to convert inner integrals into the Siegel-Weil Eisenstein series $E(\psi)$ and $E(\phi_0)$, reaching Rankin-Selberg integral

$$P = \int_X F(\tau)E(\psi)E(\phi_0)d\mu_\tau = L\text{–value}.$$

If $\dim Z = 1$, $E(\psi)$ is a half integral theta series as we already remarked, and by Shimura's calculation of the Rankin product, the adjoint L-values (twisted by a character) shows up independently of (the size of) V as long as V has even dimension ≥ 4. We hope to deal with the general V in future. Waldspurger [W85] applied in the early 1980s this scheme to $V = D$ for a quaternion algebra $D_{/\mathbb{Q}}$ with a splitting $V := D \cong E \oplus E$ for a quadratic field $E = \mathbb{Q}[\sqrt{\Delta}]$ over $\mathbb{Q}$ and expressed the period by the central critical value of the Hecke L-function $L(s, F \otimes \chi_E)$ for $\chi_E := \left(\frac{\Delta}{}\right)$.

In this paper, we apply the principles (S) and (R) to the simplest case of an arbitrary anisotropic 4-dimensional quadratic space over $\mathbb{Q}$ which produces the quaternionic Doi–Naganuma lift to the quadratic extension $E = \mathbb{Q}[\sqrt{\Delta}]$ of $\mathbb{Q}$ (including $\mathbb{Q} \times \mathbb{Q}$) as a theta left, and we compute the period in terms of the adjoint L-value $L(s, Ad(F) \otimes \chi_E) = L(s, Ad(\rho_F) \otimes \chi_E)$ at $s = 1$. We calculate fully the period integral without any ambiguous factors. To prove the integral period factorization predicted in [DHI, Conjectures 1.3 and 1.5] (not just rational one as conjectured, for example, in [Y95]) of the canonical period of the Doi-Naganuma lift into canonical elliptic period factors, an absolutely precise computation (as in Theorem 0.1) involving the size of the congruence module of F as the error factor of the period and the Petersson inner product. This point will be discussed in our subsequent papers once the results in this paper are generalized to Hilbert modular forms. Note that the Galois side of [DHI, Conjecture 1.5] is proven in [EMI, Theorem 8.3.7] and [H22, Theorems A and C] and that the exact computation of the mass by Shimura in [Sh99] for an arbitrary lattice in W_0 is useful in this integral point of view (though some mathematicians living purely in an abstract ghostly world may not like such a tedious computation).

Let D be a quaternion algebra over $\mathbb{Q}$, and we put $D_E = D \otimes_{\mathbb{Q}} E$ on which $1 \neq \sigma \in \mathrm{Gal}(E/\mathbb{Q})$ acts through the factor E. Then the Doi–Naganuma lift of elliptic cusp forms to $D_E^\times$ is realized as a theta lift with respect to the quadratic space over $\mathbb{Q}$:

$$D_\sigma = D_\sigma^\pm := \{v \in D_E | v^\sigma = \pm v^\iota\}$$

for the involution ι given by $v + v^\iota = \mathrm{Tr}_{D_E/E}(v)$. The quadratic form on $D_\sigma^\pm$ is induced by $v \mapsto vv^\sigma = \pm N(v)$ for the reduced norm $N : D_E \to E$ which has values in $\mathbb{Q}$ over $D_\sigma^\pm$. In the late 1970s to the early 1980s, the quadratic $\mathbb{Q}$-space D_σ for $D = M_2(\mathbb{Q})$ is studied to establish the Doi-Naganuma lift for E real in [K78] and for E imaginary in [A78] and [F83], though the investigation of base-change via theta lift has been somehow forgotten after the general theory of base-change lift by Langlands via trace formulas. Doi-Naganuma lift via theta correspondence is not just an abstract trace (or representation) identity but contains more arithmetic information as it computes the lift (ascent) and the descent explicitly. This paper is perhaps the first attempt of computing fully explicitly the period integral of the base-change over Shimura subvariety $D^\times\backslash D^\times_{\mathbb{A}}$ inside the ambient quaternionic automorphic variety $D^\times_E\backslash D^\times_{E_{\mathbb{A}}}$.

Let $D_0 = D_0^\pm := \{v \in D_\sigma^\pm | v^\iota = -v\}$ and $Z = Z^\pm := D_\sigma^\pm \cap E$ inside D_E. Then $\dim_{\mathbb{Q}} Z^\pm = 1$ and $\dim_{\mathbb{Q}} D_0^\pm = 3$, and we have a splitting $D_\sigma^\pm = Z^\pm \oplus D_0^\pm$ of quadratic spaces. Write $\mathbf{w} = \mathbf{w}_V$ for the Weil representation of the metaplectic cover $\pi_{\mathbb{A}} : \mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$ realized on Schwartz-Bruhat functions ϕ on $V_{\mathbb{A}}$ for $V = Z, D_0, D_\sigma$ and $\mathrm{O}_V(\mathbb{A})$ (resp. $\mathrm{SO}_V(\mathbb{A})$) for the (resp. special) orthogonal group of V acting from the right on $V_{\mathbb{A}}$. Though the metaplectic group Mp is not an algebraic group, we write $\mathrm{Mp}(A)$ for $A = \mathbb{A}, \mathbb{R}, \mathbb{Q}_v$ and so on as if it were an algebraic group. For all possible coefficients A, we write π_A for the projection $\pi_A : \mathrm{Mp}(A) \to \mathrm{SL}_2(A)$. Note that O_{D_σ} (resp. O_{D_0}) is close to $D_E^\times$ (resp. $D^\times$). We have the theta kernel $\theta_V(\phi)(g,h) = \sum_{\alpha\in V}(\mathbf{w}_V(g)\phi)(\alpha h)$ with $g \in \mathrm{Mp}(\mathbb{A})$ and $h \in \mathrm{O}_V(\mathbb{A})$. Since $Z^\pm$ is definite, we have theta series $\theta_Z(\psi) = \sum_{n\in\mathbb{Z}} \psi(n) n^k \mathbf{e}(n^2\tau)$ of Z with $\mathbf{e}(\tau) := \exp(2\pi\sqrt{-1}\tau)$ for the variable τ in the Poincaré half plane $\mathfrak{H}$ (on the metaplectic side) and a Dirichlet character ψ regarded as a Schwartz–Bruhat function on $Z_{\mathbb{A}}$. The Eichler order $R(N)$ (of level N) in D gives rise to a lattice $L \subset D_0^\pm$, and we consider a Schwartz–Bruhat function ϕ_0 on $D_{0,\mathbb{A}}$ with a good choice of infinity part and with a finite part given by a modified characteristic function of $\widehat{L}$. Then we apply Waldspurger's technique to $\theta_{D_\sigma}(\psi\otimes\phi_0) = \theta_Z(\psi)\theta_{D_0}(\phi_0)$. Since $Z^\pm$ is definite with negligible orthogonal group, we apply Siegel–Weil formula to $\theta_{D_0}(\phi_0)$ which produces an Eisenstein series of half integral weight $E(\phi_0)$. The formula for our use is not in the convergent range Weil studied and is proven for a division algebra D in Sweet's thesis [Sw90, §3.3] computed by the method of Kudla and Rallis [MSS, §5.3]. Because of this, we need to assume that D is a division algebra for the period formula. In any case, the formula (F) below is proven for $D = M_2(\mathbb{Q})$ in my earlier work [H99] (by a different method) which deals with $D = M_2(E_+)$ for a general number field E_+ and a quadratic extension $E_{/E_+}$.

The even Clifford group of D_σ (resp. D_0) is almost $D_E^\times$ (resp. $D^\times$); so, the period integral is over a Shimura subvariety associated to $D^\times$ inside the Shimura variety (or the automorphic manifold) of $D_E^\times$. Assume that F is a Hecke eigenform with an appropriate level determined by ϕ_0 and ψ with Neben character $\psi^{-1}\chi_E$, and write ρ_F for the compatible system of Galois representations of F. Define $L(s, Ad(F)) = L(s, Ad(\rho_F))$ for the adjoint representation $Ad(\rho_F)$ of ρ_F on $\mathfrak{sl}(2)$ on which the Galois group acts via conjugation by ρ_F. The final formula has the following form:

$$P = c \cdot \mathfrak{m}_1 \cdot L(1, Ad(F) \otimes \chi_E) \tag{F}$$

for an explicit (rather trivial) constant c involving some Euler factors and the mass factor $\mathfrak{m}_1$ described below. As is well known, the analytic continuation of $L(s, Ad(F))$ was first given by Shimura in [Sh75] as a Rankin product of $\theta_Z(\psi)$ and F over a modular curve, and therefore it is natural to have the adjoint L-value as the period over the Shimura subvariety associated to $D^\times$.

Here is a description of why we get the mass factor $0 < \mathfrak{m}_1 \in \mathbb{Q}$ in our formula. For the Riemann zeta function $\zeta(s)$, the mass factor is the ratio

$$\mathfrak{m}_1 := \pi^{\epsilon_D}\mathfrak{m}/\zeta(2) \tag{0.1}$$

for the mass $\mathfrak{m}$ of Siegel–Shimura [AQF, §37.1] for the quadratic space D_0 with respect to the lattice L for the level N of $\theta(\phi)$ (and F). Here $\epsilon_D = 1, 2$ accordingly to $D \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}), \mathbb{H}$. The Siegel–Weil formula by Weil [W65, no.52] is formulated with respect to the Tamagawa measure $d\omega$ for the orthogonal group O_{D_0} while the period is computed with respect to another canonical measure $d\mu_h$ with volume 1 over the image of $\widehat{R(N)}^\times$ in $\mathrm{O}_{D_0}(\mathbb{A}^{(\infty)})$. One can define the mass $\mathfrak{m}$ by the ratio $d\mu_h/\frac{1}{2}d\omega$ (we need to have $\frac{1}{2}d\omega$ as the Tamagawa number of O_{D_0} is equal to 2; see [MSS, (5.3.3)]). For ternary quadratic forms, apart from some initial (partial) results by Siegel, the exact form of the mass $\mathfrak{m}$ was not known until Shimura's determination for all quadratic spaces over a totally real field in [Sh99], and by his formula, $\mathfrak{m}$ in our case is a multiple of $\zeta(2)$ by a simple (but hard to determine) constant $\mathfrak{m}_1$, though, as Siegel discovered, $\mathfrak{m}$ is a product of local factors. Our formula resembles the Siegel–Shimura mass formula (with $L(1, Ad(F) \otimes \chi_E)$ in place of $\zeta(2)$), particularly when D is definite (see §5.4).

Shintani [S75] and Waldspurger [W81] had another idea of computing the Fourier expansion on the metaplectic side of the theta descent (the adjoint map of the theta lift) from O_{D_σ}, and the period of the form P is expected to show up as a Fourier coefficient of the Doi-Naganuma descent from O_{D_σ}. More precisely the coefficient in $\mathbf{e}(\pm N(\alpha)\tau)$ (or $\mathbf{e}(\pm N(\alpha)\overline{\tau})$) for $\alpha \in D_\sigma^\pm \cap D_E^\times$ with the variable $\tau \in \mathfrak{H}$ on the metaplectic side is the period with respect to an orthogonal group O_α depending on α times the adjoint mass factor $\mathfrak{m}_\alpha$ dependent on α (see Theorems 4.10, 5.9, 7.3 and 8.4), where $\mathfrak{m}_\alpha = \mathfrak{m}_1$ if α is a scalar in Z. The even Clifford group of O_α is the multiplicative group of a quaternion subalgebra $D_\alpha \subset D_E$ over $\mathbb{Q}$ dependent on α and $D_\alpha = D$ if $0 \neq \alpha \in Z$. When E is imaginary, those α appearing in the q-expansion must satisfy $D_\alpha \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, while if E is real, all quaternion subalgebras over $\mathbb{Q}$ of D_E appear, and therefore we expect that these periods depending on α have the same rationality (see Remark 4.11) as predicted by a conjecture of Tate (since Shimura curves of different quaternion algebras have motivic factors with identical Hasse–Weil L-functions by the Jacquet–Langlands correspondence). The computation of the Fourier expansion of the theta descent will be done for all quaternion algebra including $D = M_2(\mathbb{Q})$.

The result in this paper should be generalized to a general base field and even dimensional quadratic spaces V. The author plans to do all computations in near future, as he hopes to make progress in the integral period relations predicted in [DHI] and [H99] of the above type of periods, where in the latter paper, the computation of P was done for D given by the 2×2 matrix algebra over the base field E_+ with $E = E_+ \times E_+$ without recourse to the theta correspondence, and the result in [H99] has been used in the study of period relations in [TU22].

Contents

1. Theta correspondence

We describe theta correspondences between the metaplectic cover Mp of SL(2) and orthogonal groups O_V for quadratic spaces V over $\mathbb{Q}$. Write U for the upper unipotent subgroup of SL(2) and T for the diagonal torus of SL(2) with $B = TU \subset \mathrm{SL}(2)$ (the upper triangular Borel subgroup).

1.1. **Weil representation.** Let (V,Q) be a quadratic space over $\mathbb{Q}$ with $\dim V = m$. The quadratic form $V \ni x \mapsto Q(x) \in \mathbb{Q}$ produces a $\mathbb{Q}$-bilinear symmetric pairing $s(x,y) = Q(x+y) - Q(x) - Q(y)$. We choose a basis $v_1,\dots,v_m$ of V over $\mathbb{Q}$ and define $S = (s(v_i,v_j))_{i,j} \in M_m(\mathbb{Q})$. The discriminant $d(V) := \det(S)$ modulo $(\mathbb{Q}^\times)^2$ is well defined independent of the choice of the basis. We also denote this class modulo square as $d(V)$. In particular, $k_V := \mathbb{Q}[\sqrt{(-1)^{m(m-1)/2}\det(Q)}]$ is $\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$ independent of the choice of the basis. If $m = \dim V$ is even, we put χ_V for the character $\left(\frac{k_V/\mathbb{Q}}{}\right)$. If m is odd, χ_V is the "Neben" character of the standard theta series weight $\frac{m}{2}$ (we will specify it in §3.1 for a specific V). Define $\mathbf{e} = \mathbf{e}_\mathbb{A} : \mathbb{A}/\mathbb{Q} \to S^1$ be the standard additive character such that $\mathbf{e}_\mathbb{A}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{Q}_\infty = \mathbb{R}$.

Write $\mathbb{A}$ for the adele ring of $\mathbb{Q}$ and $\mathbb{A}^{(v)} = \mathbb{A} \cap \prod_{w\neq v}\mathbb{Q}_w$ for a place v of $\mathbb{Q}$. The group $\mathrm{SL}_2(k)$ for a field k is generated by $J := \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, diagonal matrices and upper unipotent matrices [PAF, Lemma 4.46]; so, the density of $\mathrm{SL}_2(\mathbb{Q}) \subset \mathrm{SL}_2(\mathbb{A}^{(v)})$ (removing one place v) by strong approximation tells us that $\mathrm{SL}_2(\mathbb{A}^{(v)})$ is topologically generated by these elements. By Iwasawa decomposition, $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}^{(\infty)}) \times \mathrm{SL}_2(\mathbb{R})$ is generated by $B(\mathbb{A})\mathrm{SO}_2(\mathbb{R})$ and J. Write $\mathcal{S}(V_X)$ for the space of Schwartz-Bruhat functions on $V_X = V \otimes_\mathbb{Q} X$ for $X = \mathbb{Q}_p, \mathbb{R}, \mathbb{A}, \mathbb{A}^{(\infty)}$.

Let $B \subset \mathrm{SL}(2)$ be the upper triangular Borel subgroup with its unipotent radical U and the diagonal torus T so that $B = T \ltimes U$. Weil defined in [W64, no.13] an action of $U(X)$, $T(X)$ and $J = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ on $\mathcal{S}(V_X)$ as follows:

$$\mathbf{r}\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)\phi(v) = \mathbf{e}_\mathbb{A}(Q(v)u)\phi(v),\ \mathbf{r}\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)\phi(v) = |a|_\mathbb{A}^{m/2}\phi(av) \text{ and } \mathbf{r}(J)\phi(v) = \widehat{\phi}(v), \tag{1.1}$$

where $\phi \mapsto \widehat{\phi}$ is the Fourier transform with respect to $\mathbf{e}_{\mathbb{A}}(s(x,y))$ normalized so that $\widehat{\widehat{\phi}}(x) = \phi(-x)$. By computation, $\mathbf{r}(-1)$ commutes with $\mathbf{r}(g)$ for all g as above, and $\mathbf{r}(-1)\phi(v) = \phi(-v)$.

The metaplectic group $\mathrm{Mp}(X)$ is defined to be the subgroup of the continuous $\mathbb{C}$-linear automorphism group $\mathrm{Aut}(\mathcal{S}(V_X))$ generated by $\mathbf{r}(g)$ for g as in (1.1). It appears that $\mathrm{Mp}(X)$ depends on the quadratic space V. As we will see later, there is a construction of $\mathrm{Mp}(X)$ independent of V; so, there exists a unique group $\mathrm{Mp}(X)$ acting on $\mathcal{S}(V_X)$ for all V. However the action $\mathbf{r}$ depends on V; so, if necessary, we write $\mathbf{r}_V$ to indicate dependence. For example, $\mathbf{r}_V \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) = \left(\frac{\Delta}{a}\right) \mathbf{r}_{V'} \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ for the quadratic character $\left(\frac{\Delta}{}\right) : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ if $X = \mathbb{A}$, $V = (\mathbb{Q}, x^2)$ and $V' = (\mathbb{Q}, \Delta x^2)$ with $\Delta \in \mathbb{Q}^\times$ (Neben type change as in [Sh73, Proposition 1.3]). Since it is known that $\mathbf{r}(gg') = \mathbf{r}(g)\mathbf{r}(g')$ up to scalars $z \in \mathbb{C}$ with $|z| = 1$, we have a canonical exact sequence of locally compact groups:

$$1 \to S^1 \to \mathrm{Mp}(X) \xrightarrow{\pi_X} \mathrm{SL}_2(X) \to 1,$$

where $S^1 = \{z \in \mathbb{C}^\times | z\overline{z} = 1\}$ acts on ϕ by scalar multiplication $\phi \mapsto z\phi$. The center of $\mathrm{Mp}(X)$ is isomorphic to $S^1 \times \{\pm 1\}$ generated by S^1 and $\mathbf{r}(-1)$ ($(z, -1) \in S^1 \times \{\pm 1\}$ acts as $\phi(v) \mapsto z\phi(-v)$).

Let $\Omega = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{A}) | c \in \mathbb{A}^\times\}$. Then, we have $\mathbf{r}(b)\mathbf{r}(\sigma)\mathbf{r}(b') = \mathbf{r}(v\sigma b')$ for $b, b' \in B(X)$ and $\sigma \in \Omega$ [W64, no.32]. Weil made a canonical section $\mathbf{r}_{\mathbb{Q}} : \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{Mp}(\mathbb{A})$ which coincides with $\mathbf{r}$ on Ω [W64, no.40]. We identify $\mathrm{SL}_2(\mathbb{Q})$ with the image of $\mathbf{r}_{\mathbb{Q}}$ in $\mathrm{Mp}(\mathbb{A})$. The action of $\mathrm{Mp}(X)$ on $\mathcal{S}(V_X)$ gives rise to the Weil representation $\mathbf{w} = \mathbf{w}_V$ of the metaplectic group $\mathrm{Mp}(X)$ into $\mathrm{Aut}(\mathcal{S}(V_X))$.

Let O_V be the orthogonal group for V over $\mathbb{Q}$ and GO_V be its similitude group (acting from the right on V); so,

$$\mathrm{GO}_V(A) = \left\{\alpha \in GL(V \otimes_{\mathbb{Q}} A) | Q(x\alpha) = \nu_V(\alpha)Q(x) \quad \text{with } \nu_V(\alpha) \in A^\times\right\}$$

and $\mathrm{SO}_V = \mathrm{SL}(V) \cap \mathrm{O}_V$. We let $g \in \mathrm{GO}_V(\mathbb{A})$ acts on $\mathcal{S}(V_{\mathbb{A}})$ by

$$L(g)\phi(v) = |\nu_V(g)|_{\mathbb{A}}^{-m/2}\phi(vg).$$

1.2. **Siegel–Weil theta series.** The action $\mathbf{w}$ and L commutes on $\mathrm{Mp}(\mathbb{A}) \times \mathrm{O}_V(\mathbb{A})$; so, we may regard $\mathbf{w} \otimes L$ as a representation of $\mathrm{Mp}(\mathbb{A}) \times \mathrm{O}_V(\mathbb{A})$. The following result is [W64, Théorème 6].

Theorem 1.1. *The generalized theta series of Siegel–Weil*

$$\boldsymbol{\theta}(\phi)(x; g) = \sum_{v \in V} (\mathbf{w}(x)L(g)\phi)(v) \quad (\textit{for each } \phi \in \mathcal{S}(V_{\mathbb{A}}))$$

gives an automorphic form defined as a function on $(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})) \times (\mathrm{O}_V(\mathbb{Q})\backslash\mathrm{O}_V(\mathbb{A}))$.

1.3. **Explicit form of metaplectic groups.** The extension $S^1 \hookrightarrow \mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$ actually descends down to $\mu_2 \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$. In other words, the 2-cocycle: $\mathrm{SL}_2(\mathbb{A}) \to S^1$ giving rise to the extension $\mathrm{Mp}(\mathbb{A})$ is cohomologous to another one $\kappa : \mathrm{SL}_2(\mathbb{A}) \to \mu_2$ with values in μ_2 (as we will describe it later in this subsection), and we have the following commutative diagram:

$$\begin{array}{ccccc} \mu_2 & \xrightarrow{\hookrightarrow} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) & \xrightarrow{\twoheadrightarrow} & \mathrm{SL}_2(\mathbb{A}) \\ \cap\downarrow & & \cap\downarrow & & \downarrow\cap \\ S^1 & \xrightarrow[\hookrightarrow]{} & \mathrm{Mp}(\mathbb{A}) & \xrightarrow[\twoheadrightarrow]{} & \mathrm{SL}_2(\mathbb{A}). \end{array} \tag{1.2}$$

The representation $\mathbf{w}$ descends (non-canonically) to $\mathrm{SL}_2(\mathbb{A})$ if $m = \dim V$ is even, and in this case, we can replace $\widetilde{\mathrm{SL}}_2$ by SL_2 choosing a descent.

For integers $a, b \neq 0$, we define Shimura's symbol $\left(\frac{a}{b}\right)$ in [Sh73, page 442] by

(1) $\left(\frac{a}{b}\right) = 0$ if $(a, b) \neq 1$ (where (a, b) is the GCD of a and b),
(2) If b is an odd prime, $\left(\frac{a}{b}\right)$ is the Legendre symbol (i.e., it is less one than the number of solutions of $x^2 \equiv a \mod b$),
(3) If $b > 0$, $a \mapsto \left(\frac{a}{b}\right)$ is a character modulo b,
(4) If $a \neq 0$, $b \mapsto \left(\frac{a}{b}\right)$ is a character modulo $4a$ whose conductor is the conductor of $\mathbb{Q}[\sqrt{a}]_{/\mathbb{Q}}$,
(5) $\left(\frac{a}{-1}\right) = 1$ or -1 according as $a > 0$ or $a < 0$,
(6) $\left(\frac{0}{\pm 1}\right) = 1$.

Consider $\theta : \mathfrak{H} \to \mathbb{C}$ given by $\theta(\tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}(n^2\tau)$, where $\mathfrak{H} = \mathfrak{H}^+$ and

$$\mathfrak{H}^{\pm} := \{z \in \mathbb{C} | \pm \mathrm{Im}(z) > 0\}, \tag{1.3}$$

which are isomorphic to $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ by $g \mapsto g(\pm\sqrt{-1})$ respectively.

Define for $\gamma \in \Gamma_0(4)$, $h(\gamma, \tau) := \theta(\gamma(\tau))/\theta(\tau)$. Then by [Sh73, (1.10)]

$$h(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau)^{1/2} \ \text{ with } \ j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d, \tag{1.4}$$

where $z^{1/2} = \sqrt{|z|}\exp(\pi i\theta)$ if $z = |z|\mathbf{e}(\theta)$ with $-\pi < \theta \le \pi$ and $\varepsilon_d = \sqrt{-1}$ or 1 according as $d \equiv 3$ or $1 \mod 4$.

We can realize $\widetilde{\mathrm{SL}}_2(\mathbb{R}) = \{(g, J(g, \tau)) | g \in \mathrm{SL}_2(\mathbb{R}), J(g, \tau)^2 = j(g, \tau)\}$ with multiplication given by $(g, J(g, \tau))(h, J(h, \tau)) = (gh, J(g, h(\tau))J(h, \tau))$. Here $\tau \mapsto J(g, \tau)$ is supposed to be a holomorphic function on $\mathfrak{H}$. Thus we have the central extension $\mu_2 \overset{i}{\hookrightarrow} \widetilde{\mathrm{SL}}_2(\mathbb{R}) \overset{\pi}{\twoheadrightarrow} \mathrm{SL}_2(\mathbb{R})$ with $i(-1) = (1_2, -1)$ and $\pi(g, J) = g$. The center of $\widetilde{\mathrm{SL}}_2$ is given by $\mu_2 \times \mu_2$.

Define a set-theoretic section $s : \mathrm{SL}_2(\mathbb{R}) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R})$ by $g \mapsto (g, j(g, \tau)^{1/2})$. Then $s(g)s(h) = (gh, a_\infty(g, h)j(gh, \tau)^{1/2})$ for $a_\infty(g, h) = j(g, h(\tau))^{1/2}j(h, \tau)^{1/2}/j(gh, \tau)^{1/2} \in \{\pm 1\}$, since $j(g, \tau)$ is an automorphic factor. As a_∞ is the coboundary of the $\mathrm{SL}_2(\mathbb{R})$ 1-chain $g \mapsto j(g, \tau)$ (under the pull-back action of $\mathrm{SL}_2(\mathbb{R})$ on the $\mathrm{SL}_2(\mathbb{R})$-module of holomorphic functions, a_∞ is a factor set for $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ whose value can be made explicit as we will see below. The factor set defines the metaplectic covering $\mu_2 \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{R}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{R})$. As Weil showed, we can extend this covering to $\mu_2 \overset{i}{\hookrightarrow} \widetilde{\mathrm{SL}}_2(\mathbb{A}) \overset{\pi_{\mathbb{A}}}{\twoheadrightarrow} \mathrm{SL}_2(\mathbb{A})$ so that the covering is trivial on $\mathrm{SL}_2(\mathbb{Q})$ as described in §1.1. Again we write $\pi_A : \widetilde{\mathrm{SL}}_2(A) \twoheadrightarrow \mathrm{SL}_2(A)$ for the projection. The map $\Gamma_0(4) \ni \gamma \mapsto (\gamma, h(\gamma, \tau)) \in \mathrm{Mp}(\mathbb{R})$ is a group embedding.

There is an explicit description of the 2-cocycle $a_v(x, y)$ given by T. Kubota [K67] for the covering of $\mathrm{SL}_2(\mathbb{Q}_v)$. For a local field $\mathbb{Q}_v$, it is expressed by Hilbert's symbol $(\cdot, \cdot)$ with values in μ_2 [BNT, XIII.5], and the product formula [BNT, Proposition XIII.5.8] of Hilbert's symbol provides the splitting over $\mathrm{SL}_2(\mathbb{Q})$ (see [WRS, §2]). Here writing $x(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = c$ or d according as $c \neq 0$ or $c = 0$,

$$a_v(\gamma, \delta) = (x(\gamma), x(\delta))(-x(\gamma)^{-1}x(\delta), x(\gamma\delta)).$$

For $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, we write $\widehat{\Gamma}$ for the closure of Γ in $\mathrm{SL}_2(\mathbb{A}^{(\infty)})$, and put $\widehat{\Gamma}_?(N) = \widehat{\Gamma_?(N)}$ for $? = 0, 1$. To describe a splitting of $\mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$ over an open compact subgroup of $\mathrm{SL}_2(\mathbb{A}^{(\infty)})$, we define $s_p : \mathrm{SL}_2(\mathbb{Q}_p) \to \mu_2$ for a prime p by

$$s_p(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \begin{cases} (c, d) & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \equiv 1 \mod 2, \\ 1 & \text{otherwise,} \end{cases}$$

and for $\mathbb{Q}_\infty = \mathbb{R}$, $s_\infty := 1$ (the constant function). Then we modify a_v as

$$\kappa_v(\gamma, \delta) = a_v(\gamma, \delta)s_v(\gamma)s_v(\delta)s_v(\gamma\delta)^{-1}.$$

The 2-cocycle κ_v defines an isomorphic central extension $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v)$ of $\mathrm{SL}_2(\mathbb{Q}_v)$ by μ_2. This modification makes κ_v trivial on $\widehat{\Gamma}_1(4)$ [WRS, Proposition 2.8] and $\kappa(\gamma, \delta) = \prod_v \kappa_v(\gamma_v, \delta_v)$ makes sense for $(\gamma, \delta) \in \mathrm{SL}_2(\mathbb{A})$. Thus the product cocycle gives the metaplectic extension $\widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A}^{(\infty)})$. We extend this to $\mathrm{SL}_2(\mathbb{A})$ by

$$\widetilde{\mathrm{SL}}_2(\mathbb{A}) = \{(g^{(\infty)}, \epsilon^{(\infty)}J(g_\infty, \tau)) | (g^{(\infty)}, \epsilon^{(\infty)}) \in \mathrm{SL}_2(\mathbb{A}^{(\infty)}) \times \mu_2, (g_\infty, J(g_\infty, \tau)) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})\} \tag{1.5}$$

with the product given by

$$(g, \epsilon^{(\infty)}J(g_\infty, \tau))(h, \zeta^{(\infty)}J(h_\infty, \tau)) = (gh, \kappa(g, h)\epsilon^{(\infty)}\zeta^{(\infty)}J(g_\infty, h_\infty(\tau))J(h_\infty, \tau)).$$

This definition is an amalgamation of Shimura's definition of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ in [Sh73] and that of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ by Kubota–Gelbart [WRS]. Shimura's group is isomorphic to the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ defined in [WRS, §2] by the cocycle κ_∞ via sending the pair $(g, J(g, \tau))$ to $(g, J(g, \tau)/(cz + d)^{1/2}) \in \mathrm{SL}_2(\mathbb{R}) \times \mu_2$, since $(g, h) \mapsto j(g, h(\tau))^{1/2}j(h, \tau)^{1/2}/j(gh, \tau)^{1/2}$ is the 2-cocycle κ_∞ by (1.4). Indeed, by (1.4), κ coincides with

$$\kappa^{(\infty)}(g, h)j(g_\infty, h_\infty(\tau))^{1/2}j(h_\infty, \tau)^{1/2}/j(g_\infty h_\infty, \tau)^{1/2}$$

on $\Gamma_1(4)$, and by strong approximation and triviality on $\mathrm{SL}_2(\mathbb{Q}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A})$, they coincide on $\widehat{\Gamma}_1(4)\mathrm{SL}_2(\mathbb{Q})$ whose image in $\mathrm{SL}_2(\mathbb{R})$ is dense. The commutativity of the diagram (1.2) is clear from the above definition of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ adapted to theta series automorphic factors.

Sending the set theoretic product $\mathrm{SL}_2(\mathbb{Q}_v) \times \mu_2$ into $\mathrm{SL}_2(\mathbb{A}) \times \mu_2$ produces group homomorphism $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_v) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A})$. In particular, we have commutative diagrams

$$\begin{array}{ccccc} \mu_2 & \xrightarrow[\hookrightarrow]{} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) & \xrightarrow[\twoheadrightarrow]{\pi_{\mathbb{A}}} & \mathrm{SL}_2(\mathbb{A}) \\ \| \uparrow & & \uparrow & & \uparrow \cup \\ \mu_2 & \xrightarrow[\hookrightarrow]{} & \widetilde{\mathrm{SL}}_2(\mathbb{R}) & \xrightarrow[\twoheadrightarrow]{} & \mathrm{SL}_2(\mathbb{R}) \end{array} \tag{1.6}$$

and

$$\begin{array}{ccccc} \mu_2 & \xrightarrow[\hookrightarrow]{} & \widetilde{\mathrm{SL}}_2(\mathbb{A}) & \xrightarrow[\twoheadrightarrow]{} & \mathrm{SL}_2(\mathbb{A}) \\ \| \uparrow & & \uparrow & & \uparrow \cup \\ \mu_2 & \xrightarrow[\hookrightarrow]{} & \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) & \xrightarrow[\twoheadrightarrow]{} & \mathrm{SL}_2(\mathbb{A}^{(\infty)}). \end{array} \tag{1.7}$$

Then we have $\widetilde{\mathrm{SL}}_2(\mathbb{A}) = (\widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) \times \widetilde{\mathrm{SL}}_2(\mathbb{R}))/\Delta(\mu_2)$ for the diagonal embedding $\Delta : \mu_2 \to \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) \times \widetilde{\mathrm{SL}}_2(\mathbb{R})$.

We write the projection to $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ as $g_\infty = (\pi_{\mathbb{A}}(g)_\infty, J(\pi_{\mathbb{A}}(g)_\infty, \tau))$ for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$ ($\pi_{\mathbb{A}}(g) \in \mathrm{SL}_2(\mathbb{A})$). The map $\mathrm{SL}_2(?) \ni g \mapsto (g,1) \in \widetilde{\mathrm{SL}}_2(?)$ (for $? = \mathbb{Q}_v, \mathbb{A}, \mathbb{A}^{(\infty)}$) is not a group homomorphism. We simply write $J(g_\infty, \tau)$ for $J(\pi_{\mathbb{A}}(g)_\infty), \tau)$ and $g_\infty(\sqrt{-1}) = \pi_{\mathbb{A}}(g)_\infty(\sqrt{-1}) \in \mathfrak{H}$. We put $C_\infty = \pi_{\mathbb{A}}^{-1}(\mathrm{SO}_2(\mathbb{R})) \subset \mathrm{Mp}(\mathbb{A})$ for the projection $\pi_{\mathbb{A}} : \mathrm{Mp}(\mathbb{A}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{A})$.

We have a splitting $\mathbf{r}_{\mathbb{Q}}$ over $\mathrm{SL}_2(\mathbb{Q})$ into $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ as described in §1.1. Thus $\Gamma_1(4) \subset \mathrm{SL}_2(\mathbb{Q}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A})$ and a commutative diagram:

$$\begin{array}{ccc} \Gamma_1(4) & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{A}) \\ \| \uparrow & & \uparrow \cup \\ \Gamma_1(4) & \xrightarrow{\gamma \mapsto (\gamma, h(\gamma,\tau))} & \widetilde{\mathrm{SL}}_2(\mathbb{R}). \end{array}$$

This inclusion of $\Gamma_1(4)$ extends to $\widehat{\Gamma}_1(4) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)})$ [WRS, Proposition 2.8], and we have a commutative diagram:

$$\begin{array}{ccc} \widehat{\Gamma}_1(4) & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)}) \\ \cup \uparrow & & \uparrow \cup \\ \Gamma_1(4) & \longrightarrow & \mathrm{SL}_2(\mathbb{Q}). \end{array}$$

Since $h(\gamma, \tau)$ in (1.4) involves $\varepsilon_d \in \mu_4$ and $h(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau)^2 = \left(\frac{-1}{d}\right)(cz+d)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, we can embed $\Gamma_0(4) \hookrightarrow \mathrm{Mp}(\mathbb{R})$ by $\gamma \mapsto \widetilde{\gamma}_\infty := (\gamma_\infty, h(\gamma_\infty, \tau))$. Since $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{Mp}(\mathbb{A})$, by $\Gamma_0(4) \ni \gamma \mapsto \gamma\widetilde{\gamma}_\infty^{-1}$, we get an embedding $\Gamma_0(4) \hookrightarrow \mathrm{Mp}(\mathbb{A}^{(\infty)})$. Since $\widehat{\Gamma}_0(4) = \widehat{\Gamma}_1(4)\Gamma_0(4) \subset \mathrm{Mp}(\mathbb{A}^{(\infty)})$,

(1.8) *we can lift the inclusion* $\widehat{\Gamma}_1(4) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}^{(\infty)})$ *to* $\widehat{\Gamma}_0(4) \hookrightarrow \mathrm{Mp}(\mathbb{A}^{(\infty)})$.

Define $\phi \in \mathcal{S}(\mathbb{A})$ by $\phi(x) = \begin{cases} 0 & \text{if } x^{(\infty)} \notin \widehat{\mathbb{Z}}, \\ \mathbf{e}(\sqrt{-1}x_\infty^2) & \text{if } x^{(\infty)} \in \widehat{\mathbb{Z}}. \end{cases}$ Let $\boldsymbol{\theta}(g) := \sum_{n \in \mathbb{Q}} \mathbf{w}(g)\phi(g)$ for $g \in \mathrm{Mp}(\mathbb{A})$ taking the quadratic space $(\mathbb{Q}, x^2)$. By definition, for $\tau = \xi + \eta\sqrt{-1} \in \mathfrak{H}$ and $g_\tau = \eta^{-1/2}\left(\begin{smallmatrix} \eta & \xi \\ 0 & 1 \end{smallmatrix}\right)$,

$$\boldsymbol{\theta}(\tau) = \boldsymbol{\theta}((g_\tau, \eta^{-1/4})) = \eta^{1/4} \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau).$$

For $\gamma \in \Gamma_0(4)$,

$$\begin{aligned} \boldsymbol{\theta}((g_\tau, \eta^{-1/4})\gamma^{(\infty)}) &= \boldsymbol{\theta}(\gamma\gamma_\infty^{-1}(g_\tau, \eta^{-1/4})) \\ &= \boldsymbol{\theta}((\gamma_\infty^{-1}, \theta(\gamma^{-1}(\tau))/\theta(\tau))(g_\tau, \eta^{-1/4})) = \boldsymbol{\theta}(g_{\gamma^{-1}(\tau)})(\theta(\gamma^{-1}(\tau))/\theta(\tau))^{-1} = \boldsymbol{\theta}(g_\tau). \end{aligned}$$

Since we made splitting of $\pi_{\mathbb{R}}:\mathrm{Mp}(\mathbb{R})\twoheadrightarrow \mathrm{SL}_2(\mathbb{R})$ over $\Gamma_0(4)$ by setting $h(\gamma,\tau):=\theta(\gamma(\tau))/\theta(\tau)$, the above argument is a tautology. Anyway, since $\widehat{\Gamma}_0(4)$ is a closure of $\Gamma_0(4)$ in $\mathrm{Mp}(\mathbb{A}^{(\infty)})$, the stability of $\boldsymbol{\theta}$ under $\Gamma_0(4)$ implies

Lemma 1.2. *The above theta function $\boldsymbol{\theta}$ is fixed by the right multiplication by $\widehat{\Gamma}_0(4)$ embedded in* $\mathrm{Mp}(\mathbb{A})$.

2. Rankin convolution

We adapt to SL(2) the adelic Rankin convolution in [LFE, Chapter 9] described for GL(2).

2.1. **Adelic Fourier expansion, cusp forms of integral weight.** Let $F\in S_\kappa(\Gamma_0(C),\varphi)$ be a cusp form of weight κ for $0<\kappa\in\mathbb{Z}$, where φ is a Dirichlet character modulo C and $\varphi(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))=\varphi(d)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(C)$. Then $F(\gamma(\tau))=\varphi(\gamma)F(\tau)j(\gamma,\tau)^\kappa$ for $j(\gamma,\tau)$ as in (1.4). The character $\varphi:\Gamma_0(C)\to\mathbb{C}^\times$ extends to $\varphi:\widehat{\Gamma}_0(C)\to\mathbb{C}^\times$ by continuity. Then $\varphi(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))=\varphi(d_C \bmod C\mathbb{Z}_C)$ for $\mathbb{Z}_C=\prod_{l|C}\mathbb{Z}_l$ and the projection d_C of d to $\mathbb{Q}_C:=\prod_{l|C}\mathbb{Q}_l$.

Since $\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_+^\times\cong\widehat{\mathbb{Z}}^\times\to\mathbb{C}^\times$, we extend φ to a character $\varphi^*:\mathbb{A}^\times/\mathbb{Q}^\times\to\mathbb{C}^\times$ so that $\varphi^*(l)=\varphi(l)$ for primes $l\nmid C$ regarded in $\mathbb{Q}_l^\times\subset\mathbb{A}^\times$. We lift F to $\mathbf{F}:\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})\to\mathbb{C}$ by putting

$$\mathbf{F}(\alpha u)=\varphi^*(u)^{-1}F(u_\infty(\sqrt{-1}))j(u_\infty,\sqrt{-1})^{-\kappa}\tag{2.1}$$

for $\alpha\in\mathrm{SL}_2(\mathbb{Q})$ and $u\in\widehat{\Gamma}_0(C)\mathrm{SL}_2(\mathbb{R})$ [H10, §1.1], where $\varphi^*(u)^{-1}=\varphi(u)$.

Define an idele character $\boldsymbol{\varphi}:\mathbb{A}^\times/\mathbb{Q}^\times\to\mathbb{C}^\times$ by $\boldsymbol{\varphi}(a)=\varphi^*(a)|a|_{\mathbb{A}}^{-\kappa}$. Write the Fourier expansion of F as $F(\tau)=\sum_{n=1}^\infty a_n(F)\mathbf{e}(n\tau)$. For $g=\left(\begin{smallmatrix}a&b\\0&a^{-1}\end{smallmatrix}\right)\in B(\widehat{\mathbb{Z}})B(\mathbb{R})$ with $a\in\widehat{\mathbb{Z}}^\times\mathbb{R}_+$, we find, for $\tau=g_\infty(i)=a_\infty^2 i+a_\infty b_\infty$,

$$\mathbf{F}(g)=\varphi^*(a^{(\infty)})^{-1}a_\infty^\kappa\sum_{n=1}^\infty a_n(F)\mathbf{e}(n\tau)=\boldsymbol{\varphi}^{-1}(a)\sum_{n=1}^\infty a_n(F)\exp(-2\pi n a_\infty^2)\mathbf{e}(na_\infty b_\infty).\tag{2.2}$$

Recall the character $\mathbf{e}:\mathbb{Q}\backslash\mathbb{A}\to\mathbb{C}^\times$ defined by $\mathbf{e}(x)=\prod_v\mathbf{e}(x_v)$ with $\mathbf{e}(x_v)=\exp(-2\pi\sqrt{-1}[x]_l)$ if v is a prime l for the fraction part $[x_l]_l$ for the l-adic expansion of x and $\mathbf{e}(x_\infty)=\exp(2\pi\sqrt{-1}x_\infty)$. Let $\upsilon(u)=\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)\in U(\mathbb{A})$. Then for $b=\upsilon(u)\operatorname{diag}[a,a^{-1}]\in B(\mathbb{A})$, write $\mathbf{F}(a,u):=\mathbf{F}(b)$. Since $\mathbf{F}(a,u+\alpha)=F(\upsilon(\alpha)b)=\mathbf{F}(a,u)$ for $\alpha\in\mathbb{Q}$, we have the adelic Fourier expansion of $\mathbf{F}(a,u)$ over $u\in\mathbb{A}$:

$$\mathbf{F}(a,u)=\sum_{\alpha\in\mathbb{Q}}a_{\mathbf{F}}(\alpha;a)\mathbf{e}(\alpha u).$$

By $\operatorname{diag}[\beta,\beta^{-1}]\upsilon(u)\operatorname{diag}[a,a^{-1}]=\upsilon(\beta^2u)\operatorname{diag}[\beta a,(\beta a)^{-1}]$, we have

$$\sum_{\alpha\in\mathbb{Q}}a_{\mathbf{F}}(\alpha;a)\mathbf{e}(\alpha u)=\mathbf{F}(a,u)=\mathbf{F}(\beta a,\beta^2u)=\sum_{\alpha\in\mathbb{Q}}a_{\mathbf{F}}(\alpha;\beta a)\mathbf{e}(\alpha\beta^2u).$$

From the uniqueness of the Fourier expansion, we conclude

$$\begin{aligned}&a_{\mathbf{F}}(\alpha;a)=a_{\mathbf{F}}(\alpha\beta^{-2},\beta a)\ \text{ for }\beta\in\mathbb{Q}^\times\text{ and }\alpha\in\mathbb{Q},\\&a_{\mathbf{F}}(\alpha;a)=\boldsymbol{\varphi}^{-1}(a)a_\alpha(F)\exp(-2\pi\alpha a_\infty^2)\ \text{ if }0<\alpha\in\mathbb{Z}\text{ and }a\in\widehat{\mathbb{Z}}^\times.\end{aligned}\tag{2.3}$$

By $\upsilon(u)\operatorname{diag}[a,a^{-1}]\operatorname{diag}[t,t^{-1}]=\upsilon(u)\operatorname{diag}[ta,(ta)^{-1}]$, for $t\in\widehat{\mathbb{Z}}^\times$, we get

$$a_{\mathbf{F}}(\alpha;at)=\boldsymbol{\varphi}^{-1}(t)a_{\mathbf{F}}(\alpha,a).\tag{2.4}$$

If $x\in\mathbb{Q}^\times(\mathbb{A}^\times)^2\cap\mathbb{A}^\times$, choose $\alpha\in\mathbb{Q}^\times$ and $a\in\mathbb{A}^\times$ so that $x=\alpha a^2$, define,

$$\mathbf{a}_{\mathbf{F}}(x):=\boldsymbol{\varphi}(a)a_{\mathbf{F}}(\alpha,a)\exp(2\pi\alpha_\infty a_\infty^2),\tag{2.5}$$

which is equal to $a_\alpha(F)$ if $0<\alpha\in\mathbb{Z}$ and $a\in\widehat{\mathbb{Z}}^\times$, and $\mathbf{a}_{\mathbf{F}}(x)=0$ if $x\notin\mathbb{Z}(\widehat{\mathbb{Z}}^\times\mathbb{R}^\times)^2$. If $\alpha a^2(\widehat{\mathbb{Z}}^\times)^2=\beta b^2(\widehat{\mathbb{Z}}^\times)^2$, then writing $\alpha a^2=\beta b^2t^2$ for $t\in\widehat{\mathbb{Z}}^\times$, we have

$$\begin{aligned}\mathbf{a}_{\mathbf{F}}(\alpha a^2):=\boldsymbol{\varphi}(a)\mathbf{a}_{\mathbf{F}}(\alpha,a)\exp(2\pi\alpha_\infty a_\infty^2)&=\boldsymbol{\varphi}(a)a_{\mathbf{F}}(\alpha\beta^{-2},\beta a)\exp(2\pi(\alpha_\infty\beta_\infty^{-2})(\beta a)_\infty^2)\\&=\boldsymbol{\varphi}(\beta a)a_{\mathbf{F}}(\alpha\beta^{-2},\beta a)\exp(2\pi(\alpha_\infty\beta_\infty^{-2})(\beta a)_\infty^2),\end{aligned}$$

since $\boldsymbol{\varphi}(\beta a) = \boldsymbol{\varphi}(a)$. By (2.3), this shows that $\mathbf{a_F}(x)$ is well defined independent of the choice of the expression $x = \alpha a^2$ with $a \in \mathbb{A}^\times$ and $\alpha \in \mathbb{Q}^\times$.

By (2.5), we have

$$\begin{aligned} &\mathbf{a_F}(x) = a_\alpha(F) = \mathbf{a_F}(xt^2) \text{ for } t \in \widehat{\mathbb{Z}}^\times\mathbb{R}^\times \text{ and } x = \alpha a^2, \\ &\mathbf{F}(a,u) = \mathbf{F}(\upsilon(u)\operatorname{diag}[a,a^{-1}]) = \boldsymbol{\varphi}^{-1}(a)\sum_{\alpha\in\mathbb{Q}} \mathbf{a_F}(\alpha a^2)\mathbf{e}(\alpha_\infty a_\infty^2\sqrt{-1})\mathbf{e}(\alpha u). \end{aligned} \tag{2.6}$$

Since $\mathbf{a_F}$ is supported over $\mathbb{A}_+^\times = (\mathbb{A}^{(\infty)})^\times\mathbb{R}_+^\times$, $\mathbf{a_F}$ only depends on the finite part of the idele.

2.2. **Adelic Fourier expansion, cusp forms of half integral weight.** We describe in detail the adelic Fourier expansion in case of half integral. Writing the level of a half integral weight modular form as M; so, $4|M$. Let $f \in S_{k/2}(\Gamma_0(M),\psi_1)$ be a cusp form of weight $\frac{k}{2}$ for odd k, where ψ_1 is an even Dirichlet character modulo M. Then $f(\gamma(\tau)) = \psi_1(\gamma)f(\tau)h(\gamma,\tau)^k$ for $\gamma \in \Gamma_0(M)$ and $h(\gamma,\tau)$ in (1.4), and as before $\psi_1(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) = \psi_1(d)$ for $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(M)$. The cusp form f has its Fourier expansion: $f(\tau) = \sum_{n=1}^\infty a_n(f)\mathbf{e}(n\tau)$. As before, we extend ψ_1 to a character of $\psi_1^* : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ and then ψ_1^* to $\widehat{\Gamma}_0(M)$ so that $\psi_1^*(u) = \psi_1(u)^{-1}$. We lift f to $\mathbf{f} : \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ by putting

$$\mathbf{f}(\alpha(u,\zeta J(u_\infty,\tau))) = \psi_1^*(u)f(u_\infty(\sqrt{-1}))\zeta^k J(u_\infty,i)^{-k} \tag{2.7}$$

for $\alpha \in \mathrm{SL}_2(\mathbb{Q}) \subset \mathrm{Mp}(\mathbb{A})$ and $(u,\zeta J(u_\infty,\tau)) \in \widehat{\Gamma}_0(M)\mathrm{Mp}(\mathbb{R})$ ($\zeta \in S^1$ and $u_\infty \in \mathrm{SL}_2(\mathbb{R})$) regarding $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A}) \subset \mathrm{Mp}(\mathbb{A})$ by (1.6).

Since $B(\mathbb{A})$ is canonically lifted into $\mathrm{Mp}(\mathbb{A})$ by $\mathbf{r}$ and this lifting coincides with the splitting $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{Mp}(\mathbb{A})$ over $B(\mathbb{Q})$ as already remarked, we regard $B(\mathbb{A}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A}) \subset \mathrm{Mp}(\mathbb{A})$ and think of $\mathbf{f}|_{B(\mathbb{A})}$. We get back to $f(\tau)$ by reversing the process:

$$f(\tau) := \mathbf{f}(g_\tau, j(g_\tau,\tau)^{1/2})j(g_\tau,i)^{k/2} = \mathbf{f}(g_\tau,\eta^{-1/2})\eta^{-k/4} \text{ for } g_\tau = \eta^{-1/2}\left(\begin{smallmatrix} \eta & \xi \\ 0 & 1\end{smallmatrix}\right),$$

which is holomorphic in $\tau = \xi + \eta\sqrt{-1}$.

Write $\boldsymbol{\theta} : \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ for the lift of $\theta(\tau) = \sum_{n\in\mathbb{Z}} \mathbf{e}(n^2\tau)$. We have an inclusion $\widehat{\Gamma}_0(4) \hookrightarrow \mathrm{Mp}(\mathbb{A})$ as in (1.8). Since $B(\widehat{\mathbb{Z}}) \subset \widehat{\Gamma}_0(4)$, we regard $B(\widehat{\mathbb{Z}}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A})$. This inclusion coincides over $B(\widehat{\mathbb{Z}})$ with the one induced by $\mathbf{r}$ and hence matches with the inclusion $B(\mathbb{Q}) \hookrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Define an idele character $\boldsymbol{\psi}_1 : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ by $\boldsymbol{\psi}_1(a) = \psi_1^*(a)|a|_{\mathbb{A}}^{-k/2}$. Thus for $g = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) \in B(\widehat{\mathbb{Z}})B(\mathbb{R}) \subset \widetilde{\mathrm{SL}}_2(\mathbb{A})$, from (2.7), we find for $\tau = a_\infty(a_\infty i + b_\infty)$

$$\mathbf{f}(g) = \boldsymbol{\psi}_1^{-1}(a)\sum_{n=1}^\infty a_n(f)\mathbf{e}(n\tau) = \boldsymbol{\psi}_1^{-1}(a)\sum_{n=1}^\infty a_n(f)\exp(-2\pi n a_\infty^2)\mathbf{e}(na_\infty b_\infty). \tag{2.8}$$

Let $\upsilon(u) = \left(\begin{smallmatrix} 1 & u \\ 0 & 1\end{smallmatrix}\right) \in U(\mathbb{A})$. Then we consider for a general $b = \upsilon(u)\operatorname{diag}[a,a^{-1}] \in B(\mathbb{A})$. Write $\mathbf{f}(a,u) := \mathbf{f}(b)$. Then $\mathbf{f}(a,u+\alpha) = \mathbf{f}(\upsilon(\alpha)b) = \mathbf{f}(b)$ if $\alpha \in \mathbb{Q}$. Thus $\mathbf{f}(a,u)$ has a Fourier expansion over $u \in \mathbb{A}$ of the form

$$\mathbf{f}(a,u) = \sum_{\alpha\in\mathbb{Q}} a_{\mathbf{f}}(\alpha;a)\mathbf{e}(\alpha u). \tag{2.9}$$

By $\operatorname{diag}[\beta,\beta^{-1}]\upsilon(u)\operatorname{diag}[a,a^{-1}] = \upsilon(\beta^2 u)\operatorname{diag}[\beta a,(\beta a)^{-1}]$, we have

$$\sum_{\alpha\in\mathbb{Q}} a_{\mathbf{f}}(\alpha;a)\mathbf{e}(\alpha u) = \mathbf{f}(a,u) = \mathbf{f}(\beta a,\beta^2 u) = \sum_{\alpha\in\mathbb{Q}} a_{\mathbf{f}}(\alpha;\beta a)\mathbf{e}(\alpha\beta^2 u).$$

By the uniqueness of the Fourier expansion, we get

$$a_{\mathbf{f}}(\alpha;a) = a_{\mathbf{f}}(\alpha\beta^{-2},\beta a) \text{ and } a_{\mathbf{f}}(\alpha;a) = \begin{cases} \boldsymbol{\psi}_1^{-1}(a)a_\alpha(f)\exp(-2\pi\alpha a_\infty^2) & \text{if } 0 < \alpha \in \mathbb{Z}, \\ 0 & \text{otherwise.}\end{cases}$$

By $\upsilon(u)\operatorname{diag}[a,a^{-1}]\operatorname{diag}[t,t^{-1}] = \upsilon(u)\operatorname{diag}[ta,(ta)^{-1}]$, for $t \in \widehat{\mathbb{Z}}^\times$, we get

$$a_{\mathbf{f}}(\alpha;at) = \boldsymbol{\psi}_1^{-1}(t)a_{\mathbf{f}}(\alpha,a).$$

Define

$$\mathbf{a_f}(\alpha a^2) := \begin{cases} \boldsymbol{\psi}_1(a) a_{\mathbf{f}}(\alpha, a)\exp(2\pi\alpha_\infty a_\infty^2) = a_\alpha(f) & \text{if } \alpha a^2 \in \mathbb{Z}(\widehat{\mathbb{Z}}\mathbb{R}_+)^2 \cap \mathbb{A}^\times, \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.10}$$

Thus $\mathbf{a_f}$ is supported over $\mathbb{Z}(\widehat{\mathbb{Z}}\mathbb{R}^\times)^2 \cap \mathbb{A}^\times$. If $\alpha a^2(\widehat{\mathbb{Z}}^\times)^2 = \beta b^2(\widehat{\mathbb{Z}}^\times)^2$, then writing $\alpha a^2 a^2 = \beta b^2 t^2$ for $t \in \widehat{\mathbb{Z}}^\times$, we have

$$\begin{aligned}\mathbf{a_f}(\alpha a^2) := \boldsymbol{\psi}_1(a) a_{\mathbf{f}}(\alpha, a)\exp(2\pi\alpha_\infty a_\infty^2) &= \boldsymbol{\psi}_1(a) a_{\mathbf{f}}(\alpha\beta^{-2}, \beta a)\exp(2\pi(\alpha_\infty\beta_\infty^{-2})(\beta a)_\infty^2) \\ &= \boldsymbol{\psi}_1(\beta a) a_{\mathbf{f}}(\alpha\beta^{-2}, \beta a)\exp(2\pi(\alpha_\infty\beta_\infty^{-2})(\beta a)_\infty^2),\end{aligned}$$

since $\boldsymbol{\psi}_1(\beta a) = \boldsymbol{\psi}_1(a)$. This shows that $\mathbf{a_f}(x)$ is well defined independent of the choice of the expression $x = \alpha a^2$ with $a \in \mathbb{A}^\times$ and $\alpha \in \mathbb{Q}^\times$.

By (2.10), we have

$$\mathbf{a_f}(x) = a_\alpha(f) = \mathbf{a_f}(xt^2) \text{ for } t \in \widehat{\mathbb{Z}}^\times\mathbb{R}^\times \text{ and } x = \alpha a^2. \tag{2.11}$$

Since $\mathbf{a_f}$ is supported over $\mathbb{A}_+^\times = \mathbb{A}^{(\infty)}\mathbb{R}_+^\times$, $\mathbf{a_f}$ only depends on the finite part of the idele. Thus we can recover

$$\mathbf{f}(a, u) = \mathbf{f}(v(u)\,\mathrm{diag}[a, a^{-1}]) = \boldsymbol{\psi}(a)^{-1} \sum_{0<\alpha\in\mathbb{Q}} \mathbf{a_f}(\alpha a^2)\exp(-2\pi\alpha_\infty a_\infty^2)\mathbf{e}(\alpha u). \tag{2.12}$$

Pick a Dirichlet character χ modulo N with $\chi(-1) = (-1)^{\epsilon(\chi)}$ ($\epsilon(\chi) \in \{0, 1\}$) and consider the adelic form $\boldsymbol{\theta}_{\epsilon(\chi)}(\chi)$ corresponding to

$$\theta_{\epsilon(\chi)}(\chi)(\tau) = \sum_{n\in\mathbb{Z}} \chi(n) n^{\epsilon(\chi)}\mathbf{e}(n^2\tau).$$

Note that $\theta_{\epsilon(\chi)}(\chi) \in S_{\epsilon(\chi)+\frac{1}{2}}(\Gamma_0([4, N^2]), \chi_1)$ [Sh73, Proposition 2.2], where $\chi_1(d) = \chi(d)\left(\frac{-1}{d}\right)^{\epsilon(\chi)}$ and $[4, N^2]$ is the LCM of 4 and N^2. Then $\boldsymbol{\theta}_{\epsilon(\chi)}(\chi)$ is right invariant under $\widehat{\Gamma}_0([4, N^2])$.

(2.13) *Taking $a \equiv n \mod N$, we find $\mathbf{a}_{\boldsymbol{\theta}_{\epsilon(\chi)}(\chi)}(a^2) = \chi(n)|n|^{\epsilon(\chi)}$ and $\mathbf{a}_{\boldsymbol{\theta}_{\epsilon(\chi)}(\chi)}(x) = 0$ if $x \notin (\widehat{\mathbb{Z}})^2$.*

Define for an integer $r \geq 0$ and $d = q\frac{d}{dq} = 2\pi\sqrt{-1}\frac{\partial}{\partial\tau}$

$$\theta_j(\chi)(\tau) = \sum_{n\in\mathbb{Z}} \chi(n) n^j \mathbf{e}(n^2\tau) \text{ and } \theta_j(\chi)(\tau) = d^r\theta_{\epsilon(\chi)}(\chi)(\tau) \text{ if } j = \epsilon(\chi) + 2r. \tag{2.14}$$

If the parity of j and $\epsilon(\chi)$ does not match, $\theta_j(\chi) = \boldsymbol{\theta}_j(\chi) = 0$.

2.3. **Eisenstein series.** For the projection $\pi_X : \mathrm{Mp}(X) \twoheadrightarrow \mathrm{SL}_2(X)$, write $C_\infty := \pi_{\mathbb{R}}^{-1}(\mathrm{SO}_2(\mathbb{R})) \subset \mathrm{Mp}(\mathbb{R}) \subset \mathrm{Mp}(\mathbb{A})$ and $\mathrm{Mp}(\widehat{\mathbb{Z}}) := \pi_{\mathbb{A}}^{-1}(\mathrm{SL}_2(\widehat{\mathbb{Z}}))$. We have $B(\mathbb{A})\mathrm{Mp}(\widehat{\mathbb{Z}})C_\infty = \mathrm{Mp}(\mathbb{A})$ by Iwasawa decomposition. Note that $B(\mathbb{Q})\backslash B(\mathbb{A}) = (\mathbb{A}^\times/\mathbb{Q}^\times) \times (\mathbb{A}/\mathbb{Q})$. Since $\widehat{\mathbb{Z}} \times [0, 1)$ is the fundamental domain of the translation action of $\mathbb{Q}$ on $\mathbb{A}$, $\mathbb{A}/\mathbb{Q}$ is compact. Since $\mathbb{A}^\times = \widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times\mathbb{Q}^\times$, we find $\mathbb{A}^\times/\mathbb{Q}^\times \cong \widehat{\mathbb{Z}}^\times \times \mathbb{R}_+$. Thus $B(\mathbb{Q})\backslash B(\mathbb{A}) \cong \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}}^\times \times \Gamma_\infty\backslash\mathfrak{H}$ for $\Gamma_\infty := \{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) | m \in \mathbb{Z}\}$.

Write $S^1 = \{z \in \mathbb{C}^\times \,|\, |z| = 1\}$ for the center of $\mathrm{Mp}(X)$ (independent of the choice of X). Let $\Phi : B(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ be a continuous function such that

(1) if $z \in S^1$, $\Phi(zg) = \chi(z)\Phi(g)$ for a continuous character $\chi : S^1 \to \mathbb{C}^\times$,
(2) For u in an open subgroup $\widehat{\Gamma} \subset \widehat{\Gamma}_0(4)$, $\Phi(gu) = \phi(u)\Phi(g)$ for a character $\phi : \widehat{\Gamma} \to \mathbb{C}^\times$,
(3) $\Phi(g(u_\infty, \zeta J(u_\infty, \tau)) = \Phi(g)\zeta^\ell J(u_\infty, i)^\ell$ for $(u_\infty, \zeta J(u_\infty, \tau)) \in C_\infty$, where ℓ is an integer.

If $\zeta \in S^1$, the above conditions imply $\chi(\zeta) = \zeta^\ell$ as long as $\Phi \neq 0$. Thus $\chi(z) = z^\ell$ can be a good choice. In the same manner, we have $\phi(\zeta) = \chi(\zeta)$ for $\zeta \in S^1 \cap \widehat{\Gamma}$.

Assuming an absolute and local uniform convergence, we define

$$E(\Phi)(g) := \sum_{\gamma\in B(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{Q})} \Phi(\gamma g) \quad (g \in \mathrm{Mp}(\mathbb{A})).$$

Then, as a function of $g \in \mathrm{Mp}(\mathbb{A})$, $E(\Phi)$ satisfies $E(\Phi)(\zeta\gamma g u) = \chi(\zeta)\phi(u^{(\infty)})E(\Phi)(g)J(u_\infty, i)^\ell$ for $\zeta \in S^1$, $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in \widehat{\Gamma}C_\infty$ with infinity part $(u_\infty, \zeta J(u_\infty, \tau))$.

Writing $\pi_{\mathbb{A}}(g) = bu$ ($b \in B(\mathbb{A})$ and $u \in \mathrm{SL}_2(\widehat{\mathbb{Z}})\mathrm{SO}_2(\mathbb{R})$) for $g \in \mathrm{Mp}(\mathbb{A})$, we define $|a(g)|_{\mathbb{A}}$ by $|a|_{\mathbb{A}}$ if $b = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$. This function does not depend on the choice of the decomposition and factors through $B(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$. For $s \in \mathbb{C}$, we define $\Phi_s(g) = \Phi(g)|a(g)|^s_{\mathbb{A}}$. We assume that

(A) $E(\Phi_s)$ converges absolutely and locally uniformly if $\mathrm{Re}(s) \gg 0$ and is continued to a meromorphic function of s over the entire $\mathbb{C}$.

The meromorphy in (A) means that for any $s_0 \in \mathbb{C}$, there exists an open simply connected neighborhood U_{s_0} of s_0 and a non-zero holomorphic function $h : U_{s_0} \to \mathbb{C}$ such that $s \mapsto h(s)E(\Phi_s)(g)$ is a holomorphic function of $s \in U_{s_0}$ for each g.

2.4. **Rankin product, integral weight versus half integral weight.** We keep assuming $4|M$. We pick $C|M$. As in §2.1, let F be an integral weight cusp form in $S_\kappa(C, \varphi)$, and using $\mathrm{SL}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\widehat{\Gamma}_0(C)\mathrm{SL}_2(\mathbb{R})$, lift it to $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$ as in §2.1 by

$$\mathbf{F}(\alpha u) = \varphi^*(u^{(\infty)})F(g_\infty(\sqrt{-1}))j(u_\infty, \sqrt{-1})^{-\kappa} \quad \text{for } u \in \widehat{\Gamma}_0(C)\mathrm{SL}_2(\mathbb{R}).$$

Let $\widetilde{F}(\tau) = F(-\overline{\tau})$. Then $\widetilde{F}$ is an anti-holomorphic cusp form in $S^-_\kappa(C, \varphi)$, where $S^-_\kappa(C, \varphi)$ is made up of anti-holomorphic cusp forms $G : \mathfrak{H} \to \mathbb{C}$ satisfying $G(\gamma(\tau)) = \varphi(\gamma)G(\tau)j(\gamma, \overline{\tau})^\kappa$ for $\gamma \in \Gamma_0(C)$. Lift $\widetilde{F}$ to $\mathrm{SL}_2(\mathbb{A})$ by $\widetilde{\mathbf{F}}(g) = \varphi^*(u)\widetilde{F}(g_\infty(\sqrt{-1}))j(u_\infty, -\sqrt{-1})^{-\kappa}$. Since $\widetilde{F}(\frac{a\tau+b}{c\tau+d}) = \varphi(d)\widetilde{F}(\tau)(c\overline{\tau}+d)^k$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(C)$, we have

$$\widetilde{\mathbf{F}}(\gamma g u) = \varphi^*(u)\widetilde{\mathbf{F}}(g)j(u_\infty, -\sqrt{-1})^{-\kappa} \quad \text{for } u \in \widehat{\Gamma}_0(C)\mathrm{SO}_2(\mathbb{R}). \tag{2.15}$$

Pick a half integral cusp form $f \in S_{k/2}(\Gamma_0(M), \psi)$ and lift it to $\mathbf{f} : \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ as in §2.2; so, we have

$$\mathbf{f}(\gamma g(u, \zeta J(u_\infty, \tau))) = \psi^*(u^{(\infty)})\mathbf{f}(g)\zeta^k J(u_\infty, -\sqrt{-1})^{-k} \quad \text{for } u \in \widehat{\Gamma}_0(C)\mathrm{SO}_2(\mathbb{R}). \tag{2.16}$$

Take a continuous $\Phi : B(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ and consider

(Φ1) $\Phi(x\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = (\varphi^*\psi_1^*)^{-1}(d_M)\Phi(x)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(M)$,
(Φ2) $\Phi(x(u_\infty, \zeta J(u_\infty, \tau))) = \Phi(x)\zeta^{-k}J(u_\infty, \sqrt{-1})^k j(u_\infty, -\sqrt{-1})^\kappa$ for $(u_\infty, \zeta J(u_\infty, \tau)) \in C_\infty$,
(Φ3) $\Phi|_{B(\mathbb{A})}(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = (\varphi^*\psi_1^*)(a)|a|^{2s}_{\mathbb{A}}$ for $a \in \mathbb{A}^\times$, $b \in \mathbb{A}$ and $s \in \mathbb{C}$.

For the moment, we suppose (Φ1–2). Then we have

$$\mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g) = \mathbf{f}(gu)\widetilde{\mathbf{F}}(gu)\Phi(gu) \quad \text{for all } u \in \widehat{\Gamma}_0(M)C_\infty. \tag{2.17}$$

This follows from the above properties (Φ1–2) of Φ and (2.15) and (2.16).

Take an open compact subgroup $\widehat{\Gamma} \supset \widehat{\Gamma}_0(M)$ inside $\mathrm{Mp}(\mathbb{A}^{(\infty)})$. Consider the space $\mathcal{F}_\infty(\widehat{\Gamma}) := B(\mathbb{Q})\backslash B(\mathbb{A})\{\pm 1\}\widehat{\Gamma}C_\infty/\{\pm 1\}\widehat{\Gamma}C_\infty$. Since $B(\mathbb{A})\{\pm 1\}\widehat{\Gamma}C_\infty/\{\pm 1\}\widehat{\Gamma}C_\infty = B(\mathbb{A})/B(\mathbb{A}) \cap \{\pm 1\}\widehat{\Gamma}C_\infty$,

$$\mathcal{F}_\infty(\widehat{\Gamma}) = B(\mathbb{Q})\backslash B(\mathbb{A})/B(\mathbb{A}) \cap \{\pm 1\}\widehat{\Gamma}C_\infty = B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}}) \cap \{\pm 1\}\widehat{\Gamma} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}^\times_+ \cong [0, 1) \times \mathbb{R}^\times_+.$$

Let $\widehat{\Gamma}_\infty := B(\mathbb{A}) \cap \widehat{\Gamma}$. Defining $\mathrm{Tr}_{\widehat{\Gamma}_\infty/\widehat{\Gamma}_0(M)_\infty}\Phi(g) := \sum_{u \in \widehat{\Gamma}_\infty/\widehat{\Gamma}_0(M)_\infty} \Phi(gu)$ if $\Phi(gu)$ is invariant under $\widehat{\Gamma} \supset \widehat{\Gamma}_0(M)$, we find $\int_{\mathcal{F}_\infty(\widehat{\Gamma}_0(M))} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g = \int_{\mathcal{F}_\infty(\widehat{\Gamma})} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\mathrm{Tr}_{\widehat{\Gamma}_\infty/\widehat{\Gamma}_0(M)_\infty}\Phi(g)d\mu_g$. Here $d\mu_g$ is a measure on $\mathcal{F}_\infty(\widehat{\Gamma}_0(M))$ invariant under right multiplication by $\widehat{\Gamma}$ induced by a Haar measure on $\mathrm{Mp}(\mathbb{A})$ with volume 1 on $\widehat{\Gamma}_0(M)\{\pm 1\}C_\infty$, which can be specified to be $y^{-2}dxdy$ for $x \in [0, 1)$ and $y \in \mathbb{R}^\times_+$ for the Lebesgue measure dx and dy (although the above identity holds once we fix a such a measure on $\mathcal{F}_\infty(\widehat{\Gamma}_0(M))$ not necessarily this choice). Similarly $\Phi\mathbf{f}$ is right-invariant under $\widehat{\Gamma} \supset \widehat{\Gamma}_0(M)$, we have

$$\int_{\mathcal{F}_\infty(\widehat{\Gamma}_0(M))} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g = \int_{\mathcal{F}_\infty(\widehat{\Gamma})} \mathbf{f}(g)(\mathrm{Tr}_{\widehat{\Gamma}_\infty/\widehat{\Gamma}_0(M)_\infty}\widetilde{\mathbf{F}}(g))\Phi(g)d\mu_g.$$

Therefore, as long as $\widehat{\Gamma}_\infty = \widehat{\Gamma}_0(M)_\infty$, we do not worry much about the group fixing $\mathbf{f}$ and $\mathbf{F}$. In particular, since $\widehat{\Gamma}_0(C)_\infty = \widehat{\Gamma}_0(M)_\infty$, as long as $C|M$, the convolution integral is well defined.

Lemma 2.1. *The natural map $\pi_1 : B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \to \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$ is an isomorphism.*

This is an adelic analogue of $B(\mathbb{R}) \twoheadrightarrow B(\mathbb{R})/\mathbb{R}^\times = \mathfrak{H} \twoheadrightarrow \Gamma\backslash\mathfrak{H} = \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/\widehat{\Gamma} \cdot \mathrm{SO}_2(\mathbb{R})$ for any subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with closure $\widehat{\Gamma} \subset \mathrm{SL}_2(\mathbb{A}^{(\infty)})$.

Proof. Since $\mathrm{SL}_2(\mathbb{Q})K = \mathrm{SL}_2(\mathbb{A}^{(\infty)})$ for any open subgroup K of $\widehat{\Gamma}_0(4)$ (strong approximation) and $B(\mathbb{R})C_\infty = \mathrm{Mp}(\mathbb{R})$ (Iwasawa decomposition), we have $\mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})KC_\infty = \mathrm{Mp}(\mathbb{A})$. Thus we have a natural continuous surjection:

$$\pi_K : B_K := B(\mathbb{Q})\backslash B(\mathbb{A})KC_\infty \twoheadrightarrow \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}) = \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})KC_\infty$$

induced by sending $b \in B(\mathbb{Q})\backslash B(\mathbb{A})$ to its class in $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})$. Thus we have a continuous map

$$\pi_1 : B_1 = \varprojlim_{K \supset B(\widehat{\mathbb{Z}})} B(\mathbb{Q})\backslash B(\mathbb{A})KC_\infty \to \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A}).$$

For an open subgroup $K' \subset K$, $\pi_{K',K} : B(\mathbb{Q})\backslash B(\mathbb{A})K'C_\infty \to B(\mathbb{Q})\backslash B(\mathbb{A})KC_\infty$ is a finite map, thus for each compact subset X of B_K, $\pi_{K',K}^{-1}(X)$ is compact surjecting down to X and hence $\pi_1^{-1}(X) = \varprojlim_{K'} \pi_{K',K}^{-1}(X)$ is compact and non-empty. Thus the projection $B_1 \to B_K$ is onto.

If X is open compact, $\pi_1^{-1}(X)$ is open compact; so, B_1 is locally compact. Since $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty$ is locally compact with dense image in the locally compact space B_1, we see that $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \twoheadrightarrow B_1$. Since $B(\mathbb{Q})$ is discrete in $B(\mathbb{A})C_\infty$, we find $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \cong B_1$. Identifying B_1 and $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty$, we find $\pi_1 : B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty \to \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})$ is a continuous morphism with dense image of locally compact spaces; so, π_1 is onto. If $\pi_1(b) = \pi_1(b')$ for $b, b' \in B(\mathbb{A})$, then $\gamma b = b'u$ for $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ and $u \in B(\widehat{\mathbb{Z}})C_\infty$. By projecting down to $\mathrm{SL}_2(\mathbb{A})$, we find $\gamma b = b'\pi_{\mathbb{A}}(u)$, comparing the finite part, we conclude $\gamma \in B(\mathbb{Q})$; so, we find π_1 is an isomorphism. □

Remark 2.2. *For an open compact subgroup K of $\mathrm{Mp}(\mathbb{A}^{(\infty)})$ and regarding $K \subset \mathrm{Mp}(\mathbb{A})$ by the natural inclusion $\mathrm{Mp}(\mathbb{A}^{(\infty)}) \hookrightarrow \mathrm{Mp}(\mathbb{A})$, $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/KC_\infty$ is a modular curve $\Gamma_K\backslash\mathfrak{H}$ for $\Gamma_K = \pi_{\mathbb{A}}(K) \cap \mathrm{SL}_2(\mathbb{Q})$. Similarly $B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty/(K \cap B(\mathbb{A})C_\infty) = (\Gamma_K \cap B(\mathbb{Q}))\backslash\mathfrak{H})$, which is an infinite covering of $\Gamma_K\backslash\mathfrak{H}$.*

By Lemma 2.1, choose a fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/\widehat{\Gamma}_0(M)C_\infty =: X_0(M)$ so that

$$\mathcal{F} \subset B(\mathbb{Q})\backslash C_\infty B(\mathbb{A})/(C_\infty B(\mathbb{A}) \cap C_\infty \widehat{\Gamma}_0(M)) = B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}}) \cong [0,1) \times \mathbb{R}_+^\times \subset \mathfrak{H}. \tag{2.18}$$

Note $\mathrm{SL}_2(\mathbb{A}) = B(\mathbb{A})\mathrm{SL}_2(\mathbb{Q})\widehat{\Gamma}_0(M)C_\infty$. Consider $\bigcup_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \gamma\mathcal{F}$ which is a fundamental domain of $B(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})/C_\infty\widehat{\Gamma}_0(M)$. Since $\mathrm{SL}_2(\mathbb{Q}) = B(\mathbb{Q}) \sqcup B(\mathbb{Q})JB(\mathbb{Q})$, we find

$$B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})\widehat{\Gamma}_0(M)C_\infty/\widehat{\Gamma}_0(M)C_\infty = (B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})) \sqcup (T(\mathbb{Q})\backslash JB(\mathbb{A})/B(\widehat{\mathbb{Z}})).$$

Recalling the Haar measure $d\mu = d\mu_g$ on $\mathrm{Mp}(\mathbb{A})$ inducing the Dirac measure on each point in $\mathrm{SL}_2(\mathbb{Q})$ and $\int_{\widehat{\Gamma}_0(M)C_\infty} d\mu = 1$, we have

$$\begin{aligned}\int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)E(\Phi)(g)d\mu_g &= \int_{\mathcal{F}} \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \mathbf{f}(\gamma g)\widetilde{\mathbf{F}}(\gamma g)\Phi(\gamma g)d\mu_g \\ = \int_{\cup_\gamma \gamma\mathcal{F}} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g &= \int_{B(\mathbb{Q})\backslash B(\mathbb{A})} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g + \int_{T(\mathbb{Q})\backslash JB(\mathbb{A})} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g.\end{aligned} \tag{2.19}$$

We assume

(V) $\int_{T(\mathbb{Q})\backslash JB(\mathbb{A})} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g = 0$ $(\Leftarrow \Phi|_{JB(\mathbb{A})} = 0)$.

The Rankin convolution is often computed when F is integrated against the Eisenstein series of the infinity cusp. The integral in (V) corresponds F integrated against the Eisenstein series of the zero cusp, which could be converted into the standard Rankin product by applying the Weil involution $W = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$ if F is on $\Gamma_0(N)$ up to a (supposed-to-be) complicated constant; so, without the assumption (V), the outcome would be a sum of L-values (possibly the counter-part of functional equation in good cases). To avoid this complication, we assume (V) which will be verified in our case, choosing Φ well.

Taking $d\mu(b) = d\mu\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right) = d^\times a^{(\infty)} \otimes |a_\infty|^{-1} da_\infty \otimes dx^{(\infty)} \otimes dx_\infty$ for the Lebesgue measure da_∞ and dx_∞ on $\mathbb{R}$ with $\int_{\widehat{\mathbb{Z}}} dx^{(\infty)} = 1$ and $d^\times a^{(\infty)}$ with $\int_{\widehat{\mathbb{Z}}^\times} d^\times a^{(\infty)} = 1$, we have

$$L(s) := \int_{B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty\widehat{\Gamma}_0(M)} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)\Phi(g)d\mu_g = \int_{B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})} \mathbf{f}(b)\widetilde{\mathbf{F}}(b)\Phi(b)d\mu(b).$$

Suppose (Φ3) in addition to (Φ1–2). We continue computation:

$$L(s) \overset{(*)}{=} \sum_{\alpha,\beta\in\mathbb{Q}} \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \int_{\mathbb{A}/\mathbb{Q}} \mathbf{a}_{\widetilde{F}}(\alpha a^2)\mathbf{a}_{\mathbf{f}}(\beta a^2)\mathbf{e}((\alpha+\beta)u)\exp(-2\pi(-\alpha_\infty+\beta_\infty)a_\infty^2)du|a|_{\mathbb{A}}^{2s-1}da$$

$$= \sum_{\alpha\in\mathbb{Q}_+} \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \mathbf{a}_{\widetilde{F}}(\alpha a^2)\mathbf{a}_{\mathbf{f}}(\alpha a^2)\exp(-4\pi\alpha_\infty a_\infty^2)|a|_{\mathbb{A}}^{2s-1}da.$$

By (2.2) and (2.8) with variable change: $t = a_\infty^2$, we get

$$(2.20)\quad L(s) = 2\sum_{n=1}^\infty \int_0^\infty \mathbf{a}_{\widetilde{F}}(n)\mathbf{a}_{\mathbf{f}}(n)\exp(-4\pi nt)|t|^s t^{-1}dt$$

$$= 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty \mathbf{a}_{\widetilde{F}}(n)\mathbf{a}_{\mathbf{f}}(n)n^{-s} = 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty a_n(F)a_n(f)n^{-s}.$$

Here the identity $(*)$ involves the interchange: $\sum_{\alpha,\beta\in\mathbb{Q}}\int_{\mathbb{A}^\times/\mathbb{Q}^\times} = \int_{\mathbb{A}^\times/\mathbb{Q}^\times}\sum_{\alpha,\beta\in\mathbb{Q}}$, which is justified as long as the two sides are absolutely convergent. Thus we have

Theorem 2.3. *Let the notation be as above. Suppose* (Φ1–3) *and* (V). *Then*

$$\int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})/\widehat{\Gamma}_0(M)C_\infty} \mathbf{f}(g)\widetilde{\mathbf{F}}(g)E(\Phi)(g)d\mu_g = 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty a_n(F)a_n(f)n^{-s}.$$

2.5. **Rankin product of mixed weight.** For a function $\phi : \mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ and a compact subgroup K of $\mathrm{Mp}(\mathbb{A})$, if $\phi(gu) = \phi(g)\chi(u)$ $(u \in K)$ for a character $\chi : K \to \mathbb{C}^\times$, the character χ is called the K-type of ϕ. If there is no such character, we say that ϕ has mixed K-type. If $K = C_\infty \cap \widetilde{\mathrm{SL}}_2(\mathbb{A})$, K is a two-fold covering of $\mathrm{SO}_2(\mathbb{R})$ isomorphic to S^1; so, for $c \in K$, we have $c^2 = r(\theta)$ with $r(\theta) = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$. If $\chi(c^2) = e^{k\theta\sqrt{-1}}$ for an integer k, we say that ϕ has weight $k/2$ if it has K-type given by χ as above. The integer k is odd if χ is non-trivial on $\mathrm{Ker}(\pi_{\mathbb{A}} : K \to \mathrm{SO}_2(\mathbb{R})) \cong \{\pm1\}$. This definition of K-type χ naturally extends to absolutely irreducible representations ρ of K. If the functions $\{\pi(u)\phi(g) = \phi(gu)\}_{u\in K}$ generate a K-representation space π which contains ρ as a direct constituent, we say ϕ has K-type ρ. The following remark will not be referred to in this paper.

Remark 2.4. The theta series $\theta_j(\chi)$ has mixed C_∞-type if $j > 1$. We can decompose $\theta_j(\chi)$ as a sum of non-holomorphic modular forms with C_∞-type using a formula relating Shimura–Maass differential operators and Ramanujan differential operators $q\frac{d}{dq}$ [LFE, Chapter 10, (3)]. Since we do not need the exact formula, we do not recall the formula.

For an open subgroup $\widehat{\Gamma}$ of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ and a right $\widehat{\Gamma}$-invariant function $f : \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\widehat{\mathbb{Z}}) \to \mathbb{C}$, we define the trace $\mathrm{Tr}_{\widehat{\Gamma}}(f)(g) = \sum_{u\in\mathrm{SL}_2(\widehat{\mathbb{Z}})/\widehat{\Gamma}} f(gu)$ which is a right $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-invariant function. We apply also this operator to $f : \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A}) \to \mathbb{C}$ as long as f factors through the quotient $\mathrm{SL}_2(\mathbb{A})$. For two open compact subgroups $\widehat{\Gamma}$ and $\widehat{\Gamma}'$, we have

$$(\mathrm{SL}_2(\widehat{\mathbb{Z}}) : \widehat{\Gamma})^{-1}\mathrm{Tr}_{\widehat{\Gamma}}(f) = (\mathrm{SL}_2(\widehat{\mathbb{Z}}) : \widehat{\Gamma}')^{-1}\mathrm{Tr}_{\widehat{\Gamma}'}(f)$$

as long as the two sides are well defined; so, we just write the condition $\mathrm{Tr}_{\widehat{\Gamma}}(f) = 0$ as $\mathrm{Tr}(f) = 0$ choosing $\widehat{\Gamma}$ sufficiently small.

Now suppose that we have a finite set of cusp forms $\{\mathbf{f}_j\}$ on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$ which have $\widehat{\Gamma}_0(M)$-type ψ^* but may have a mixed C_∞-type. We assume the index set of j is integers $[0,k]\cap\mathbb{Z}$ for $0 < k \in \mathbb{Z}$. We assume that the right translations of $\mathbf{f}_j$ by C_∞ span a finite dimensional space of functions on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$. Thus $\mathbf{f}_j$ is a finite sum of cusp forms of different C_∞-types. For example, $\mathbf{f}_j$ can be $\boldsymbol{\theta}_{\chi,j}$ for $0 \le j$. We suppose to have $\{\Phi_j\}_j$ satisfying (V), (Φ1) and (Φ3) but possibly not (Φ2). Instead we suppose

- (Φ0) $\mathrm{Tr}(\widetilde{\mathbf{F}}\sum_j \mathbf{f}_j E(\Phi_j)) \neq 0$ (i.e., the $\mathrm{SL}_2(\mathbb{A})$-representation generated by the right translations of $\widetilde{\mathbf{F}}\sum_j \mathbf{f}_j E(\Phi_j)$ by $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ contains the trivial representation of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$),
- (F) $\mathbf{f}_0|_{B(\mathbb{A})}$ has Fourier expansion as in (2.9) for the index 0.

The condition ($\Phi 0$) means the following matching condition:

(M) the irreducible cuspidal automorphic representation $\pi_{\mathbf{F}}$ generated by $\widetilde{\mathbf{F}}$ has $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-type which is a contragredient of one of the $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-types of the automorphic representation generated by the set $\{\mathbf{f}_j E(\Phi_j)\}_j$.

Example 2.1. Here is an example of $\{\mathbf{f}_j\}_j$ and $\{\Phi_j\}_j$ satisfying (Φ0–1) and (Φ3) with (F). Let (V,Q) be a quadratic space with a decomposition $(V,Q)=(\mathbb{Q},x^2)\oplus(W,Q')$ as a quadratic space. Take a Schwartz–Bruhat function ϕ on $V_{\mathbb{A}}$ and suppose that the Siegel–Weil theta series $\theta(\phi)$ has $\widehat{\Gamma}_0(M)$-type ψ^* and a unique C_∞-type k.

(1) $\phi=\phi^{(\infty)}\otimes\phi_\infty$ for a Bruhat function $\phi^{(\infty)}$ and a Schwartz function ϕ_∞,
(2) $\phi_\infty=\sum_j \phi_{\mathbb{Q},j}\otimes\phi_{W,j}$ for Schwartz functions $\phi_{\mathbb{Q},j}$ and $\phi_{W,j}$ on $\mathbb{R}$ and $W(\mathbb{R})$, respectively,
(3) $\phi^{(\infty)}=\phi^{(\infty)}_{\mathbb{Q}}\otimes\phi^{(\infty)}_W$ for Bruhat functions on $\mathbb{A}^{(\infty)}$ and $W(\mathbb{A}^{(\infty)})$, respectively,
(4) $\phi_{\mathbb{Q},j}(x)=x^j\exp(-\pi x^2)$.

Let $\phi^X_j:=\phi^{(\infty)}_X\otimes\phi_{X,j}$ for $X=\mathbb{Q},W$. We have a natural diagonal embedding of $\mathrm{O}_{\mathbb{Q}}\times\mathrm{O}_W$ into O_V. Note that $\mathrm{O}_{\mathbb{Q}}=\{\pm1\}$; so, we forget about it. Then we write $\theta(\phi)(g,h)=\sum_{j=0}^k\theta(\phi^{\mathbb{Q}}_j)(g)\theta(\phi^W_j)(g,h)$ for $g\in\mathrm{Mp}(\mathbb{A})$ and $h\in\mathrm{O}_W(\mathbb{A})$. Consider

$$\int_{\mathrm{O}_W(\mathbb{Q})\backslash\mathrm{O}_W(\mathbb{A})}\theta(\phi^W_j)(g,h)d\mu(h).$$

By the Siegel–Weil formula, for $\Phi_j(g)=(\mathbf{w}_W(g)\phi^W_j)(0)$ as a function on $B(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$, this integral is proportional to the Eisenstein series $E(\Phi_j)$. Ignoring the proportion,

$$\begin{aligned}\Theta(g)&:=\int_{\mathrm{O}_W(\mathbb{Q})\backslash\mathrm{O}_W(\mathbb{A})}\theta(\phi)(g,h)d\mu(h)\\&=\sum_j\theta(\phi^{\mathbb{Q}}_j)(g)\int_{\mathrm{O}_W(\mathbb{Q})\backslash\mathrm{O}_W(\mathbb{A})}\theta(\phi^W_j)(g,h)d\mu(h)=\sum_j\theta(\phi^{\mathbb{Q}}_j)(g)E(\Phi_j)(g)\end{aligned}$$

is a modular form whose $\widehat{\Gamma}_0(M)$-type is given by ψ^* and has weight k as C_∞-type. However $\theta(\phi^{\mathbb{Q}}_j)$ with $j>1$ does not have a C_∞-type. We compute $\sum_j\theta(\phi^{\mathbb{Q}}_j)(gu)E(\Phi_j)(gu)$ for $u\in C_\infty$:

$$\begin{aligned}\sum_j\theta(\phi^{\mathbb{Q}}_j)(gu)E(\Phi_j)(gu)&=\int_{\mathrm{O}_W(\mathbb{Q})\backslash\mathrm{O}_W(\mathbb{A})}\theta(\phi)(gu,h)d\mu(h)\\&=\int_{\mathrm{O}_W(\mathbb{Q})\backslash\mathrm{O}_W(\mathbb{A})}\theta(\phi)(g,h)j(u,i)^{-k}d\mu(h)=\sum_j\theta(\phi^{\mathbb{Q}}_j)(g)E(\Phi_j)(g)j(u,i)^{-k}.\end{aligned}$$

Thus choosing a cusp form $\mathbf{F}$ with the inverse $\widehat{\Gamma}_0(M)C_\infty$-type and putting $\mathbf{f}_j=\theta(\phi^{\mathbb{Q}}_j)$, the pair $\{\mathbf{f}_j,\Phi_j\}_j$ satisfies the required conditions (Φ0) and (F).

If the automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ generated by the theta descent

$$\theta_*(f):=\int_{\mathrm{O}_V(\mathbb{Q})\backslash\mathrm{O}_V(\mathbb{A})}\theta(\phi)(g,h)f(h)d\mu(h)$$

of any cusp form f on $D^\times_{\mathbb{A}}$ does not contain the automorphic representation $\pi_{\mathbf{F}}$, the matching condition (M) could fail. For example, if $\mathbf{F}$ has level 1 (i.e., principal everywhere) and $V=D$ for a division quaternion ramified at a prime q with a Bruhat function $\phi^{(\infty)}$ of $\widehat{R}$ for a maximal order R of D, irreducible factors of the representation generated by $\theta_*(f)$ is special at q, and by the new form theory, it does not have the identity representation as $\mathrm{SL}_2(\mathbb{Z}_q)$-type; so, $\mathrm{Tr}(\widetilde{\mathbf{F}}\theta_*(f))=0$. Since any $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-type of $\sum_j\theta(\phi_j)E(\Phi_j)$ would appear as one of the $\mathrm{SL}_2(\widehat{\mathbb{Z}})$-type of the theta descent $\theta_*(f)$ for some f, perhaps this implies the failure of the matching condition (M); i.e., the theta lift of $\pi_{\mathbf{F}}$ by $\theta(\phi)$ would vanish, and also the Rankin product (2.21) by the identity at (2). Another caution is that, to have (V), we need to choose ϕ_j carefully.

Noting the fundamental domain $\mathcal{F}$ is chosen in $B(\mathbb{R})$, we reverse the earlier computation in (2.19):

$$\int_{B(\mathbb{Q})\backslash B(\mathbb{A})/(B(\mathbb{A})\cap\widehat{\Gamma}_0(M)C_\infty)} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)\Phi_j(g)d\mu_g = \int_{\cup_\gamma \gamma\mathcal{F}} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)\Phi_j(g)d\mu_g \tag{2.21}$$

$$= \int_{\mathcal{F}} \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \widetilde{\mathbf{F}}(\gamma g)\sum_j \mathbf{f}_j(\gamma g)\Phi_j(\gamma g)d\mu_g = \int_{\mathcal{F}} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g) \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} \Phi_j(\gamma g)d\mu_g$$

$$\overset{(1)}{=} \int_{X_0(M)} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)E(\Phi_j)(g)d\mu_g \overset{(2)}{=} \int_{X_0(1)} \mathrm{Tr}(\widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)E(\Phi_j)(g))d\mu_g.$$

Before reaching identity (1), the computation is done inside $B(\mathbb{A})$, and we do not need left C_∞-invariance of $\widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)\Phi_j(g)$. The integral is extended from $B(\mathbb{A})$ to entire $\mathrm{Mp}(\mathbb{A})$ by the left $\mathrm{SL}_2(\widehat{\mathbb{Z}})C_\infty$-invariance of $\mathrm{Tr}(\widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)E(\Phi_j)(g))$ in (Φ0), and we replace $\mathcal{F}$ by the isomorphic

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})/\widehat{\Gamma}_0(M)C_\infty = \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})/\widehat{\Gamma}_0(M)\mathrm{SO}_2(\mathbb{R}) = X_0(M)$$

at the identity (1).

Theorem 2.5. *In addition to* (Φ0–1), (V) *for all* Φ_j *and* (F), *assume*

(Key) $\Phi_0|_{B(\mathbb{A})}(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)) = \chi(a)|a|_{\mathbb{A}}^{2s}$ *with a character* $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ *for* $a \in \mathbb{A}^\times$ *and* $b \in \mathbb{A}$ *and* $\Phi_j|_{B(\mathbb{A})} = 0$ *if* $j \neq 0$.

Then, assuming the Fourier expansion of $\mathbf{f}_0(g_\tau)$ *has the following form:* $\sum_{n=0}^\infty a_n(\mathbf{f}_0)\mathbf{e}(nz)$, *we have*

$$\int_{X_0(M)} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)E(\Phi_j)(g)d\mu_g = 2(4\pi)^{-s}\Gamma(s)\sum_{n=1}^\infty a_n(F)a_n(\mathbf{f}_0)n^{-s}.$$

Without the assumption (Key), the convolution integral would produce a sum of L-values (possibly evaluation points shifted). Thus (Key) is a purity condition in the mixed weight case.

Proof. By (2.21), we compute $\int_{B(\mathbb{Q})\backslash B(\mathbb{A})/(B(\mathbb{A})\cap\widehat{\Gamma}_0(M)C_\infty)} \widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_j(g)\Phi_j(g)d\mu_g$. By (Key), this integral is reduced to $\int_{B(\mathbb{Q})\backslash B(\mathbb{A})/(B(\mathbb{A})\cap\widehat{\Gamma}_0(M)C_\infty)} \widetilde{\mathbf{F}}(b)\mathbf{f}_0(b)|a(b)|_{\mathbb{A}}^{2s}d\mu(b)$. Then by the same computation as in (2.20), replacing $\mathbf{f}$ in (2.20) by $\mathbf{f}_0$, we get the desired formula. □

Note here $\mathbf{f}_0$ in the above proof can have a mixed C_∞-type. It can be a finite sum of cusp forms with different irreducible C_∞-type as in Remark 2.4.

3. Quadratic space over $\mathbb{Q}$

Let V be a finite dimensional $\mathbb{Q}$-vector space with a quadratic form $Q : V \to \mathbb{Q}$. The corresponding symmetric bilinear form s is given by $s(x, y) = Q(x + y) - Q(x) - Q(y)$ or equivalently $2Q(v) = s(v, v) =: s[v]$. For a $\mathbb{Q}$-algebra A, we write $V_A := V \otimes_{\mathbb{Q}} A$ as a quadratic space over A. We let the orthogonal group O_V of V act on V from the right.

Let $E_{/\mathbb{Q}}$ be a semi-simple quadratic extension $\mathbb{Q}[\sqrt{\Delta}]$ with integer ring O_E and D be a quaternion algebra over $\mathbb{Q}$. If $E = \mathbb{Q}\times\mathbb{Q}$, we take $\sqrt{\Delta} = (1, -1)$ and hence $\Delta = 1 \in \mathbb{Q}$. Write $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$ for the non-trivial $\mathbb{Q}$-algebra automorphism of $E_{/\mathbb{Q}}$. Put $D_E = D \otimes_{\mathbb{Q}} E$, and extend σ to D_E by $\sigma(d \otimes e) = d \otimes e^\sigma$ for $d \in D$ and $e \in E$. We write $N : D_E \to E$ (resp. $\mathrm{Tr} : D_E \to E$) for the reduced norm (resp. trace) map which induces the reduced norm (resp. trace) map $D \to \mathbb{Q}$. The main involution of D_E is denoted by $x \mapsto x^\iota$; so, $x^\iota = \mathrm{Tr}(x) - x = N(x)x^{-1}$. Define $D_\sigma = D_\sigma^\pm := \{x \in D_E | x^\sigma = \pm x^\iota\}$. We fix a pair of maximal orders $R \subset D$ and $R_E \subset D_E$ such that $R \otimes_{\mathbb{Z}} O_E \subset R_E$. Let K be a quadratic subfield of D. We write $\mathbb{H}$ for the Hamilton quaternion algebra over $\mathbb{R}$.

3.1. List of quadratic spaces we study.

We study the following low dimensional quadratic spaces $(V_{/\mathbb{Q}}, Q)$. Here we write as before m for the dimension of the quadratic space V over $\mathbb{Q}$.

($\mathrm{D}^\pm$) Let $V = D$ and $Q(x) = \pm xx^\iota = \pm N(x)$ (for the reduced norm $N : D \to \mathbb{Q}$ and the main involution ι). Then $s(x, y) = \pm\mathrm{Tr}(xy^\iota)$. In this case, we have $m = 4$, and usually we assume the sign to be + (we write $D^\pm$ for D if we need to indicate the sign). More explicitly, we have an expression $D = \left\{\left(\begin{smallmatrix} a & b \\ \partial b^\varsigma & a^\varsigma\end{smallmatrix}\right) | a, b \in K\right\}$ with $0 \neq \partial \in \mathbb{Z}$ for a quadratic field $K/\mathbb{Q}$ with

$\langle\varsigma\rangle = \mathrm{Gal}(K/\mathbb{Q})$. Here ∂^2 is the discriminant of $(D_{/\mathbb{Q}}, Q)$. The maximal order R of D we fixed is assumed to satisfy $R \hookrightarrow M_2(O_K)$ by the above embedding. We let $(\alpha,\beta) \in D^\times \times D^\times$ act from the right on D by $x \mapsto \alpha^{-1}x\beta$; so, $N(\alpha^{-1}x\beta) = N(\alpha)^{-1}N(\beta)N(x)$. Let $G_D^+(\mathbb{Q}) := \{(\alpha,\beta) \in D^\times \times D^\times | N(\alpha) = N(\beta)\}$. Then by this action $x \mapsto \alpha^{-1}x\beta$ and $N(\alpha) = N(\beta)$, we have a morphism $\varrho = \varrho_{\mathbb{Q}\times\mathbb{Q}} : G_D^+ \to \mathrm{O}_D$ for the orthogonal group O_D of (D,Q). Sometimes, we also use a slightly different action $x \mapsto \alpha^\iota x\beta$.

($\mathrm{D}_\sigma^\pm$) Let $V = D_\sigma^\pm := \{x \in D_E | x^\sigma = \pm x^\iota\}$ and $Q(x) = xx^\sigma = \pm xx^\iota = \pm N(x) \in \mathbb{Q}$ $(x \in D_\sigma^\pm)$; then, $s(x,y) = s_\pm(x,y) = \pm\mathrm{Tr}_{D_E/E}(xy^\iota) = \mathrm{Tr}_{D_E/E}(xy^\sigma) \in \mathbb{Q}$. We have $m = 4$. Indeed, over $\mathbb{C}$, we can identify $D_E \otimes_{\mathbb{Q}} \mathbb{C} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ with σ interchanging the components $M_2(\mathbb{C})$, and we have $D_\sigma \otimes_{\mathbb{Q}} \mathbb{C} = \{(X, \pm X^\iota) \in M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) | X \in M_2(\mathbb{C})\}$ which has dimension 4 over $\mathbb{C}$. We may let $\alpha \in D_E^\times$ act on $D_\sigma^\pm$ by $x \mapsto \alpha^\iota x\alpha^\sigma$, as

$$(\alpha^\iota x\alpha^\sigma)^\sigma = \alpha^{\iota\sigma}x^\sigma\alpha = \pm\alpha^{\iota\sigma}x^\iota\alpha = \pm(\alpha^\iota x\alpha^\sigma)^\iota.$$

This action preserves V and Q up to scalar $N(\alpha)N(\alpha)^\sigma \in \mathbb{Q}$. If $N(\alpha) \in \mathbb{Q}$, we can modify slightly the action by

$$(\alpha^{-1}x\alpha^\sigma)^\sigma = N(\alpha)^{-1}\alpha^{\iota\sigma}x^\sigma\alpha = \pm N(\alpha)^{-1}\alpha^{\iota\sigma}x^\iota\alpha = \pm(\alpha^{-1}x\alpha^\sigma)^\iota.$$

Let $G_{D_\sigma}^+(\mathbb{Q}) := \{\alpha \in D_E^\times | N(\alpha) \in \mathbb{Q}^\times\}$. By $x \mapsto \alpha^{-1}v\alpha^\sigma$ and $N(\alpha) = N(\alpha)^\sigma$ ($\Leftrightarrow N(\alpha) \in \mathbb{Q}^\times$), we have a morphism $\varrho_E : G_{D_\sigma}^+(\mathbb{Q}) \to \mathrm{O}_{D_\sigma}$ of algebraic groups. Regard $\mathrm{Gal}(KE/E) = \langle\varsigma\rangle$ and $\mathrm{Gal}(KE/K) = \langle\sigma\rangle$. We have $\mathrm{Gal}(KE/\mathbb{Q}) = \langle\sigma\rangle \times \langle\varsigma\rangle$. Write L for the fixed field of $\sigma\varsigma$ and $K = \mathbb{Q}[\sqrt{d(K)}]$. Then $L = \mathbb{Q}[\sqrt{d(K)\Delta}]$. Then $D_\sigma^+ = \left\{\left(\begin{smallmatrix} a & b \\ \partial b^\varsigma & a^\varsigma \end{smallmatrix}\right) | b \in \sqrt{\Delta}K, a \in L\right\}$ and $D_\sigma^- = \left\{\left(\begin{smallmatrix} a & b \\ \partial b^\varsigma & a^\varsigma \end{smallmatrix}\right) | b \in K, a \in \sqrt{\Delta}L\right\}$. Since σ and ι are involutions commuting each other, they act on $D_\sigma^\pm$.

($\mathrm{D}_0^\pm$) Let $D_0^\pm = \{x \in D_\sigma^\pm | \mathrm{Tr}(x) = x + x^\iota = 0\}$ and $Q^\pm(x) = xx^\sigma = \pm N(x)$. We have $D_0^+ = \sqrt{\Delta}D_0^- \subset D_E$. Then $s(x,y) = \pm\mathrm{Tr}(xy^\iota)$. In this case, we have $m = 3$. We let $D^\times$ act on D_0 by $x \mapsto \alpha^{-1}x\alpha$. By this action, we have a morphism $\varrho_0 : G_{D_0}^+(\mathbb{Q}) := D^\times \to \mathrm{O}_{D_0}$ of algebraic groups.

($\mathrm{Z}^\pm$) Write Δ_- for the square-free part of Δ, and put $\Delta_+ = 1$. Here we assume that $\Delta_- > 0$ if $\Delta > 0$ and $\Delta_- < 0$ if $\Delta < 0$. Let $Z^\pm = \{x \in D_\sigma^\pm | x^\iota = x\} = \delta_\pm\mathbb{Q}$ for $\delta_\pm = \sqrt{\Delta_\pm}$ with $s^\pm(\delta_\pm x, \delta_\pm y) = \mathrm{Tr}(\delta_\pm x(\delta_\pm y)^\sigma) = \pm 2\delta_\pm^2 xy$ (so, $Q^\pm(\delta_\pm x) = \pm\delta_\pm^2 x^2$). Then $Z^+ = \mathbb{Q} \subset D_\sigma^+$ and $Z^+ = \mathbb{Q}\sqrt{\Delta} \subset D_\sigma^-$. Here $Z^\pm = D_\sigma \cap Z(D_E)$ for the center $Z(D_E)$ of D_E. The space (Z^+, Q^+) is positive definite, and (Z^-, Q^-) is either positive definite or negative definite according to whether E is imaginary or real.

The list include all isomorphism classes of 4-dimensional quadratic spaces over $\mathbb{Q}$. We record

$$\delta = \delta_+ = 1 \text{ for } D_\sigma^+ \text{ and } \delta = \delta_- = \sqrt{\Delta_-} \in E \text{ for } D_\sigma^-. \tag{3.1}$$

Cases $\mathrm{D}^\pm$ and $\mathrm{D}_\sigma^\pm$ are not disjoint. Indeed, if we take $E = \mathbb{Q} \times \mathbb{Q}$, we find $D_E = D \times D$ with σ interchanging two simple components, and $D_\sigma^\pm = \{(x, \pm x^\iota) | x \in D\} \cong D$ by $(x, \pm x^\iota) \mapsto x$. This identification is an identification of quadratic spaces $D_\sigma^\pm \cong D^\pm$. Because of this overlap, in this paper, we deal with $D_\sigma^\pm$, $D_0^\pm$ and $Z^\pm$ without losing the case $D^\pm$.

Here is the list of the signature of D_σ. We say E is real if $E = \mathbb{Q}[\sqrt{\Delta}]$ with $0 < \Delta \in \mathbb{Z}$ and this case include $E = \mathbb{Q} \times \mathbb{Q}$ with $\Delta = 1$ and $\sqrt{\Delta} = (1,-1)$. Otherwise, we say E is imaginary.

(RI) If E is real and D is indefinite, we find

$$D_\sigma^\pm \otimes_{\mathbb{Q}} \mathbb{R} = \{(X, \pm X^\iota) \in M_2(\mathbb{R}) \oplus M_2(\mathbb{R}) | X \in M_2(\mathbb{R})\} \cong M_2(\mathbb{R}).$$

Thus it has signature $(2,2)$.

(RD) If E is real and D is definite, then

$$D_\sigma \otimes_{\mathbb{Q}} \mathbb{R} = \{(X, \pm X^\iota) \in \mathbb{H} \oplus \mathbb{H} | X \in \mathbb{H}\} \cong \mathbb{H}$$

for Hamilton quaternion algebra $\mathbb{H}$; so, it has signature $(4,0)$ for $(D_\sigma^\pm, N)$ or $(0,4)$ for $(D_\sigma^\pm, -N)$.

(ID) If E is imaginary and $D\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{H}$, let $H:=\{x\in M_2(\mathbb{C})|xJ=J\overline{x}\}$. Then by computation, we have $H=\left\{\left(\begin{smallmatrix}a&b\\-\overline{b}&\overline{a}\end{smallmatrix}\right)\right\}\cong\mathbb{H}$. Defining $\sigma_J(x):=Jx^\sigma J^{-1}$, σ_J gives an involution of $M_2(\mathbb{C})$ such that $H^0(\langle\sigma_J\rangle,M_2(\mathbb{C}))=\mathbb{H}$. Then

$$D^{\pm}_{\sigma_J}=\left\{\begin{pmatrix}\sqrt{\pm1}a&b\\\pm\overline{b}&\sqrt{\pm1}d\end{pmatrix}\Big|a,d\in\mathbb{R},b\in\mathbb{C}\right\}\tag{3.2}$$

and $N\begin{pmatrix}\sqrt{\pm1}a&\pm b\\\overline{b}&\sqrt{\pm1}d\end{pmatrix}=\pm ad\mp\overline{b}b$. Thus $Q^\pm$ has signature $(1,3)$ on $D^\pm_{\sigma_J}$.

(II) If E is imaginary and $D_{\mathbb{R}}:=D\otimes_{\mathbb{Q}}\mathbb{R}\cong M_2(\mathbb{R})$, moving by an inner automorphism of $M_2(\mathbb{C})$, we may assume $D_{\mathbb{R}}=M_2(\mathbb{R})\subset M_2(\mathbb{C})$. Thus $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^\sigma=\left(\begin{smallmatrix}\overline{a}&\overline{b}\\\overline{c}&\overline{d}\end{smallmatrix}\right)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in M_2(\mathbb{C})=D_{\mathbb{C}}$. When we need to distinguish the action of $\mathrm{Gal}(E/\mathbb{Q})$ in the two cases ID and II, we write the action of σ as σ_1 in Case II and σ_J in Case ID. Then we find

$$D^\pm_\sigma\otimes_{\mathbb{Q}}\mathbb{R}=\{\overline{X}=\pm X^\iota|X\in M_2(\mathbb{C})\}=\left\{\begin{pmatrix}a&\sqrt{\mp1}b\\\sqrt{\mp1}c&\pm\overline{a}\end{pmatrix}\Big|a\in\mathbb{C},b,c\in\mathbb{R}\right\}.\tag{3.3}$$

Thus the signature is $(3,1)$ for $(D^\pm_\sigma,\pm N)$ and $(1,3)$ for $(D^\pm_\sigma,\mp N)$.

In Case II, on $D_\sigma\otimes_{\mathbb{Q}}\mathbb{R}$, $g\in(D\otimes_{\mathbb{Q}}\mathbb{R})^\times\cong\mathrm{GL}_2(\mathbb{C})$ with $N(g)\in\mathbb{R}$ acts by $x\mapsto g^{-1}xg^\sigma$ which preserves $s_\pm$. Thus $\mathrm{PSL}_2(\mathbb{C})$ is isomorphic to $\mathrm{SO}(3,1)$.

Let $A^\pm(V)=A^\pm(V,Q)$ for the set of even or odd elements of the Clifford algebra $A(V)$ of V as defined in [AQF, §23]. Write the graded Clifford algebra as $A^\bullet(V):=A^+(V)\oplus A^-(V)$. Put

$$G_V=\{\alpha\in A(V)^\times|\alpha^{-1}V\alpha=V\}$$

as an algebraic group, and set $G_V^\pm=G_V\cap A^\pm(V)$. By sending $\alpha\in G_V$ to $\varrho_V(\alpha)\in\mathrm{Aut}(V)$ given by $\varrho_V(\alpha)(v)=\alpha^{-1}v\alpha$, we have a morphism $\varrho_V:G_V\to\mathrm{O}_V$ by the construction of the Clifford algebra [AQF, (24.1c)]. Then $G_V^\bullet=G_V^+\sqcup G_V^-$ is a subgroup of G_V.

In the following statements, let A denote either a field extension of $\mathbb{Q}$ or the (finite or full) adele ring of a number field. We have (cf. [MSS, §8.5.3], [HMI, Proposition 2.65] and [AQF, §24–25])

- If $(V,Q)=(D,N)$ for the reduced norm $N:D\to\mathbb{Q}$, $\chi_D=1$ and $A^+(D)=D\times D$ [AQF, Theorems 23.8 and 25.4]. The morphism ϱ_D induces surjections $G_D^\bullet(A)\twoheadrightarrow\mathrm{O}_D(A)$ and $G_D^+(A)\twoheadrightarrow\mathrm{SO}_D(A)$ [AQF, Theorem 24.6]. In this case $d(D)=\partial^2=1$ modulo square.
- If $(V,Q)=(D^\pm_\sigma,\pm N)$, writing the discriminant of D (resp. E,K) as ∂^2 (resp. Δ,d_K) with $0<\partial,\Delta,d_K\in\mathbb{Z}$ (so, $E=\mathbb{Q}[\sqrt{\Delta}]$), $\chi_{D_\sigma}=\left(\frac{\mathbb{Q}[\sqrt{\Delta\partial}]/\mathbb{Q}}{}\right)$. Indeed, by the expression of D_σ as in (D_σ), for the fixed field $L=\mathbb{Q}[\sqrt{\Delta d_K}]$ in KE of $\sigma\varsigma$, $(D^+_\sigma,N)\cong(K,-\partial\Delta N_{K/\mathbb{Q}})\oplus(L,N_{L/\mathbb{Q}})$ and $(D^-_\sigma,-N)\cong(K,\partial N_{K/\mathbb{Q}})\oplus(L,-\Delta N_{L/\mathbb{Q}})$. Thus $d(D_\sigma)=\Delta^3d_K^2\partial^2=\Delta$ modulo square. Thus $\chi_{D_\sigma}=\chi_E$. We also have $A^+(D_\sigma)=D_E$ [AQF, Theorem 23.8]. The morphism ϱ_{D_σ} induces surjections $G^\bullet_{D_\sigma}(A)\twoheadrightarrow\mathrm{O}_{D_\sigma}(A)$ and $G^+_{D_\sigma}(A)\twoheadrightarrow\mathrm{SO}_{D_\sigma}(A)$ with kernel given by the center [AQF, Theorem 24.6], $\mathrm{O}_{D_\sigma}=\mathrm{SO}_{D_\sigma}\sqcup\mathrm{SO}_{D_\sigma}\iota$, and $G^+_{D_\sigma}(\mathbb{Q})=\{\alpha\in D_E^\times|N(\alpha)\in\mathbb{Q}\}$ is the even Clifford group of D_σ [AQF, Theorem 24.6].
- If $(V,Q)=(Z^\pm,\pm N)$, we put $\chi_Z\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\epsilon_d^{-1}\left(\frac{c}{d}\right)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(4)$ (e.g., [S75, Lemma 1.2]). where $\left(\frac{c}{d}\right)$ is the Legendre symbol as in §1.3 and $\epsilon_d=1$ if $d\equiv1\mod 4$ and $\sqrt{-1}$ if $d\equiv3\mod 4$. We have $d(Z^\pm)=\pm(\delta_\pm)^2$ and $\chi_{Z^-}=\chi_{Z^+}\left(\frac{\Delta_-}{}\right)=\left(\frac{\Delta_-}{}\right)$ [Sh73, Proposition 1.3].
- Let $(V,Q)=(D_0^\pm,\pm N)$. We have $A^+(D_0)=D$ [AQF, Lemma 25.2], and the morphism $\varrho_0=\varrho_{D_0}$ induces a surjection $G^+_{D_0}(A)\twoheadrightarrow\mathrm{SO}_{D_0}(A)$ [AQF, Theorem 24.6] whose kernel is the center of $G^+_{D_0}=D^\times$. We have $d(D_0^\pm)=d(D_\sigma)d(Z^\pm)^{-1}=\pm\Delta_-$ modulo square.

Remark 3.1. When m is odd, we find $d(V,zQ)=z^md(V,Q)$ for $z\in\mathbb{Q}^\times$. Since GO_V, O_V, SO_V and G_V^+ are equal for all scalar multiple of Q, for the statement concerning these groups, if $m=\dim V$ is odd, we may assume that $d(V)=1$ modulo square.

3.2. **Choices of D for a fixed D_E.** The Shimura variety of $D^\times_{/\mathbb{Q}}$ gives rise to a subvariety of the Shimura variety or the automorphic manifold of $D^\times_{E/\mathbb{Q}}:=\mathrm{Res}_{E/\mathbb{Q}}D_E^\times$. If we fix (the isomorphism class of) D_E, there are many choices of D which gives rise to the given D_E. They produce different cycles over which we compute the period. We fix D_E and study quaternion sub-algebras of D_E.

We discuss slightly more generally for a while. Let D be a quaternion algebra over a field E_+ of characteristic 0 and for a semi-simple quadratic extension E/E_+, we put $D_E = D \otimes_{E_+} E$. Write σ for the generator of $\mathrm{Aut}(E/E_+)$ and let it act on D_E through the factor E. Thus if $E = E_+ \times E_+$, $\sigma(x, y) = (y, x)$ for $x, y \in E_+$. Define $D^{\pm}_{\sigma} = \{x \in D_E | x^{\sigma} = \pm x^{\iota}\}$.

If we do not need to refer to the sign ϵ defining $D^{\pm}_{\sigma}$, we just write D_σ for $D^{\pm}_{\sigma}$. Pick $\alpha \in D_\sigma \cap D^{\times}_E$. Consider $x^{\sigma_\alpha} = \alpha x^{\sigma}\alpha^{-1}$. Then $(x^{\sigma_\alpha})^{\sigma_\alpha} = \alpha(\alpha^{\sigma}(x^{\sigma})^{\sigma}\alpha^{-\sigma})\alpha^{-1} = \alpha(\epsilon\alpha^{\iota}x\epsilon\alpha^{-\iota})\alpha^{-1} = x$. Thus we get a new action of $\mathrm{Gal}(E/E_+)$ on D_E. Then $D_\alpha := H^0(E/E_+, D_E)$ under this new action is a quaternion algebra over $\mathbb{Q}$, and plainly $D_E = D_\alpha \otimes_{E_+} E$. Often $D_\alpha \not\cong D$. Thus the Shimura variety Sh associated to $D^{\times}_E$ has the Shimura subvariety Sh_α associated to $D^{\times}_\alpha$. If $x \in D_\alpha$, then $x^{\sigma_\alpha} = \alpha x^{\sigma}\alpha^{-1} = x$. Thus we have the following expression

$$D_\alpha := \{x \in D_E | x\alpha = \alpha x^{\sigma}\} \text{ and } D^{\times}_\alpha := \{x \in D^{\times}_E | x\alpha x^{\sigma\iota} = N(x)\alpha\}.$$

Lemma 3.2. *Let the notation be as above.*

1. *If B is a central-simple E_+-subalgebra of D_E of dimension 4, then there exists $\alpha \in D_\sigma \cap D^{\times}_E$ such that $B = D_\alpha$.*
2. *We have $\alpha = \xi\beta\xi^{\iota\sigma}$ for $\beta \in D_\sigma \cap D^{\times}_E$ and $\xi \in D^{\times}_E$ if and only if $D_\alpha \cong D_\beta$ as a quaternion algebra over E_+, and in this case, we have $D_\alpha = \xi D_\beta \xi^{-1}$ inside D_E.*
3. *We have $D_\alpha = D$ if and only if $\alpha \in E^{\times} \cap D_\sigma$.*

Proof. Pick a quaternion E_+-subalgebra $B \subset D_E$. Then we have an action of $\sigma \in \mathrm{Gal}(E/E_+)$ on D_E such that $H^0(E/E_+, D_E) = B$. Identify $D_E = B \otimes_{E_+} E$ and write the Galois action of this expression as σ_α; i.e., we have $(b \otimes e)^{\sigma_\alpha} = b \otimes e^{\sigma}$ for $b \in B$ and $e \in E$. Then $x \mapsto (x^{\sigma})^{\sigma_\alpha}$ is an E-linear automorphism of D_E, which is inner. Thus there exists $\alpha \in D^{\times}_E$ such that $x^{\sigma_\alpha} = \alpha x^{\sigma}\alpha^{-1}$ for all $x \in D^{\times}_E$. Since $(x^{\sigma_\alpha})^{\sigma_\alpha} = x$, we find $\alpha^{\sigma}\alpha$ is in the center of D_E and hence $\alpha\alpha^{\sigma} \in E^{\times}$. Therefore $(\alpha^{\sigma}\alpha)\alpha = \alpha(\alpha^{\sigma}\alpha)$, and dividing by α from the right, we conclude α^{σ} commutes with α. Then $(\alpha\alpha^{\sigma})^{\sigma} = \alpha^{\sigma}\alpha = \alpha\alpha^{\sigma}$. This shows $\alpha\alpha^{\sigma} \in E_+$. Thus $\alpha^{\sigma} = z\alpha^{\iota}$ for $z \in E^{\times}_+$ and hence $\alpha^{\sigma\iota} = \alpha z = z\alpha$. Therefore α in the E_+-vector space D_E is an eigen vector of the E_+-linear map $\sigma\iota$ with eigenvalue z. Since $\sigma\iota$ has order 2, we have $z = \pm 1$ and $\alpha^{\sigma} = \pm\alpha^{\iota}$.

Since $E = E_+[\sqrt{\Delta}]$ for $\Delta \in E^{\times}_+$, if the sign of z does not match with the sign of $D^{\pm}_{\sigma}$, for $\sqrt{\Delta} \in E^{\times}$ with $\sqrt{\Delta}^{\sigma} = -\sqrt{\Delta}$, we have $\sqrt{\Delta}\alpha$ has matching parity. Replacing α by $\sqrt{\Delta}\alpha$, we may assume that $\alpha^{\sigma} = \pm\alpha^{\iota}$, and we have $B = D_\alpha$. In this way, every quaternion E_+-subalgebra of D_E appears as D_α. This proves (1).

We now prove (2). We may assume that $\beta \in D_\sigma \cap E^{\times}$. Plainly $D_\beta = D$. Suppose $\alpha = \xi\beta\xi^{\iota\sigma}$ for $\xi \in D^{\times}_E$ for $0 \neq \beta \in E^{\times} \cap D_\sigma$. Then

$$\gamma \in D_\alpha \Leftrightarrow \gamma\alpha = \alpha\gamma^{\sigma} \Leftrightarrow \gamma\xi\beta\xi^{\iota\sigma} = \xi\beta\xi^{\iota\sigma}\gamma^{\sigma} \Leftrightarrow \xi^{-1}\gamma\xi\beta = \beta\xi^{\iota\sigma}\gamma^{\sigma}\xi^{-\iota\sigma} \Leftrightarrow \xi^{-1}\gamma\xi\beta = \beta(\xi^{-1}\gamma^{\sigma}\xi)^{\sigma}.$$

This shows that $D_\alpha = \xi D_\beta \xi^{-1}$.

To see the converse, suppose we have an isomorphism $i : D \cong B = D_\alpha$ of E_+-algebras. Then we can identify $D_E = B \otimes_{E_+} E$; so, i is induced by $x \mapsto \xi x \xi^{-1}$ for $\xi \in D_E$.

The assertion (3) just follows from the definition. □

We state Lemma 3.2 (2) in a different way, whose proof we leave to the reader.

Corollary 3.3. *Let $\sigma_\alpha(x) = \alpha x^{\sigma}\alpha^{-1}$ and $D^{\pm}_{\sigma_\alpha} = \{v \in D_E | v^{\sigma_\alpha} = \pm v^{\iota}\}$. The following three conditions are equivalent:*

- *We have $\alpha = \xi\beta\xi^{\iota\sigma}$ for $\beta \in D_\sigma \cap D^{\times}_E$ and $\xi \in D^{\times}_E$;*
- *$D^{\pm}_{\sigma_\alpha} \cong D^{\pm}_{\sigma_\beta}$ as a quadratic space over E_+, and in this case, we have $D^{\pm}_{\sigma_\alpha} = \xi D^{\pm}_{\sigma_\beta}\xi^{-1}$;*
- *$D^{\pm}_{\alpha,0} \cong D^{\pm}_{\beta,0}$ as a quadratic space over E_+ for $D^{\pm}_{\alpha,0} = \{v \in D^{\pm}_{\sigma_\alpha} | v + v^{\iota} = 0\}$.*

Here is an adelized version. We assume that E_+ is a finite extension of $\mathbb{Q}$ (i.e., a number field). Let $a \in D^{\pm}_{\sigma}$ with $N(a) \neq 0$. An $E_{+\mathbb{A}}$-subalgebra $B_{E_{+\mathbb{A}}}$ of $D_{E_{\mathbb{A}}}$ of rank 4 is called an adelic quaternion algebra over $E_{+\mathbb{A}}$ if its projection to D_{E_v} is a central simple quaternion algebra over E_{+v} for all places v of E_+ and is isomorphic to $M_2(E_{+v})$ for almost all v. Then define $D_a := \{x \in D_{E_{\mathbb{A}}} | xa = ax^{\sigma}\}$.

Lemma 3.4. (1) *If $B_{E_{+\mathbb{A}}}$ is an adelic quaternion $E_{+\mathbb{A}}$-subalgebra of $D_{E_{\mathbb{A}}}$, then there exists $a \in D_{\sigma,E_{+\mathbb{A}}}$ such that $B_{E_{+\mathbb{A}}} = D_a$.*

(2) *We have* $a = xbx^{\iota\sigma}$ *for* $0 \neq b \in D_\sigma$ *and* $x \in D_{E_{\mathbb{A}}}^\times$ *if and only if* $D_a \cong D_b$ *as an adelic quaternion algebra over* $E_{+\mathbb{A}}$*, and in this case, we have* $D_a = xD_bx^{-1}$*.*
(3) *We have* $D_a = D_{E_{+\mathbb{A}}}$ *if and only if* $a \in E_{\mathbb{A}}^\times \cap D_\sigma$*.*

The proof follows from Lemma 3.2 place by place. We leave it to the reader.

Assume $E_+ = \mathbb{Q}$. Let $V = D_{\alpha,0}^{\pm}$ with quadratic form $\pm N(v) = vv^\sigma = \pm vv^\iota$. Let

$$\mathrm{O}_\alpha = \mathrm{O}_{D_{\alpha,0}^{\pm}} \text{ and } \mathrm{SO}_\alpha = \mathrm{SO}_{D_{\alpha,0}^{\pm}} \text{ which are independent of the sign.} \tag{3.4}$$

If $z \in \mathbb{Q}$ is a scalar in O_α, then $z^2 = 1$ by $\pm z^2N(v) = \pm N(zv) = \pm N(v)$ for all $v \in V$. Thus the center $Z(\mathrm{O}_\alpha)(A) = \mu_2(A)$. Since V has dimension 3, $\det(z) = z^3$. Thus $\mathrm{SO}_\alpha(A) \cap Z(\mathrm{O}_\alpha) = \{1\}$.

Taking the associated symmetric matrix S for the symmetric form $s^\pm(x, y)$ on V, we find for $\alpha \in \mathrm{O}_\alpha$, ${}^t\alpha S\alpha = S$; so, $\det(\alpha)^2 = 1$. Thus we find $\mathrm{O}_\alpha/\mathrm{SO}_\alpha$ is embedded into μ_2 by det. Since $-1 \in Z(\mathrm{O}_\alpha)$ has determinant -1, we find $\mathrm{O}_\alpha \cong \mu_2 \times \mathrm{SO}_\alpha$.

We identify SO_{D_σ} as a quotient of $G_{D_\sigma}^+$ by ϱ_{D_σ}. If $D_E \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(E_{\mathbb{R}})$, we identify the two $E_{\mathbb{R}}$-algebras and let $G_{D_\sigma}^+(\mathbb{R})$ act on $\mathfrak{H}^2$ by $(z, w) \mapsto (\frac{az+b}{cz+d}, \frac{a^\sigma w+b^\sigma}{c^\sigma w+d^\sigma})$. The stabilizer of $(\sqrt{-1}, \sqrt{-1}) \in \mathfrak{H}^2$ gives rise to a maximal compact subgroup $C_\infty(\mathrm{SO}_{D_\sigma}(\mathbb{R}))$. If D_E is definite, we put $C_\infty(\mathrm{SO}_{D_\sigma}(\mathbb{R})) = \mathrm{SO}_{D_\sigma}(\mathbb{R})$. For an open compact subgroup U of $\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})$, we define

$$Sh_\alpha = Sh_{\alpha,U} := \mathrm{SO}_\alpha(\mathbb{Q})\backslash \mathrm{SO}(\mathbb{A})/UC_\alpha = \mathrm{SO}_\alpha(\mathbb{Q})\backslash \mathrm{SO}(\mathbb{A})/UZ(\mathrm{O}_\alpha)(\mathbb{A})C_\alpha \tag{3.5}$$

with the maximal compact subgroup $C_\alpha = \mathrm{SO}_\alpha(\mathbb{R}) \cap C_\infty(\mathrm{SO}_{D_\sigma}(\mathbb{R}))$. Since $\varrho_0 = \varrho_{D_{\alpha,0}} : D^\times \to \mathrm{SO}_\alpha$ has kernel $\mathbb{G}_{m/\mathbb{Q}}$ and is surjective over $\mathbb{Q}$, $\mathbb{A}$ and $\mathbb{Q}_v$ for all places v of $\mathbb{Q}$, we find

$$Sh_\alpha \cong D^\times\backslash D_{\mathbb{A}}^\times/\varrho_0^{-1}(U)\mathbb{A}^\times\varrho_0^{-1}(C_\alpha) \tag{3.6}$$

which is a quaternionic Shimura variety (strictly speaking, is the Shimura variety of the quaternionic group $D_\alpha^\times$ modulo the center).

4. Indefinite D_σ with E real.

For any (V, Q) as in §3.1, recall $V_A = V \otimes_{\mathbb{Q}} A$; e.g., $D_{\sigma,\mathbb{A}}^\pm = D_\sigma^\pm \otimes_{\mathbb{Q}} \mathbb{A}$ and $D_{E_{\mathbb{A}}} = D \otimes_{\mathbb{Q}} E_{\mathbb{A}} = D_E \otimes_{\mathbb{Q}} \mathbb{A}$. We apply the principle (S) and (R) of Waldspurger described in the introduction to the splitting $D_\sigma^\pm = Z^\pm \oplus D_0^\pm$ and compute the period $\int_{Sh_\alpha} \theta^*(F)(h)d\mu_h$ for the theta lift $\theta^*(F)(h) = \int_{X_0(M)} \mathbf{F}(g)\theta(\phi)(g, h)dg$ ($g \in \mathrm{Mp}(\mathbb{A})$, $h \in D_{E_{\mathbb{A}}}^\times$) for a well-chosen Siegel-Weil theta series $\theta(\phi)(g, h)$. Depending on the choice of D, E, ϕ and the level M, $\theta^*(F)$ could vanish. In this paper, we hereafter assume, for our choice of ϕ and the level M determined by ϕ,

($\Phi'0$) $G_{D_\sigma}^+(\mathbb{A}) \ni h \mapsto \theta^*(F)(h) = \int_{X_0(M)} \mathbf{F}(g)\theta^*(\phi)(g, h)d\mu_g \neq 0$ as a function of h.

This condition ($\Phi'0$) is almost equivalent to ($\Phi0$) for the following reason (so, this is not an additional restriction). By our choice, $\phi^{(\infty)} = \phi_Z \otimes \phi_{D_0}$ and $\phi_\infty = \sum_j \phi_{k-j}^Z \otimes \phi_j^{D_0}$ ($j \in \mathbb{Z} \cap [0, k]$) for the weight k of F with $\phi_V \in \mathcal{S}(V_{\mathbb{A}^{(\infty)}})$ and $\phi_i^V \in \mathcal{S}(V_{\mathbb{R}})$ ($V = Z, D_0$). Set $\mathbf{f}_j := \boldsymbol{\theta}(\phi_{k-j}^Z)$ and $\Phi_j(g) := \mathbf{w}(g)(\phi_{D_0} \otimes \phi_j^{D_0}))(0)$ for $g \in \mathrm{Mp}(\mathbb{A})$. Then $\{\mathbf{f}_j, \Phi_j\}_j$ satisfies ($\Phi0$) under ($\Phi'0$) (except for a rare case of $a_{\Delta_-|}(F) = 0$ for D_σ^- in Theorem 4.7). Indeed, by Siegel–Weil formula, for the Shimura subvariety S of $D^\times$ in the automorphic manifold of $D_E^\times$, we have $\mathfrak{m}E(\Phi_j) = \int_S \theta(\phi_0 \otimes \phi_j^{D_0})(g, h)d\mu_h$ ($h \in D_{\mathbb{A}}^\times$) for the mass $\mathfrak{m} \in \mathbb{C}^\times$ in [AQF, §37.1], and $\mathrm{Tr}(\widetilde{\mathbf{F}}(g)\sum_j \mathbf{f}_jE(\Phi_j)(g)) = \mathfrak{m}\mathrm{Tr}(\widetilde{\mathbf{F}}(g)\int_S \theta(\phi)(g, h)d\mu_h)$ cannot vanish as a function of g as its S-period given by the adjoint L-value does not vanish (so, otherwise $\theta^*(F)(h)$ in ($\Phi'0$) has to vanishes except for the rare case described above).

In this section, we assume $D_{\mathbb{R}} \cong M_2(\mathbb{R})$ and $E_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$ including the case $E = \mathbb{Q} \times \mathbb{Q}$. Under this setting, We write $S_\kappa(\Gamma, \varphi) = S_k^+(\Gamma, \varphi)$ for the space of holomorphic elliptic cusp forms of weight k on $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with character φ. We write $S_k^-(\Gamma, \varphi)$ for the anti-holomorphic version of $S_k(\Gamma, \varphi)$.

4.1. **Explicit form of Siegel–Weil theta series.** Let ϕ be a Schwartz-Bruhat function on $D_{\sigma,\mathbb{A}}$. As in [H06, (2.17)], define a Schwartz function Ψ_k on $D_{\sigma,\mathbb{R}}$ for $(\tau, z, w) \in \mathfrak{H} \times (\mathbb{C} - \mathbb{R})^2$ and $0 \le k \in \mathbb{Z}$,

$$\Psi_k(\tau; z, w)(v) = \mathrm{Im}(\tau)\frac{[v; \overline{z}, w]^k}{(z - \overline{z})^k(w - \overline{w})^k}\mathbf{e}(\pm N(v)\tau + \sqrt{-1}\frac{\mathrm{Im}(\tau)}{2|\mathrm{Im}(z)\mathrm{Im}(w)|}|[v; z, \overline{w}]|^2), \tag{4.1}$$

where for $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$[v; z, w] = -\mathrm{Tr}_{D_E/E}(v^\iota p(z,w)) = -(w,1)Jv^\iota \left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right) = (z,1)Jv \left(\begin{smallmatrix} w \\ 1 \end{smallmatrix}\right) = wcz - aw + dz - b \tag{4.2}$$

for $p(z,w) := {}^t(z,1)(w,1)J = \left(\begin{smallmatrix} -z & zw \\ -1 & w \end{smallmatrix}\right)$ with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In [H06, (2.17)], we have $\mathrm{Im}(z)\,\mathrm{Im}(w)$ in place of $|\,\mathrm{Im}(z)\,\mathrm{Im}(w)|$. This is because in [H06], we have chosen the connected component of the hermitian symmetric domain $\mathfrak{H}^2$ for the orthogonal group given by $\mathrm{Im}(z) > 0$. If we insert $\overline{z}$ or $\overline{w}$ in place of z or w, we need to work the lower half plane; so, we need to replace $\mathrm{Im}(z)\,\mathrm{Im}(w)$ by $|\,\mathrm{Im}(z)\,\mathrm{Im}(w)|$. We choose a Bruhat function $\phi^{(\infty)} : D^{(\infty)}_{\sigma,\mathbb{A}} \to \mathbb{C}$ and put

$$\phi = \phi_k = \phi^{(\infty)} \otimes \Psi_k \quad \text{and} \quad \theta(\phi_k) = \theta(\phi_k)(\tau; z, w) = \sum_{v \in D_\sigma} \phi(v). \tag{4.3}$$

Write $\mathrm{O}(A) := \mathrm{O}_{D_\sigma}(A)$ and $\mathrm{SO}(A) = \{g \in \mathrm{GO}(A) |\det(g) = 1\}$ as an algebraic group over $\mathbb{Q}$. Here det is taken regarding g as an automorphism: $v \mapsto g^{-1}vg^\sigma$ of D_σ. Since

$$G^+(\mathbb{Q}) = \{\alpha \in D_E^\times | N(\alpha) \in \mathbb{Q}\}$$

is the even Clifford group of D_σ [AQF, Theorem 24.6], we have $\mathrm{SO}(A) = G^+(A)/Z_{G^+}(A)$ for the center Z_{G^+} of G^+. The action of $g \in D_E^\times$ on D_σ as an element of O (and GO) is given by $v \mapsto g^{-1}vg^\sigma$. Since $x \mapsto x^\iota = \pm x^\sigma$ preserves $N(x)$, $\iota \in \mathrm{O}(A)$ and $\mathrm{O}(A) = \mathrm{SO}(A) \sqcup \iota\mathrm{SO}(A) = \mathrm{SO}(A) \sqcup \sigma\mathrm{SO}(A)$.

We have for $g \in \mathrm{GO}(\mathbb{R})$

$$\begin{gathered}
[g^\iota v g^\sigma; z, w] = [v; g(z), g^\sigma(w)]j(g,z)j(g^\sigma, w), \\
\frac{[v; \overline{g(z)}, g^\sigma(w)]}{\mathrm{Im}(g(z))\,\mathrm{Im}(g^\sigma(w))} = N(gg^\sigma)^{-1} j(g,z)j(g^\sigma,\overline{w}) \frac{[g^\iota v g^\sigma; \overline{z}, w]}{\mathrm{Im}(z)\,\mathrm{Im}(w)}, \\
\frac{|[g^\iota v g^\sigma; z, \overline{w}]|^2}{\mathrm{Im}(z)\,\mathrm{Im}(w)} = N(gg^\sigma) \frac{|[v; g(z), \overline{g^\sigma(w)}]|^2}{\mathrm{Im}(g(z))\,\mathrm{Im}(g^\sigma(w))},
\end{gathered} \tag{4.4}$$

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = c\tau + d$. These formulas tells us, if $\gamma \in D_E^\times$ with $N(\gamma) = 1$ and $\phi^{(\infty)} \circ \gamma = \phi^{(\infty)}$,

$$\theta(\phi_k)(\gamma(z), \gamma^\sigma(w)) = \theta(\phi_k)(z,w)j(\gamma,z)^k j(\gamma^\sigma, \overline{w})^k. \tag{4.5}$$

Over $\mathbb{R}$, if E is real, $(D_{\sigma,\mathbb{R}}, s) \cong (D_\mathbb{R}, s)$; so, $\mathrm{O}_{D_\sigma}(\mathbb{R}) \cong \mathrm{O}_D(\mathbb{R})$ and $\mathrm{SO}_{D_\sigma}(\mathbb{R}) \cong \mathrm{SO}_D(\mathbb{R})$.

$$\begin{aligned}
\mathrm{SO}_{D_\sigma}(\mathbb{R}) \cong \{(x,y) \in (\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}))/Z_{G^+}(\mathbb{R}) | \det(x)/\det(y) = 1\} \\
\hookrightarrow \mathrm{O}_{D_\sigma}(\mathbb{R}) = \mathrm{SO}_{D_\sigma}(\mathbb{R}) \sqcup \mathrm{SO}_{D_\sigma}(\mathbb{R})\sigma,
\end{aligned}$$

regarding $\sigma \in \mathrm{Aut}(D_{\sigma/\mathbb{Q}})$ as an element of $\mathrm{O}_{D_\sigma}(\mathbb{Q})$ [AQF, §25.3]. Here $Z_{G^+}(\mathbb{R})$ is the diagonal image of $\mathbb{R}$ in $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. The special orthogonal group $\mathrm{SO}_{D_\sigma}(\mathbb{R})$ has two connected components

$$\mathrm{SO}^+_{D_\sigma}(\mathbb{R}) = \{(x,y) \in \mathrm{SO}_{D_\sigma}(\mathbb{R}) | \det(x) > 0\}, \quad \mathrm{SO}^-_{D_\sigma}(\mathbb{R}) = \{(x,y) \in \mathrm{SO}_{D_\sigma}(\mathbb{R}) | \det(x) < 0\} \quad \text{and}$$

$$\mathrm{SO}^+_{D_\sigma}(\mathbb{R}) \cong (\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{\pm 1\}. \tag{4.6}$$

4.2. **Differential form coming from theta series.** We write the Shimura subvariety associated to D_α as Sh_α hereafter. We just write $Sh = Sh_\delta$ for δ as in (3.1); so, $D_\delta = D$ (since $\delta \in D_\sigma \cap E^\times$). See (4.13) for a precise definition of Sh_α. Since we want to compute the integral of the theta lift over the Shimura subvariety Sh, we describe the differential form associated to the theta series of degree $2 = \dim_\mathbb{R} Sh$.

Suppose that $\theta(\phi)$ is automorphic on $\Gamma_\tau \subset \mathrm{SL}_2(\mathbb{Z})$. Since $\theta(\phi)$ has three variables (τ, z, w), we use the symbol Γ_τ for the level group for the metaplectic variable τ (as in [H05, Proposition 2.3]). Let $L_E(n, A)$ be the space of homogeneous polynomials for a pair (X, Y) and (X', Y') of variables of degree n with coefficients in an E-algebra A. Suppose that $D_E \otimes_\mathbb{Q} A \cong M_2(A) \times M_2(A)$ for two projections inducing identity and σ. Let D_E acts on $P((X, Y; X', Y')) \in L_E(n; A)$ by $\gamma P(X, Y; X', Y') = P((X, Y)^t\gamma^\iota, (X', Y')^t\gamma^{\iota\sigma})$. Then

$$\Theta(z, w) = \Theta(\tau; z, w) := \theta(\phi_k)(\tau; z, w)(X - zY)^{k-2}(X' - \overline{w}Y')^{k-2} dz \wedge d\overline{w} \tag{4.7}$$

for $\theta(\phi_k)$ in (4.3) is a C^∞-differential form with values in $L_E(k-2; \mathbb{C})$. Then we have

$$\gamma^*\Theta = \Theta(\gamma(z), \gamma^\sigma(w)) = \theta(\tau; \gamma(z), \gamma^\sigma(w))(X - \gamma(z)Y)^{k-2}(X' - \gamma^\sigma(\overline{w})Y')^{k-2} d\gamma(z) \wedge d\gamma(\overline{w}) = \gamma \cdot \Theta(z)$$

and $\gamma \cdot \Theta$ is the action of γ on the value in $L_E(k-2; \mathbb{C})$. We write $\Theta(\tau; z) := \Theta(\tau; z, z)$.

Recall the maximal order $R \subset D$ we fixed in (D) in §1.1, we consider $R \otimes_{\mathbb{Z}} A$-module $L(n;A) = L_{\mathbb{Q}}(n;A)$ made of homogeneous polynomials of degree n in (S,T) with coefficients in A. We assume that $R \otimes_{\mathbb{Z}} A \hookrightarrow M_2(A)$ and $\gamma \in R$ acts on $P(S,T) \mapsto \gamma P(S,T) := P((S,T)^t\gamma^\iota)$.

We would like to project the locally constant sheaf of $L_E(n;A)$ restricted to the Shimura subvariety Sh of $D^\times$ to compute the integral of $L_E(n;\mathbb{C})$-valued harmonic forms over Sh. The $D^\times$-module $L_E(n;A)|_{D^\times}$ has the following decomposition into irreducible factors

$$L_E(n;A)|_{D^\times} \cong L_{\mathbb{Q}}(n;A) \otimes L_{\mathbb{Q}}(n;A) \cong \bigoplus_{j=0}^{n} L_{\mathbb{Q}}(2n-2j;A).$$

As in [H94, page 498], write $\pi : L_E(n;A) = L_E(n,0;A) \to A = L(0,n;A)$ for the SL(2)-equivariant projection given by

$$\pi(P(X,Y;X',Y')) = n!^{-2}\nabla^n P \text{ for } \nabla = \frac{\partial^2}{\partial X \partial Y'} - \frac{\partial^2}{\partial Y \partial X'}. \tag{4.8}$$

Here $L(n;A) = L(n,0;A)$ under the notation of [H99]. By [H99, page 141], we have

$$n!^{-2}\nabla^n X^{n-i}Y^iX'^{n-j}Y'^j = \begin{cases} (-1)^i\binom{n}{i}^{-1} & \text{if } n = i+j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Thus we get from $(X - zY)^n(X' - \overline{z}Y')^n = \sum_{i,j=0}^{n}\binom{n}{i}\binom{n}{j}(-1)^{i+j}z^i\overline{z}^jX^{n-i}Y^iX'^{n-j}Y'^j$,

$$n!^{-2}\nabla^n(X - zY)^n(X' - \overline{z}Y')^n = \sum_{j=0}^{n}\binom{n}{j}(-\overline{z})^jz^{n-j} = (z - \overline{z})^n \tag{4.10}$$

Let $n = k-2$ and write $S_k^\pm(\Gamma_\tau, \chi_{D_\sigma})$ for the space of cusp forms of weight k holomorphic in Case $+$ and anti-holomorphic in Case $-$, where we say Case $\pm$ when we deal with $D_\sigma^\pm$. Applying $n!^{-2}\nabla^n$ to $\gamma^*\Theta = \gamma \cdot \Theta(z)$, we get $n!^{-2}\nabla^n\Theta(\gamma(z)) = \det(\gamma)^n n!^{-2}\nabla^n\Theta(z) = n!^{-2}\nabla^n\Theta(z)$.

Remark 4.1. Since H^2 (and hence the integral) with non-constant (irreducible) coefficients vanishes over the Shimura subvariety we study, it appears not necessary to make explicit the projection given by a power of the differential operator ∇ to the constant sheaf. However for computational purposes, this maneuver is indispensable to reach the final explicit period formula as the operator kills very complicated redundant factors. Thus we will see many such operators in this paper. This point is already clear from the automorphic proof of the exact rational factorization formula for Deligne periods of GL(2)-automorphic forms done in [H94].

4.3. **Factoring the theta series.** Recall $E = \mathbb{Q}[\sqrt{\Delta}]$ with $0 < \Delta = -N(\sqrt{\Delta}) \in \mathbb{Z}$. Plainly $Z^\perp = D_0$ and $D_0^\perp = Z$ under $s(x,y) = \mathrm{Tr}(xy^\iota)$, writing $Z = Z^\pm$. We split the quadratic space

$$(D_\sigma^\pm, \pm N) = (Z^\pm, \pm N|_{Z^\pm}) \oplus (D_0^\pm, \pm N|_{D_0}). \tag{4.11}$$

Then $D_0 = D_0^\pm$ is 3-dimensional of signature $(1,2)$, and $Z^\pm$ has signature $(1,0)$. An element $\alpha \in D^\times$ acts on D_0 by $x \mapsto \alpha^{-1}x\alpha$ and on Z by the identity action. This action is compatible with the action of $D_E^\times$ on D_σ by (D_σ) in §1.1. We have an embedding $\mathrm{O}_Z \times \mathrm{O}_{D_0}$ into $\mathrm{O} = \mathrm{O}_{D_\sigma}$.

We apply the principles (S) and (R) to $V = D_\sigma$, $V_0 = D_0$ and $V_0^\perp = Z$. Take a Bruhat functions $\phi_Z \in \mathcal{S}(Z_{\mathbb{A}}^{(\infty)})$ and $\phi_0 \in \mathcal{S}(D_{0,\mathbb{A}}^{(\infty)})$. For $\mathfrak{z} \in Z_{\mathbb{A}}^{(\infty)}$ and $\mathfrak{x} \in D_{0,A}^{(\infty)}$, we assume

$$\textit{the tensor decomposition } \phi^{(\infty)}(\mathfrak{z} \oplus \mathfrak{x}) = (\phi_Z \otimes \phi_0)(\mathfrak{z} \oplus \mathfrak{x}) := \phi_Z(\mathfrak{z})\phi_0(\mathfrak{x}) \tag{4.12}$$

of the Bruhat function in order to factor the theta series $\theta(\phi)$.

Next we study the decomposition of the infinite part with respect to the decomposition $D_\sigma = Z \oplus D_0$. Thus we need to decompose the spherical polynomial $[\alpha; z, \overline{z}]$. Pick $\mathfrak{z} \in Z$ and $\mathfrak{x} \in D_0$. Since $s = s_Z \oplus s_0$ for the restriction s_Z and s_0 of s to Z and D_0, the signature of s_0 depends on the sign of $D_\sigma^\pm$. So we write $s_?^\pm$ to indicate the sign of $s_?$. Let $P^\pm$ be a positive majorant of $s^\pm(x,y)$ (see [HMI, §2.5.2] for majorants). When $P^\pm$ is compatible with the decomposition (4.11), we write $P^\pm = P_Z^\pm \oplus P_0^\pm$ accordingly.

Recall $p(z,\overline{w}) = \left(\begin{smallmatrix} z & -z\overline{w} \\ 1 & -\overline{w} \end{smallmatrix}\right)$. Then $s^\pm(p(z,\overline{w}), p(z,\overline{w})) = \pm 2\det p(z,\overline{w}) = 0$. Similarly,

$$s^\pm(p(z,\overline{w}), \overline{p(z,\overline{w})}) = \pm\mathrm{Tr}(\left(\begin{smallmatrix} z & -z\overline{w} \\ 1 & -\overline{w} \end{smallmatrix}\right)\left(\begin{smallmatrix} -w & \overline{z}w \\ -1 & \overline{z} \end{smallmatrix}\right)) = \pm(\overline{w} - w)(z - \overline{z}) \in \mathbb{R}_\pm.$$

Let W be the subspace of $V_{\mathbb{R}}$ generated by $\mathrm{Re}(p(z,\overline{w}))$ and $\mathrm{Im}(p(z,\overline{w}))$. Then $s^+>0$ on W and $s^-<0$ on W. Decomposing $V_{\mathbb{R}}=W\oplus W^\perp$, we have $s^+<0$ on $W^\perp$ and $s^->0$ on $W^\perp$. This means $P^\pm(x,y)=\pm s^\pm(x_W,y_W)\mp s^\pm(x_{W^\perp},y_{W^\perp})$ for the orthogonal projection $?_X$ of $?$ to $X=W,W^\perp$. Then the Schwartz function is given by

$$\Phi(v)=Q(v)\mathbf{e}(\frac{1}{2}(s^\pm[v]\xi+P^\pm[v]\eta\sqrt{-1})).$$

We compute $P^\pm[v]$. We start with

$$P^\pm[v]\pm s^\pm[v]=\pm s^\pm(x_W,y_W)\mp s^\pm(x_{W^\perp},y_{W^\perp})\pm s^\pm(v_W,v_W)\pm s^\pm(v_{W^\perp},v_{W^\perp})=\pm 2s^\pm(v_W,v_W).$$

Write $v=cp(z,\overline{w})+\overline{c}p(\overline{z},w)+x$ with $x\in W^\perp$ and $c\in\mathbb{C}$. Then

$$\begin{aligned}P^\pm[v]\pm s^\pm[v]=\pm 2s^\pm(cp(z,\overline{w})+\overline{c}p(\overline{z},w),cp(z,\overline{w})+\overline{c}p(\overline{z},w))&=\pm 4|c|^2s^\pm(p(z,\overline{w}),p(\overline{z},w))\\&=4|c|^2(\overline{w}-w)(z-\overline{z})\geq 0.\end{aligned}$$

Since $s^\pm(v,p(z,\overline{w}))=\overline{c}s^\pm(p(\overline{z},w),p(z,\overline{w}))=\pm\overline{c}(\overline{w}-w)(z-\overline{z})$, writing $[v;z,w]:=s^+(v,p(z,w))$, we have $\overline{c}=\frac{[v;z,\overline{w}]}{(\overline{w}-w)(z-\overline{z})}$. Combining all these, we get

$$P^\pm[v]=\mp s^\pm[v]+\frac{|[v;z,\overline{w}]|^2}{|\mathrm{Im}(w)\,\mathrm{Im}(z)|}.$$

Thus $\frac{1}{2}P^\pm[v]=-N(v)+|2\,\mathrm{Im}(z)\,\mathrm{Im}(w)|^{-1}|[v;z,w]|^2$ as in [H06, (2.2)]. Since $Q(v)$ is a polynomial, we can write it as $Q(v)=\sum_j Q_j^Z(\mathfrak{z})Q_j^{D_0}(\mathfrak{x})$ for $v=\mathfrak{z}\oplus\mathfrak{x}$ with $\mathfrak{z}\in Z$ and $\mathfrak{x}\in D_0$.

Recall that Case $\pm$ means that we deal with $D_\sigma^\pm$. Write $\tau^\pm:=\begin{cases}\overline{\tau} & \text{in Case }+,\\ -\tau & \text{in Case }-.\end{cases}$ Recall also

$$\Psi_k(\tau;z,w)(v)=\mathrm{Im}(\tau)\frac{[v;\overline{z},w]^k}{(z-\overline{z})^k(w-\overline{w})^k}\mathbf{e}(N(v)\tau^\pm+\sqrt{-1}\frac{\mathrm{Im}(\tau)}{2|\,\mathrm{Im}(z)\,\mathrm{Im}(w)|}|[v;z,\overline{w}]|^2).$$

Thus

$$\Psi_k(\tau;\gamma(z),\gamma^\sigma(w))(v)=\Psi_k(\tau;z,w)(\gamma^{-1}v\gamma^\sigma)j(\gamma,z)^kj(\gamma^\sigma,\overline{w})^k$$

for $\gamma\in D^\times$ with $N(\gamma)=1$ and $\theta(\phi_k)$ has positive weight (k,k) in $z,\overline{w}$, and $\theta(\phi_k)$ has automorphic factor $j(\gamma,\tau)^k$ in Case $+$ and $j(\gamma,\overline{\tau})^k$ in Case $-$ by [HMI, Theorem 2.65] as $D_\sigma^\pm$ has signature $(2,2)$.

We assume that $\Gamma_\tau=\Gamma_0(M)$ for an integer $M>0$. Define for $\alpha\in D_\sigma^\pm$

$$\begin{aligned}&D_{\alpha,0}=\{x\in D_{\sigma_\alpha}|\mathrm{Tr}_{D_E/E}(x)=0\},\\ &\widehat{\Gamma}_\alpha:=\{x\in \mathrm{O}_\alpha(\mathbb{A}^{(\infty)})|\phi^{(\infty)}(xv)=\phi^{(\infty)}(v)\ \text{ for all } v\in D_{0,\mathbb{A}^{(\infty)}}\},\\ &Sh_\alpha=Sh_{\alpha,\phi}=\mathrm{O}_\alpha(\mathbb{Q})\backslash\mathrm{O}_\alpha(\mathbb{A})/\widehat{\Gamma}_\alpha C_\alpha,\end{aligned}\tag{4.13}$$

where C_α is a maximal compact subgroup of $\mathrm{O}_\alpha(\mathbb{R})$ and $\varrho:D_\alpha^\times\twoheadrightarrow\mathrm{SO}_\alpha=\mathrm{SO}_{D_{\alpha,0}}$ is the projection from the Clifford group $D_\alpha^\times$ to the orthogonal group $\mathrm{SO}_{D_{\alpha,0}}$ described in §3.1. When D_α is definite, we have $\varrho:D_{\alpha,\mathbb{R}}^\times/\mathbb{R}^\times\cong\varrho(C_\alpha)=\mathrm{SO}_\alpha(\mathbb{R})$ with $C_\alpha=\mathrm{O}_\alpha(\mathbb{R})=\mathrm{SO}_\alpha(\mathbb{R})\sqcup\mathrm{SO}_\alpha(\mathbb{R})\iota$. If D_α is indefinite, identifying $D_{\alpha,\mathbb{R}}$ with $M_2(\mathbb{R})$, C_α is the stabilizer of $\sqrt{-1}\in\mathfrak{H}$ (though this depends on the identification $D_{\alpha,\mathbb{R}}\cong M_2(\mathbb{R})$ but the isomorphism class of Sh_α is well defined independent of the choice). We assume that

$$\phi^{(\infty)}=\prod_l\phi_l \textit{ for a local Bruhat function } \phi_l\in\mathcal{S}(D_{0,\mathbb{Q}_l}) \textit{ with } \phi_l(v_l^\iota)=\phi_l(v_l) \textit{ for all } l.\tag{4.14}$$

Choose ϕ so that $\theta(\phi)$ has Neben character $\varphi^{-1}\chi_{D_\sigma}$ (i.e., it is an easy exercise to find such a ϕ by (1.1), and see §4.6 and §4.7 for our choice). Recall $Sh=Sh_\delta$ for δ in (3.1). Note here $D_\delta=D$.

Consider the period integral for $F \in S_k^{\mp}(\Gamma_\tau, \varphi\chi_{D_\sigma})$, $\Theta(z,w)$ in (4.7) and $\theta(\phi_k)$ in (4.3):

$$\begin{aligned}(4.15)\quad P'_\delta(F) &:= \int_{Sh}\left[\int_{\Gamma_\tau\backslash\mathfrak{H}} n!^{-2}\nabla^n\Theta(\tau;z)F(\tau)\eta^{k-2}d\xi d\eta\right]dz\wedge d\overline{z}\quad (n=k-2)\\ &\overset{(4.10)}{=} \int_{Sh}\left(\int_{\Gamma_\tau\backslash\mathfrak{H}}\theta(\phi_k)(\tau;z,z)F(\tau)\eta^{k-2}d\xi d\eta\right)(z-\overline{z})^n dz\wedge d\overline{z}\\ &= -\frac{\sqrt{-1}}{2}\int_{Sh}\left(\int_{\Gamma_\tau\backslash\mathfrak{H}}\theta(\phi_k)(\tau;z,z)F(\tau)\eta^{k-2}d\xi d\eta\right)(z-\overline{z})^k y^{-2}dxdy.\end{aligned}$$

Since $s[x]=\pm 2N(x)$ on $D_\sigma^\pm$, on Z, $\mp s_Z$ is positive definite, and s_0 has signature $(1,2)$ on D_0^+ and $(2,1)$ on D_0^-. Thus $P_Z=\mp s_Z$.

Pick $\mathfrak{z}\in Z^\pm=\mathbb{Q}\delta_\pm$ and $\mathfrak{y}\in D_0$. Then we have

$$[\mathfrak{z}+\mathfrak{y};\overline{z},z]^k=([\mathfrak{z};\overline{z},z]+[\mathfrak{y};\overline{z},z])^k=\sum_{j=0}^k\binom{k}{j}[\mathfrak{z};\overline{z},z]^j[\mathfrak{y};\overline{z},z]^{k-j}=\sum_{j=0}^k\binom{k}{j}(\mathfrak{z}(\overline{z}-z))^j[\mathfrak{y};\overline{z},z]^{k-j}.$$

For $\mathfrak{y}=\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)$, noting $\mathrm{Re}\,p(\overline{z},z)\in D_0\otimes_{\mathbb{Q}}\mathbb{R}$ and $\mathrm{Im}\,p(\overline{z},z)\in Z\otimes_{\mathbb{Q}}\mathbb{R}$,

$$[\mathfrak{y};\overline{z},z]=(\overline{z},1)\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\mathfrak{y}\left(\begin{smallmatrix}z\\1\end{smallmatrix}\right)=zc\overline{z}-a\overline{z}-az-b=s(\mathfrak{y},\mathrm{Re}\,p(\overline{z},z))\in\mathbb{R},\quad [\mathfrak{z};\overline{z},z]=\mathfrak{z}(\overline{z}-z).$$

$$|[\mathfrak{z}+\mathfrak{y};z,\overline{z}]|^2=(-\mathfrak{z}(z-\overline{z})+[\mathfrak{y};z,\overline{z}])(\mathfrak{z}(z-\overline{z})+[\mathfrak{y};z,\overline{z}])=|[\mathfrak{y};z,\overline{z}]|^2-\mathfrak{z}^2(z-\overline{z})^2.$$

Write

$$\Psi_j^{D_0}(\tau,z;\mathfrak{x}):=(z-\overline{z})^{-j}[\mathfrak{x};\overline{z},z]^j\mathbf{e}(N(\mathfrak{x})\tau^\pm+\sqrt{-1}\frac{\mathrm{Im}(\tau)|[\mathfrak{x};z,\overline{z}]|^2}{2\,\mathrm{Im}(z)^2}),$$

$$\Psi_j^Z(\tau,z;\mathfrak{z}):=\mathfrak{z}^j\mathbf{e}(\mathfrak{z}^2\tau^\pm-\sqrt{-1}\frac{\mathrm{Im}(\tau)(z-\overline{z})^2\mathfrak{z}^2}{2\,\mathrm{Im}(z)^2})=\begin{cases}\mathfrak{z}^j\mathbf{e}(\mathfrak{z}^2\tau) & \text{in Case }+,\\ \mathfrak{z}^j\mathbf{e}(-\mathfrak{z}^2\overline{\tau}) & \text{in Case }-,\end{cases}$$

where $\tau^\pm:=\begin{cases}\overline{\tau} & \text{in Case }+,\\ -\tau & \text{in Case }-.\end{cases}$.

Since these functions for fixed τ and z are restriction of a Schwartz function on $D_{\sigma,\mathbb{R}}$ to $Z_{\mathbb{R}}$ and $D_{0,\mathbb{R}}$, they are Schwartz functions on the subspace. We get

$$(4.16)\qquad (z-\overline{z})^k\phi_k=\sum_{j=0}^k(-1)^j\binom{k}{j}\phi_Z\Psi_j^Z\otimes\phi_0\Psi_{k-j}^{D_0}.$$

We now interchange the order of the two integrations one over Sh and another over $X_0(M)$ in the period integral in (4.15). Then the period integral for $F\in S_k^{\mp}(\Gamma_\tau,\varphi\chi_{D_\sigma})$ now becomes

$$(4.17)\qquad P'_\delta(F)=-\frac{\sqrt{-1}}{2}\int_{\Gamma_\tau\backslash\mathfrak{H}}\int_{Sh}(z-\overline{z})^k\theta(\phi_k)(\tau;z,z)y^{-2}dxdyF(\tau)\eta^{k-2}d\xi d\eta.$$

The interchange is justified if Sh is compact (i.e., D is division) and F is a cusp form.

4.4. **Siegel–Weil formula and period integrals.** We now invoke the Siegel-Weil formula as described in (R). Since $\mathrm{O}_Z(\mathbb{R})=\{\pm1\}$, the variable $g\in\mathrm{O}_Z$ is trivial. Thus

$$\begin{aligned}(4.18)\quad \mathbf{r}_Z(g_\tau)L_Z(g)\Psi_j^Z(\sqrt{-1},\sqrt{-1};\mathfrak{z})&=\eta^{(1+2j)/4}\mathfrak{z}^j\mathbf{e}(\mathfrak{z}^2\tau^\pm-\sqrt{-1}\frac{\mathrm{Im}(\tau)(z-\overline{z})^2\mathfrak{z}^2}{2\,\mathrm{Im}(z)^2})\\ &=\eta^{(1+2j)/4}\Psi_j^Z(\tau,z;\mathfrak{z})=\begin{cases}\eta^{(1+2j)/4}\mathfrak{z}^j\mathbf{e}(\mathfrak{z}^2\tau) & \text{in Case }+,\\ \eta^{(1+2j)/4}\mathfrak{z}^j\mathbf{e}(-\mathfrak{z}^2\overline{\tau}) & \text{in Case }-.\end{cases}\end{aligned}$$

Since the Clifford algebra of D_0 is D, we find $\mathrm{SO}_{D_0}(\mathbb{R})\cong\mathrm{PGL}_2(\mathbb{R})$ by ϱ_0, and

$$\begin{aligned}(4.19)\quad \mathbf{r}_{D_0}(g_\tau)L_{D_0}(g_z)\Psi_j^{D_0}(\sqrt{-1},\sqrt{-1};\mathfrak{x})&=\eta^{(3+2j)/4}\Psi_j^{D_0}(\tau,z;\mathfrak{x})\\ &=\eta^{(3+2j)/4}(z-\overline{z})^{-j}[\mathfrak{x};z,\overline{z}]^j\mathbf{e}(N(\mathfrak{x})\tau^\pm+\sqrt{-1}\frac{\mathrm{Im}(\tau)|[\mathfrak{x};z,\overline{z}]|^2}{2\,\mathrm{Im}(z)^2}).\end{aligned}$$

Let $\phi_j^{D_0} := \phi_0\Psi_j^{D_0}$ and $\phi_j^Z := \phi_Z\Psi_j^Z$. Then $g \mapsto (\mathbf{w}_{D_0}(g)\phi_j^{D_0})(0)$ is left-invariant under $B(\mathbb{Q}) \subset \mathrm{Mp}(\mathbb{A})$ for the metaplectic cover $\widetilde{\mathrm{SL}}_2$ of SL_2. For $g \in \mathrm{Mp}(\mathbb{A})$, decompose its image $\overline{g} \in \mathrm{SL}_2(\mathbb{A})$ into bk for $k \in \mathrm{SL}_2(\widehat{\mathbb{Z}})\mathrm{SO}_2(\mathbb{R})$ and $b \in B(\mathbb{A})$. Writing $b = \left(\begin{smallmatrix} a(g) & * \\ 0 & * \end{smallmatrix}\right)$, consider the Siegel Eisenstein series

$$E(\phi_j^{D_0})(s;g) := \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} |a(\gamma g)|_{\mathbb{A}}^{s-(1/2)}(\mathbf{w}_{D_0}(\gamma g)\phi_j^{D_0})(0).$$

Assuming that D is a division algebra, this Eisenstein series has meromorphic continuation over $s \in \mathbb{C}$ and finite at $s = \frac{1}{2}$ [Sw90, Theorem 3.3.1] (or [MSS, §5.3]); so, we define $E(\phi_j^{D_0})(g) := E(\phi_j^{D_0})(\frac{1}{2};g)$. This shows $\boldsymbol{\theta}(\phi) = \eta^{1+(k/2)}\theta(\phi)$ is the sum of the product of Weil's theta series as in Theorem 1.1 of $\phi_j^{D_0}(\sqrt{-1};\mathfrak{x}) = \phi_{D_0}(\mathfrak{x}^{(\infty)})\Psi_j^{D_0}(\sqrt{-1};\mathfrak{x}_\infty)$ and $\phi_j^Z(\mathfrak{z}) = \phi_Z(\mathfrak{z}^{(\infty)})\Psi_j^Z(\sqrt{-1};\mathfrak{z}_\infty)$.

Recall the maximal order R of D. Let $O_\delta(A) = \{x \in (R\otimes_{\mathbb{Z}} A)^\times | N(x)^2 = 1\}$. Recall $\widehat{\Gamma}_\delta \subset O_\delta(\mathbb{A}^{(\infty)})$ as in (4.13) taking $\alpha = \delta$. Then we have $Sh = O_\delta(\mathbb{Q})\backslash O_\delta(\mathbb{A})/\widehat{\Gamma}_\delta C_\delta$ for the maximal compact subgroup $C_\delta = O_2(\mathbb{R})$ fixing $\sqrt{-1}$. Choose a lattice L of D_σ and assume $L = L_Z \oplus L_0 \subset D_\sigma$ for lattices $L_Z \subset Z$ and $L_0 \subset D_0^\pm$. Take the characteristic function $\phi_0 = \phi_{L_0}$ of $\widehat{L}_0 \subset D_0 \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)} = L_0^{(\infty)}$. We choose later the finite part ϕ_Z of ϕ^Z which has open support in $\widehat{L}_Z^*$. We assume that $\phi^{(\infty)} = \phi_Z \otimes \phi_{L_0}$.

As the Siegel–Weil formula is stated with respect to the theta series of variable $g \in \mathrm{O}_{D_0}(\mathbb{A})$ (not with respect to z), we lift $\theta(\phi_k)(\tau;z,z)$ to the function $\boldsymbol{\theta}(\phi_k)(\tau;g)$ in the standard way by

$$\boldsymbol{\theta}(\phi_k)(\tau;g) = \theta(\phi_k)(\tau;g(\sqrt{-1}),g(\sqrt{-1}))|j(g,\sqrt{-1})|^{-2k} \quad (|j(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right),z)| = |cz+d|)$$

(so $\boldsymbol{\theta}(\phi_k)(\tau;g_z) = \theta(\phi_k)(\tau;z,z)y^k$). The lifted theta series $\boldsymbol{\theta}(\phi_j^Z)(\tau;g) = \sum_{\mathfrak{z}\in Z}\mathbf{w}(g)\phi_j^Z(\mathfrak{z})$ is the Siegel–Weil theta series as in Theorem 1.1. We take the Haar measure $d\mu_g$ on $\mathrm{O}_\delta(\mathbb{A})$ so that it induces the discrete Dirac measure on $O_\delta(\mathbb{Q})$, measure of volume 1 on $\widehat{\Gamma}_\delta C_\delta$ and $y^{-2}dxdy$ on $\mathfrak{H} = \mathrm{O}_\delta(\mathbb{R})/C_\delta$. By the Siegel–Weil formula in [Sw90, Theorem 3.3.1] (or [MSS, (5.3.3)])

$$(4.20)\quad (2\sqrt{-1})^{-k}\int_{Sh}\eta^{1+(k/2)}(z-\overline{z})^k\theta(\phi_k)(\tau;z,z)y^{-2}dxdy$$
$$= \int_{\mathrm{O}_{D_0}(\mathbb{Q})\backslash \mathrm{O}_{D_0}(\mathbb{A})/\widehat{\Gamma}_\delta}\boldsymbol{\theta}(\phi)(\tau;g)d\mu_g = \mathfrak{m}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\boldsymbol{\theta}(\phi_j^Z)E(\phi_{k-j}^{D_0}) \text{ for a constant } \mathfrak{m} \neq 0.$$

4.5. **Mass factor $\mathfrak{m}_1$.** Let us make $\mathfrak{m}_1$ as in (0.1) explicit. Recall the Haar measure $d\mu_g$ defined above (4.20). Then we have

$$(4.21)$$
$$\eta^{1+(k/2)}\int_{Sh_\delta}\theta(\phi_k)(\tau;z,z)y^{k-2}dxdy = \eta^{1+(k/2)}\int_{\mathrm{O}_\delta(\mathbb{Q})\backslash\mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta C_\delta}\boldsymbol{\theta}(\phi_k)(\tau;g(\sqrt{-1}),g(\sqrt{-1}))d\mu_g$$
$$= \eta^{1+(k/2)}\int_{\mathrm{O}_\delta(\mathbb{Q})\backslash\mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta C_\delta}\boldsymbol{\theta}(\phi_k)(\tau;g,g)d\mu_g = \mathfrak{m}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\boldsymbol{\theta}(\phi_j^Z)E(\phi_{k-j}^{D_0})$$

for $\mathfrak{m} > 0$ such that $d\mu_g = \frac{\mathfrak{m}}{2}d\omega_{\mathrm{O}_\delta}$ for the half of the Tamagawa measure $d\omega_{\mathrm{O}_\delta}$ of O_δ. The factor $\frac{1}{2}$ is to kill the Tamagawa number $\tau(\mathrm{O}_\delta) = 2$ so that theta integral with respect to $\frac{1}{2}d\omega_{\mathrm{O}_\delta}$ and Siegel–Eisenstein series exactly match. The following result valid for indefinite and also definite D is due to Shimura [Sh99]:

Theorem 4.2. *Replace $s^\pm$ by $\varphi = es^\pm$ for $0 \neq e \in \mathbb{Z}$ with minimal $|e|$ so that the discriminant of $\varphi|_{D_0^\pm}$ is a square in $\mathbb{Q}^\times$* (cf. Remark 3.1) *Take the measure $d\mu_g$ on $\mathrm{O}(\mathbb{Q})\backslash\mathrm{O}_\delta(\mathbb{A})$ with volume 1 on $\widehat{\Gamma}_\delta C_\delta$ for a maximal compact subgroup C_δ of $\mathrm{O}_\delta(\mathbb{R})$ with invariant measure $y^{-2}dxdy$ on $\mathfrak{H}$ if $D_{\mathbb{R}} \cong M_2(\mathbb{R})$, and let $L := R \cap D_0^\pm$ with $\widehat{\Gamma}_L := \{x \in \mathrm{O}_\delta(\mathbb{A}^{(\infty)}) | x\widehat{L}x^\iota = \widehat{L}\} = \{x \in \mathrm{O}_\delta(\mathbb{A}^{(\infty)}) | x\widehat{L}x^{-1} = \widehat{L}\}$. Then, defining $(\widehat{\Gamma}_L : \widehat{\Gamma}_\delta) := (\widehat{\Gamma}_L : \widehat{\Gamma}_\delta\cap\widehat{\Gamma}_L)/(\widehat{\Gamma}_\delta : \widehat{\Gamma}_\delta\cap\widehat{\Gamma}_L)$ and assuming* (4.14)*, we have the following formula of the mass factor $\mathfrak{m}_1$ as in* (0.1)*:*

$$\mathfrak{m}_1 = \mathfrak{m}_1(L,\widehat{\Gamma}_\delta) = (\widehat{\Gamma}_L : \widehat{\Gamma}_\delta)[\widetilde{L} : L]\left[\prod_{l|\partial}2^{-1}(1+l)^{-1}(1-l^{-2})\right],$$

where $\widetilde{L} := \{x \in D_0^{\pm} | 2\varphi(x, L) \subset \mathbb{Z}\}$.

Defining $L' := \{x \in D_0^{\pm} | s^{\pm}(x, L) \subset \mathbb{Z}\}$, $\widetilde{L} = (2e)^{-1}L'$, $[\widetilde{L} : L] = 2^3e^3[L' : L]$, and $[L' : L]$ is the absolute value of the discriminant of $s|_L$. The constant e can be taken a factor of Δd_K as $d(D_0) \sim \pm\Delta d_K$. In particular, if $E \hookrightarrow D$ (i.e., $D_E \cong M_2(E)$), we can take $K = E$ and hence $e = 1$.

Proof. In [Sh99, Theorem 5.8], the volume $\int_{\mathrm{SO}_V(\mathbb{Q})\backslash \mathrm{SO}_V(\mathbb{A})/\widehat{\Gamma}^1_L C_V} d\mu'$ is computed with respect to the measure $d\mu'$ Shimura specified. For a maximal lattice $L \subset V$ (i.e., maximal among lattices with given fractional ideal $\mathfrak{M}$ generated by $Q(L)$), the measure $d\mu'$ has volume 1 over $\widehat{\Gamma}^1_L C_V$ for a maximal compact subgroup $C_V \subset \mathrm{SO}_V(\mathbb{R})$ and $\widehat{\Gamma}^1_L = \{x \in \mathrm{SO}_V(\mathbb{A}^{(\infty)}) | \widehat{L}x = \widehat{L}\}$ (note that Shimura takes a convention of right action of his orthogonal group on V). The following three facts are noteworthy:

- $Sh_\delta = \mathrm{O}_\delta(\mathbb{Q})\backslash \mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta C_\delta = \mathrm{SO}_\delta(\mathbb{Q})\backslash \mathrm{SO}_\delta(\mathbb{A})/\widehat{\Gamma}^1_\delta C_\delta$ under (4.14) (see (4.22)).
- The Tamagawa number formula $\tau(\mathrm{O}) = 2$ given by Weil [W65] does not give an exact formula of the volume, as we need to know the exact ratio of the Tamagawa measure $\frac{1}{2}d\omega$ and the more arithmetic measure $d\mu_g$ related to the L-value.
- Though the volume is calculated earlier by Shimizu [Sh65] for the algebraic group $D^\times$ when D is indefinite (here O_δ and $D^\times$ are different groups), the definite case of $V = D_0$ is elusive (because of non-validity of strong approximation outside an archimedean place), and there is only a partial computation by Siegel for O_δ and Eichler for $D^\times$. The paper [Sh99] gives an explicit form of the arithmetic measure on the symmetric space of the orthogonal group and the exact volume via Dedekind L-values for any quadratic space over a totally real field of any dimension. When the base field has a complex place, Hanke [Ha05] computed the exact volume.

We specialize Shimura's result to $\mathrm{O}_\delta = \mathrm{O}_{D_0}$ and $V = D_0$. We may assume that $\widehat{\Gamma}_\delta = \widehat{\Gamma}_L$ as the general formula follows directly from [Sh99, (5.8.1)]. If D is definite, the symmetric space for $\mathrm{SO}_\delta(\mathbb{R})$ is one point, and we take the volume 1-measure on $\widehat{\Gamma}_\phi \mathrm{SO}_{D_0}(\mathbb{R})$. Assuming $D_\mathbb{R} \cong M_2(\mathbb{R})$, we choose C_D fixing ${}^t(1, 0)$ in the symmetric domain $\mathcal{Z}$ defined in [Sh99, §4.2] and take $L := R \cap D_0$ which is a maximal lattice. To get the exact value, we need to describe $\mathcal{Z}$ and its measure. Here, as in [Sh99, §4.1], for $V = D_0$,

$$\mathcal{Z} := \{(u, v) \in \mathbb{R}^2 | 2u > -v^2/2\}.$$

Then putting $B(z) = \left(\begin{smallmatrix} u & v & u \\ 0 & -2 & v \\ -1 & 0 & 1 \end{smallmatrix}\right)$ for $z = (u, v) \in \mathcal{Z}$, $\alpha \in \mathrm{SO}_\delta(\mathbb{R})$ acts on $\mathcal{Z}$ by $\alpha B(z) = B(\alpha(z)) \operatorname{diag}[\kappa, \mu]$ for two automorphic factor $\mu \in \mathbb{C}^\times$ and $\kappa \in \mathrm{GL}_2(\mathbb{C})$. Here we regard $\mathrm{SO}_\delta = \mathrm{SO}_{D_0}$ as the special orthogonal group with respect to the anti diagonal symmetric matrix $S' = (\mathrm{Tr}(v_i v_j^\iota))_{i,j}$ for $v_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $v_2 = \operatorname{diag}[1, -1]$ and $v_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then S' is anti-diagonal with anti-diagonal entries $-1, -2, -1$ in order. Since Shimura normalizes S' via conjugation by $\mathrm{GL}_3(\mathbb{R})$ and a scalar multiplication so that the center entry is positive, we conjugate by $Ad(\operatorname{diag}[1, -1]) \in \mathrm{GL}_3(\mathbb{R})$ and multiplying by -1 to reach

$$S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

getting the center entry 2 positive which plays the role of θ_v for $v = \infty$ in [Sh99, (5.3.2)]. This is because $Ad(\mathrm{SL}_2(\mathbb{R})) = \mathrm{SO}_\delta(\mathbb{R})$, and in this way, we identify $I : \mathfrak{H} \cong \mathcal{Z}$ sending $\sqrt{-1}$ to $(1, 0)$ and $I(g(z)) = Ad(g)(I(z))$ for $g \in \mathrm{SL}_2(\mathbb{R})$.

To make the isomorphism I explicit, we choose a basis $\mathcal{B}$ of $\mathfrak{sl}(2)$ given by $\mathcal{B} := \{{}^tU, \operatorname{diag}[1, -1], U\}$ for $U = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to compute Ad. To get the isomorphism between $\mathfrak{H}$ and $\mathcal{Z}$ via $I(a^2\sqrt{-1} + ab) = I(g(\sqrt{-1})) = Ad(g)((1, 0))$ for $g = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$, we compute $Ad(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ with respect to $\mathcal{B}$. Though we conjugated by $Ad(\operatorname{diag}[1, -1])$, we can make variable change $z \mapsto -\overline{z}$ to absorb this maneuver without changing the invariant measure; so, we forget about it. Note $Ad(\alpha)(x) = \alpha^{-1}x\alpha$ with respect to $\mathcal{B}$ for $\alpha \in \mathrm{SL}_2(\mathbb{R})$ as Shimura chooses right multiplication of orthogonal group action on quadratic spaces. Then

$$Ad(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) = \begin{pmatrix} a^2 & ab & -b^2 \\ 0 & 1 & 2a^{-1}b \\ 0 & 0 & a^{-2} \end{pmatrix} \quad \text{and} \quad Ad(\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)) B((1, 0)) = \begin{pmatrix} * & * & (a^2 - b^2) \\ * & * & 2a^{-1}b \\ * & * & a^{-2} \end{pmatrix} = B((u, v)) \operatorname{diag}[\kappa, \mu].$$

Thus $\mu = a^{-2}$ and $u = a^2(a^2 - b^2)$ and $v = 2ab$. Shimura's measure corresponds to the differential form $(2^{-1}(2u + v^2/2))^{-3/2} du \wedge dv = 8a^{-2} da \wedge db = 4y^{-2} dx \wedge dy$ for $x + y\sqrt{-1} = \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)(i) = a^2 i + ab$.

Thus we need to modify Shimura's exact value by the factor 4. Noting $\mathrm{SO}_{D_0} = \mathrm{SO}_\delta$ and $C_{D_0} = C_\delta$, here is his value in [Sh99, Theorem 5.8]

$$\int_{\mathrm{SO}_\delta(\mathbb{Q})\backslash \mathrm{SO}_\delta(\mathbb{A})/\widehat{\Gamma}^1_L C_\delta} d\mu' = 2(4\pi)^{1-\epsilon_D}[\widetilde{L} : L]\left[\prod_{l|\partial} 2^{-1}(1+l)^{-1}(1-l^{-2})\right]\frac{\zeta(2)}{\pi},$$

where $\widetilde{L} := \{x \in V | 2s(x, L) \subset \mathbb{Z}\}$. We have $4d\mu_g = d\mu'$ if $D_{\mathbb{R}} \cong M_2(\mathbb{R})$ and $d\mu' = d\mu_g$ when $D_{\mathbb{R}} \cong \mathbb{H}$ by our choice. Also as remarked in [Sh99, Lemma 5.6], the volume for O_δ is twice of the one for SO_δ. Since $\widehat{\Gamma}_\delta \subset \widehat{\Gamma}_L$ is a subgroup of finite index, multiplying the index $(\widehat{\Gamma}_L : \widehat{\Gamma}_\delta)$, we get the finial formula as in the theorem. □

Remark 4.3. Take $E = \mathbb{Q} \times \mathbb{Q}$ and $\delta = (1, -1)$. Let $N_0 = 1$ and $D_{/\mathbb{Q}}$ be a definite quaternion ramifying at only one odd prime p with a maximal order R. Let $L = R \cap D^+_\sigma$. Then $R_p \cong \left\{\left(\begin{smallmatrix} a & b \\ pb^\varsigma & a^\varsigma\end{smallmatrix}\right) | a, b \in O_K\right\}$ for the unramified discrete valuation ring $O_K = \mathbb{Z}_p[\sqrt{u}]$ for a non-square p-adic unit u in $\mathbb{Z}_p$. Thus $L \cong \left\{\left(\begin{smallmatrix} a & b \\ pb^\varsigma & -a\end{smallmatrix}\right) | a \in \mathbb{Z}_p, b \in O_K\right\}$. An orthogonal basis of L over $\mathbb{Z}_p$ is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & \sqrt{u} \\ -p\sqrt{u} & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ p & 0\end{smallmatrix}\right)$. Thus $\det s_+$ with respect to this basis is given by 2^3p^2u. Thus $e = 2^{-1}$ for e as in Theorem 4.2. By [Sh99, (3.1.9)], we have $[\widetilde{L} : L] = |\det(s_+)|_p^{-1} = p^2$. Thus

$$\mathfrak{m}_1 = \mathfrak{m}_1(L, \widehat{\Gamma}_\delta) = p^2 2^{-1}(1+p)^{-1}(1-p^{-2}) = \frac{(p-1)}{2}.$$

We need to specify Γ_δ to make $\mathfrak{m}_1 = \mathfrak{m}_1(L, \widehat{\Gamma}_\delta)$ explicit down-to-earth. For $l \nmid \partial$, we identify $R_l = R \otimes_{\mathbb{Z}} \mathbb{Z}_l$ with $M_2(\mathbb{Z}_l)$. The Eichler order of level l^e of $R_l = R \otimes_{\mathbb{Z}} \mathbb{Z}_l$ is given by

$$R_l(l^e) := \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in M_2(\mathbb{Z}_l) \middle| c \in l^e\mathbb{Z}_l\right\}$$

identifying $R \otimes_{\mathbb{Z}} \mathbb{Z}_l$ with $M_2(\mathbb{Z}_l)$. Fix a level N_0 prime to ∂ with prime factorization $\prod_{l|N_0} l^{e(l)}$, and define the Eichler order $R(N_0) \subset D$ of level N_0 by $D \cap \widehat{R}(N_0)$ for $\widehat{R}(N_0) := \widehat{R}^{(N_0)} \times \prod_{l|N_0} R_l(l^{e(l)})$, where $\widehat{R}^{(N_0)} := \prod_{l\nmid N_0} R_l$.

Recall $\delta_+ = 1$ and $\delta_- = \sqrt{\Delta_-}$ for the square free part Δ_- of Δ. Note $D^\pm_0 = \delta_\mp\{v \in D | \mathrm{Tr}(v) = 0\}$. For an integer N_0 outside ∂, decomposing $N_0 = \prod_l l^{e(l)}$, we define $R_0(N_0) = D_0 \cap \delta_\mp R(N_0)$, which we call the Eichler lattice of level N_0. We take $\phi^{(\infty)}_{D_0}$ to be the characteristic function of $\widehat{R}_0(N_0) = R_0(N_0) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subset D_{0,\mathbb{A}^{(\infty)}}$. Then $\Gamma_\delta = \{\gamma \in \mathrm{SO}_\delta(\mathbb{Q}) | \gamma R_0(N_0)\gamma^{-1} \subset R_0(N_0)\} \cong R(N_0)^\times/\{\pm 1\}$. Here, as algebraic groups over $\mathbb{Q}$, we identify $D^\times/Z(D^\times)$ (for the center $Z(D^\times)$ of $D^\times$) with SO_δ by ϱ_0 in §3.1. Since $\mathrm{O}_\delta = \mathrm{SO}_\delta \sqcup \mathrm{SO}_\delta\iota$ (the involution ι regarded as an $\mathbb{Q}$-linear automorphism of D_0),

$$Sh_\delta = \mathrm{O}_\delta(\mathbb{Q})\backslash \mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta C_\infty(\mathrm{O}_\delta) \cong D^\times\backslash D^\times_{\mathbb{A}}/\mathbb{A}^\times \widehat{R}(N_0)^\times C_\infty(D^\times), \tag{4.22}$$

where $C_\infty(G)$ is a maximal compact subgroup of the identity connected component of $G(\mathbb{R})$ for a reductive group $G_{/\mathbb{Q}}$. As is well known, we have $(\widehat{R}^\times : \widehat{R}(N_0)^\times) = N_0\prod_{l|N_0}(1 + l^{-1})$. Then by Theorem 4.2 (and [Sh99, (5.8.1)]) gives the following mass factor

$$\mathfrak{m}_1 = \mathfrak{m}_1(L, \widehat{\Gamma}_\delta) = N_0[\widetilde{L} : L]\left[\prod_{l|N_0}(1 + l^{-1})\prod_{l|\partial} 2^{-1}(1+l)^{-1}(1-l^{-2})\right]. \tag{4.23}$$

We again note here that we need to assume (4.14) as Shimura's computation is done for SO and we need the result for O, and under (4.14), the formula for SO and for O is identical.

4.6. **Choice of ϕ_Z.** We make an explicit choice of ϕ (and the theta series) for $(Z^\pm, Q^\pm)$. To use the results also for imaginary E, we do not assume that E is real in this subsection.

For a lattice L of a quadratic space (V, Q), we write $\phi_L : V_{\mathbb{A}^{(\infty)}} \to \{0, 1\}$ for the characteristic function of $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. More generally, write $\phi_{L,v}$ for the characteristic function of $v + \widehat{L}$. Let $L^* = \{v \in V | s(v, L) \subset \mathbb{Z}\}$ (the dual lattice). Then we define the level $0 < M = M_L \in \mathbb{Z}$ by the minimal positive integer such that $M \cdot s[L^*] \subset 2\mathbb{Z}$ (or equivalently $M \cdot Q(L) \subset \mathbb{Z}$).

First, we deal with (Z^+, Q^+), which is always positive definite with $Q^+(x) = x^2$ and $\delta_+ = 1$. We take a lattice $L = L_+ := N\mathbb{Z}$ of Z^+. Since $L^* = (2N)^{-1}L$, the level of $L = N\mathbb{Z}$ is given by

$M_Z := M_{N\mathbb{Z}} = 4N^2$. Put $\phi_Z := \sum_{v\in\mathbb{Z}/N\mathbb{Z}} \psi(v)\phi_{L,v}$ for a primitive Dirichlet character ψ modulo N. We write this function on $Z_{\mathbb{A}^{(\infty)}} = \mathbb{A}^{(\infty)}$ as ψ_+. For an integer $k \geq 0$ with $\psi(z^{(\infty)})z_\infty^k$ is an even function on $Z_{\mathbb{A}} = \mathbb{A}$, we define $\psi_{k,\infty} : \mathfrak{H} \times \mathbb{R} \to \mathbb{C}$ by $\psi_{k,\infty}(\tau, z) := z^k\mathbf{e}(\tau z^2)$ ($\tau = \xi + \sqrt{-1}\eta \in \mathfrak{H}$). Then $\boldsymbol{\theta}(\phi_k^Z) = \eta^{(k+1)/4}\theta_k(\psi)$ for

$$\theta_k(\psi) := \theta(\psi_+\psi_{k,\infty}) = \sum_{n\in\mathbb{Z}} \psi(n)n^k\mathbf{e}(n^2\tau).$$

For $k \in \{0, 1\}$, $\theta_k(\psi)$ is a modular form on $\Gamma_0(M_{4\mathbb{Z}})$ for $M_{N\mathbb{Z}} = 4N^2$ of weight $k + \frac{1}{2}$ with Neben character $\psi_1 = \left(\frac{-1}{}\right)^k \psi$ in Shimura's sense as described in §1.3. For $k > 1$, $\theta_k(\psi)$ has mixed weight.

Second, take $D_\sigma = D_\sigma^-$ with E imaginary. Recall $\delta_- = \sqrt{\Delta_-}$ for the square-free part $\Delta_- < 0$ of Δ and $Q^-(\delta_- x) = |\Delta_-|x^2$ from ($\mathrm{Z}^\pm$) in §3.1. We take $L_- = \delta_- N\mathbb{Z}$. The dual lattice L^* of L is given by $\delta_-^{-1}(2N)^{-1}\mathbb{Z} = \delta_-\Delta_-^{-1}(2N)^{-1}\mathbb{Z}$. Thus $N(L^*) = \Delta_-^{-1}(2N)^{-2}$; so, $M_Z := M_{L_Z} = 4|\Delta_-|N^2$. For a Dirichlet character ψ modulo N, we take $\phi_Z := \sum_{v\in(\delta_-\mathbb{Z}/\delta_-N\mathbb{Z})} \psi(\delta_- v)\phi_{L_-,v}$. Then $\boldsymbol{\theta}(\phi_k^Z) = \eta^{(k+1)/4}\delta_-^k(\theta_k(\psi)|[\Delta_-])$, where $f|[\Delta_-](\tau) = f(|\Delta_-|\tau)$ and δ_-^k at the front of the left-hand-side comes from our choice of Schwartz function $\phi_\infty(x) = Q^-(x)^k\mathbf{e}(Q^-(x)\tau)$. In the half integral case, this operation changes the Neben character ψ of f to $\psi\left(\frac{\Delta_-}{}\right)$ [Sh73, Proposition 1.3].

Last, take $D_\sigma = D_\sigma^-$ with E real. Recall $\Delta_- > 0$. We take $L_- = \delta_- N\mathbb{Z}$. Then $Q^-(\delta_- n) = -\Delta_- n^2$ is negative definite. The dual lattice L^* of L is given by $\delta_-^{-1}(2N)^{-1}\mathbb{Z} = \delta_-\Delta_-^{-1}(2N)^{-1}\mathbb{Z}$. Thus $N(L^*) = \Delta_-^{-1}(2N)^{-2}$; so, $M_Z := M_{L_Z} = 4\Delta_- N^2$. For a Dirichlet character ψ modulo N, we take $\phi_Z := \sum_{v\in(\delta_-\mathbb{Z}/\delta_-N\mathbb{Z})} \psi(\delta_- v)\phi_{L_-,v}$. Then $\boldsymbol{\theta}(\phi_k^Z) = \eta^{(k+1)/4}\delta_-^k(\theta_k(\psi)|[\Delta_-])(-\overline{\tau})$, where we need to plug in $-\overline{\tau}$ by the standard choice of Schwartz function $\phi_\infty(x) = Q^-(x)^k\mathbf{e}(-Q^-(x)\overline{\tau})$.

In summary, we find

$$M_{Z^\pm} = 4|\delta_\pm^2|N^2. \tag{4.24}$$

4.7. **Verification of the assumption (V).** For the Eichler order $R(N_0)$ of level N_0, we take two lattices in D_0 which are

$$L = R_0(N_0) = \{v \in \delta_\mp R(N_0) | v + v^\iota = 0\} \subset D_0^\pm \ \text{ and } cL \subset D_0^\pm \ \text{ for a chosen } \ 0 < c \in \mathbb{Z},$$

which has a $\mathbb{Z}_l$-basis $\{\delta_\mp \operatorname{diag}[1, -1], \delta_\mp U$ with $U = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \delta_\mp N_0{}^tU\}$ for $l \nmid \partial$ whose dual basis is

$$\{2^{-1}\delta_\mp^{-1} \operatorname{diag}[1, -1], N_0^{-1}\delta_\mp^{-1}U = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \delta_\mp^{-1t}U\}.$$

If $l|\partial$, R_l^*/R_l is killed by l; hence, the level M_L of L is $4N_0\partial\delta_\mp^2$ as $D_0^+ = \delta_- D_0^-$. Similarly the level of cL is $M_{cL} = 4N_0\partial\delta_\mp^2 c^2$. Take ϕ_0 to be

$$\phi_0^{(\infty)} = c_0^{-1}(\phi_{\widehat{L}} - c^3\phi_{c\widehat{L}}) \ \text{ for } c_0 = 1 - c^3 \text{ with } \ 1 < c \in \mathbb{Z} \ \text{ and } \ \phi_0 = c_0^{-1}(\phi_{\widehat{L}} - c^3\phi_{c\widehat{L}}) \cdot \phi_\infty. \tag{4.25}$$

Then $\Gamma_\tau = \Gamma_0(M)$ for

$$M = M_\pm = [4|\delta_\pm^2|N^2, 4c^2|\delta_\mp^2|N_0\partial] \ \text{ (the LCM of } 4|\delta_\pm^2|N^2 \text{ and } 4c^2|\delta_\mp^2|N_0\partial). \tag{4.26}$$

This formula is valid for E both real and imaginary. Writing $\widehat{\Gamma}_\tau$ for the closure of Γ_τ in $\mathrm{Mp}(\mathbb{A}^{(\infty)})$, $\widehat{\Gamma}_\tau = \widehat{\Gamma}_0(M) \supset B(\widehat{\mathbb{Z}})$.

Lemma 4.4. *We have* $\mathbf{r}\left(\begin{smallmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{smallmatrix}\right)\phi_0^{(\infty)}(0) = |a|_{\mathbb{A}}^{3/2}$ *and* $\mathbf{r}(J\left(\begin{smallmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{smallmatrix}\right))\phi_0^{(\infty)}(0) = 0$ *for all* $\left(\begin{smallmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B(\mathbb{A}^{(\infty)})$. *For any Schwartz function* ϕ_∞ *on* $D_{\sigma,\mathbb{R}}^\pm$, $\Phi(g) = (\mathbf{w}(g)(\phi_0^{(\infty)}\phi_\infty))(0)$ *with* $\phi_0^{(\infty)}$ *in* (4.25) *satisfies the condition* (V) *in* §2.4.

Proof. We have $\left(\begin{smallmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$. By (1.1), $\mathbf{r}\left(\begin{smallmatrix} a & a^{-1}b \\ 0 & a^{-1} \end{smallmatrix}\right)\phi_0^{(\infty)} = |a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi^{(\infty)}(av)$. Then by taking $v = 0$, the first assertion follows. Since $\mathbf{r}(J)\phi$ is proportional to the Fourier transform $\widehat{\phi} = \mathcal{F}(\phi)$ by a constant independent of ϕ, we need to compute the finite part of the Fourier transform over $V^{(\infty)} = V_{\mathbb{A}^{(\infty)}}$:

$$c_0\mathcal{F}(|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_0^{(\infty)}(av))(0) = c_0\int_{V^{(\infty)}}|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_0^{(\infty)}(av)\mathbf{e}(s_\pm(v,w))dv|_{w=0} \tag{4.27}$$
$$= \int_{V^{(\infty)}}|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_{\widehat{L}}(av)dv - c^3\int_{V^{(\infty)}}|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_{c\widehat{L}}(av)dv$$
$$\overset{c^{-1}v\mapsto v}{=} \int_{V^{(\infty)}}|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_{\widehat{L}}(av)dv - c^3|c^{(\infty)}|_{\mathbb{A}}^3\int_{V^{(\infty)}}|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_{\widehat{L}}(av)dv = 0.$$

The value of $\Phi(g)$ for $g \in JB(\mathbb{A})$ is proportional to $\mathcal{F}(|a|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v)b)\phi_0^{(\infty)}(av))(0)$ in (4.27) times the corresponding Fourier transform $\mathcal{F}(|a_\infty|_{\mathbb{A}}^{3/2}\mathbf{e}(\pm N(v_\infty)b_\infty)\phi_\infty(a_\infty v_\infty))(0)$ at ∞ by (4.25). Then the condition (V) follows from the vanishing of (4.27). □

Remark 4.5. The convolution integral over $B(\mathbb{A})$ of Theorem 2.3 does not depend on the level M, and the integral over $B(\mathbb{A})$ with respect to $\phi_0^{(\infty)}$ and $\phi_{\widehat{L}}$ are equal, since $E(\phi^{(\infty)}\phi_\infty) = E(\phi_{\widehat{L}}\phi_\infty)$ over $B(\mathbb{A})$. Thus we hereafter forget about the part $c^3\phi_{c\widehat{L}}$ from the integral over $B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})$ with respect to $E(\phi_{\widehat{L}}\phi_\infty)$ and $\theta(\phi_{\widehat{L}}\phi_\infty)$. Hence hereafter $\phi_j^{D_0} = \phi_{\widehat{L}}\Psi_j^{D_0}$ of §4.4 in Case RI (and in the other cases, for the Schwartz function $\Psi_j^{D_0}$ defined later case by base).

4.8. **Verification of the assumption (Key).** To show the assumption (Key) in Theorem 2.5, we need to compute $\boldsymbol{\theta}(\phi_j^Z)(g_\tau)$ and $(\mathbf{r}(g_\tau)\phi_{k-j}^{D_0})(0)$. For $\Phi \in \mathcal{S}(D_{0,\mathbb{A}})$, we have

$$\begin{aligned}&\mathbf{r}_{\mathbb{Q}}(\alpha(\xi))\Phi(\mathfrak{x}) = \mathbf{e}(\frac{s[\mathfrak{x}]}{2}\xi)\Phi(\mathfrak{x}) \ \text{ for } \alpha(\xi) = \left(\begin{smallmatrix}1 & \xi\\ 0 & 1\end{smallmatrix}\right),\ \mathbf{r}_{\mathbb{Q}}(\mathrm{diag}[\eta^{1/2},\eta^{-1/2}])\Phi(\mathfrak{x}) = \eta^{3/4}\Phi(\eta^{1/2}\mathfrak{x}),\\ &\mathbf{r}_{\mathbb{Q}}(g_\tau)\Phi(\mathfrak{x}) = \mathbf{r}_{\mathbb{Q}}(\alpha(\xi)\,\mathrm{diag}[\eta^{1/2},\eta^{-1/2}])\Phi(\mathfrak{x}) = \eta^{3/4}\mathbf{e}(\frac{s[\mathfrak{x}]}{2}\xi)\Phi(\eta^{1/2}\mathfrak{x}).\end{aligned} \tag{4.28}$$

By this,

$$\mathbf{r}(g_\tau)(\phi)(0) = 0 \Leftrightarrow \phi(0) = 0 \ \text{ and } \ \mathbf{r}(g_\tau)(\phi)(0) = \eta^{3/4} \ \text{ if } \phi(0) = 1. \tag{4.29}$$

Note $\phi_{k-j}^{D_0}(0) = 0$ unless $k = j$ as its infinite part is given by (4.19). This verifies the assumption (Key) in Theorem 2.5.

4.9. **Convolution.** Recall $C_\infty = \pi_{\mathbb{A}}^{-1}(\mathrm{SO}_2(\mathbb{R})) \subset \mathrm{Mp}(\mathbb{A})$ for the projection $\pi_{\mathbb{A}} : \mathrm{Mp}(\mathbb{A}) \to \mathrm{SL}_2(\mathbb{A})$, and let $d\mu_\tau$ be the Haar measure inducing $\eta^{-2}d\xi d\eta$ on $\mathfrak{H}$, the volume one measure on $\widehat{\Gamma}_\tau C_\infty \subset \mathrm{Mp}(\mathbb{A})$ and the Dirac measure on each element of $\mathrm{SL}_2(\mathbb{Q}) \subset \mathrm{Mp}(\mathbb{A})$. Write $\overline{B} := B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})$. Thus $\overline{B} \cong U(\mathbb{Z})\backslash\mathfrak{H}$ for the unipotent radical U of B. Lift $F \in S_k^{\mp}(\Gamma_\tau, \varphi\chi_{D_\sigma})$ as in (2.1), $\theta(\phi_j^{D_0})$ and $\theta(\phi_j^Z)$ to $\mathrm{Mp}(\mathbb{A})$ as in (4.19) and (4.18). From this, we have the following identities

$$F(\tau) = \mathbf{F}(g_\tau)\eta^{-k/2},\ \theta(\phi_k^Z)(\tau)\eta^{(1+2k)/4} = \boldsymbol{\theta}(\phi_k^Z)(g_\tau),\ E(\phi_0^{D_0})(\tau)\eta^{3/4} = E(\phi_0^{D_0})(g_\tau). \tag{4.30}$$

where $E(\phi_0^{D_0})(g_\tau)$ is the Eisenstein series as a function on $\mathfrak{H}$ and $\boldsymbol{\theta}(\phi_j^V)(h) = \sum_{v\in V}(\mathbf{w}(h)\phi_j^V)(v)$ for $V = D_0$ and Z. Note the exponent of η for the lifting is the same for $\theta(\phi_0^{D_0})$ and $E(\phi_0^{D_0})$. Then $\mathbf{F}$ and $\boldsymbol{\theta}(\phi_j^Z)$ are functions on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{Mp}(\mathbb{A})$. Since $B(\mathbb{R})\mathrm{SO}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{A}^{(\infty)}) = \mathrm{SL}_2(\mathbb{Q})\widehat{\Gamma}_\tau$

by strong approximation, we have $\mathrm{SL}_2(\mathbb{Q})B(\mathbb{A})\widehat{\Gamma}_\tau C_\infty = \mathrm{Mp}(\mathbb{A})$. We have

$$\begin{aligned}(4.31)\quad &\int_{X_0(M)} \mathbf{F}(g_\tau)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\boldsymbol{\theta}(\phi_j^Z)(g_\tau)E(\phi_{k-j}^{D_0})(g_\tau)\eta^{-2}d\xi d\eta\\
&= \int_{X_0(M)}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})}\mathbf{F}(\gamma h)\boldsymbol{\theta}(\phi_j^Z)(\gamma h)\mathbf{w}(\gamma h)(\phi_{k-j}^{D_0})(0)d\mu_\tau(h)\\
&\overset{(*)}{=} \int_{\overline{B}}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\mathbf{F}(g_\tau)\boldsymbol{\theta}(\phi_j^Z)(g_\tau)\mathbf{r}(g_\tau)(\phi_{k-j}^{D_0})(0)d\mu_\tau(b)\quad \text{(Remark 4.5)}\\
&\overset{(\mathrm{Key})}{=} (-1)^k\int_{\overline{B}}(F(\tau)\eta^{k/2})(\theta(\phi_k^Z)(\tau)\eta^{(1+2k)/4})\eta^{3/4}d\mu_\tau(b)\quad (\text{as } \mathbf{r}(g_\tau)(\phi_0^{D_0})(0)=\eta^{3/4},\ (4.29))\\
&= (-1)^k\int_{\overline{B}}F(\tau)\theta(\phi_k^Z)(\tau)\eta^{k+1}\eta^{-2}d\xi d\eta.\end{aligned}$$

Remark 4.6. To have the identity at $(*)$, the conductor C of F has to be a factor of M as in (4.26).

By [HMI, Theorem 2.65], we find the Neben character of $\theta(\phi)$ is $\psi\chi$ for $\chi_{D_\sigma} = \left(\frac{\det(S)}{}\right)$. Since the product of Neben character of $\theta(\phi)$ and F has to be trivial, the cusp form F need to have Neben character $\psi^{-1}\chi_{D_\sigma}$. We put $L = L_\pm \oplus L_0$ with $L_0 \subset D_0$ such that $L_0 = R(N_0)\cap D_0$ for an Eichler order $R(N_0)$ in D_E of level N_0. Write $\tau_+ = \tau$ and $\tau_- = -\overline{\tau}$. Note $\theta(\phi_k^Z) = \delta_\pm^k \sum_{n\in\mathbb{Z}}\psi(n)n^k\mathbf{e}(|\Delta_\pm|n^2\tau_\pm)$, and write $F(\tau) = \sum_{n=1}^\infty a_n\mathbf{e}(n\tau_\mp) \in S_k^\mp(C,\psi^{-1}\chi_{D_\sigma})$. Then (4.31) is equal to

$$\begin{aligned}(4.32)\quad (-1)^k\int_{\overline{B}}F(\tau)\theta(\phi_k^Z)(\tau)\eta^{k+1}\eta^{-2}d\xi d\eta &= (-1)^k\int_0^\infty\int_0^1 F(\tau)\theta(\phi_k^Z)(g_\tau)d\xi\eta^{k-1}d\eta\\
&= (-1)^k 2\delta_\pm^k\int_0^\infty\sum_{0<n\in\mathbb{Z}}a_{|\Delta_\pm|n^2}\psi(n)n^k\exp(-4\pi|\Delta_\pm|n^2\eta)\eta^{k-1}d\eta\\
&= (-1)^k 2(4\pi)^{-k}\delta_\pm^k|\Delta_\pm|^{-k}\Gamma(k)\sum_{0<n\in\mathbb{Z}}a_{|\Delta_\pm|n^2}\psi(n)n^{-k}.\end{aligned}$$

Since $\delta_- > 0$ when E is real, $\delta_-^k|\Delta_\pm|^{-k} = \delta_-^{-k}$ (while $\delta_-^k|\Delta_\pm|^{-k} = (-\delta_-)^{-k}$ if E is imaginary).

4.10. **The period is an L-value.** We study its Euler factorization assuming F is a primitive eigenform in $S_k^\mp(C,\psi^{-1}\chi_{D_\sigma})$ of conductor C. Put

$$D(s;F,\psi) := \sum_{n=1}^\infty \psi_0(n)a_{n^2}n^{-s} \text{ for the primitive character } \psi_0 \text{ induced by } \psi.$$

Write $a_p = \alpha+\beta$ with $\alpha\beta = \psi^{-1}\chi_{D_\sigma}(p)p^{k-1}$. If it is necessary to indicate the dependence on p, we write α_p for α and β_p for β. Suppose first $\alpha\beta = \psi^{-1}\chi_{D_\sigma}(p)p^{k-1} \neq 0$. Then $a_{p^{2n}} = \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}$. Thus by a well known computation

$$\sum_{n=0}^\infty a_{p^{2n}}X^n = \sum_{n=0}^\infty \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}X^n = \frac{(1-\alpha^2\beta^2X^2)}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}.$$

Suppose $\psi_0^{-1}(p)\alpha = \psi_0^{-1}(p)a_p \neq 0$ but $\beta = 0$. Then

$$\sum_{n=0}^\infty a_{p^{2n}}X^n = \sum_{n=0}^\infty \alpha^{2n}X^n = \frac{1}{1-\alpha^2X}$$

For an Euler product $L(s) = \prod_p E_p(s)^{-1}$, we write $L^{(m)}(s) := (\prod_{p|m}E_p(s))L(s)$ (removing Euler factors at $p|m$ for an integer $m>0$). Let C_0 be the product of primes p ramified in E such that $\psi|_{\mathbb{Z}_p^\times} = \chi_E|_{\mathbb{Z}_p^\times}$ and the p-primary factor of C equals to the p-conductor of χ_E. Then for $p|C_0$, $\alpha\overline{\alpha} = p^{k-1}$ and hence the Euler p-factor of $D(s;F,\psi)$ is given by

$$(1-\alpha^2p^{-s})^{-1} = (1-\alpha\overline{\alpha}^{-1}p^{k-1-s})^{-1}$$

which is a half of the Euler p-factor $(1-\alpha\overline{\alpha}^{-1}p^{k-1-s})^{-1}(1-\alpha^{-1}\overline{\alpha}p^{k-1-s})^{-1}$ of $L(s-k+1, Ad(F)\otimes F)$. Thus one Euler p-factor of $L(s-k+1, Ad(F)\otimes\chi_E)$ is missing at $p|C_0$. This type of discrepancy does not occur at other prime factors of C. Thus writing $C_s(\psi)$ (resp. $C(\psi)$) for the product of prime factors p of C with either $a_p = 0$ or $\psi(p) = 0$ (resp. the conductor of ψ),

$$\begin{aligned}(4.33)\quad \zeta^{(C)}(2s+2-2k)D(s;F,\psi) &= L^{(C_s(\psi))}(s, \rho_F^{Sym^{\otimes 2}} \otimes \psi)\\ &= \left(\prod_{l|C_0}(1-\alpha_l^{-1}\overline{\alpha}_l l^{k-1-s})\right) L^{(C_s(\varphi))}(s-k+1, Ad(F)\otimes\chi_{D_\sigma}).\end{aligned}$$

This settles the case where $D_\sigma = D_\sigma^+$ as $\delta_+ = \Delta_+ = 1$.

Here is how to modify the computation for D_σ^-. We need to compute the Dirichlet series

$$D^-(s) := 2(4\pi)^{-k}\Gamma(k)\sum_{0<n\in\mathbb{Z}} a_{|\Delta_-|n^2}\psi(n)n^{-k}.$$

Only the Euler factor for a prime $p|\Delta_-$ matters, and it is given by

$$\begin{aligned}\sum_{0\le n\in\mathbb{Z}} a_{p^{1+2n}}X^n &= \sum_{n=0}^{\infty}\frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}X^n = \frac{1}{\alpha-\beta}\left[\alpha^2\sum_{n=0}^{\infty}\alpha^{2n}X^n - \beta^2\sum_{n=0}^{\infty}\beta^{2n}X^n\right]\\ &= \frac{1}{\alpha-\beta}\left[\frac{\alpha^2}{1-\alpha^2X}-\frac{\beta^2}{1-\beta^2X}\right] = \frac{\alpha+\beta}{(1-\alpha^2X)(1-\beta^2X)}\\ &= \frac{a_p}{(1-\alpha^2X)(1-\beta^2X)} = \frac{a_p(1-\alpha\beta X)}{(1-\alpha^2X)(1-\alpha\beta X)(1-\beta^2X)}.\end{aligned}$$

We get if $D_\sigma = D_\sigma^-$, $D^-(s) = 0$ unless $a_p \neq 0$ for all $p|\Delta_-$, and otherwise

$$\begin{aligned}(4.34)\quad &\zeta^{(C)}(2s+2-2k)D(s;F,\psi)\\ &= a^{\pm}_{|\Delta_-|}\delta_-^{-k}\left(\prod_{l|C_0}(1-\alpha_l^{-1}\overline{\alpha}_l l^{k-1-s})\right) L^{(C_s(\varphi))}(s-k+1, Ad(F)\otimes\chi_{D_\sigma}),\end{aligned}$$

where $a^{\pm}_{\Delta_-} = \begin{cases}1 & \text{if } V = D_\sigma^+,\\ a_{|\Delta_-|} & \text{if } V = D_\sigma^-.\end{cases}$

Since $\chi_{D_\sigma} = \chi_E = \left(\frac{E/\mathbb{Q}}{}\right)$, all this combined, we obtain

Theorem 4.7. *Suppose $E_{\mathbb{R}} \cong \mathbb{R}\times\mathbb{R}$ and that D is division indefinite. Let F be a primitive Hecke eigenform in $S_k^{\mp}(C,\psi^{-1}\chi_{D_\sigma})$ with $F|T(n) = a_nF$ for the conductor $C|M$ for M as in (4.26) and $f = \theta^*(F)$ be the theta lift holomorphic in z and anti-holomorphic in w:*

$$f(z,w) = \int_{\Gamma_\tau\backslash\mathfrak{H}}\theta(\phi)(\tau;z,w)F(\tau)\eta^{k-2}d\xi d\eta$$

for $\theta(\phi)$ in (4.3). Choose $\phi_Z^{(\infty)}$ associated to Dirichlet character ψ of conductor $N = C(\psi)$ as specified above and $\phi_0^{(\infty)}$ be as in (4.25). Let $\phi = \phi_Z^{(\infty)}\otimes\phi_0^{(\infty)}\Psi_k$ as a Schwartz-Bruhat function of $D^{\pm}_{\sigma,\mathbb{A}}$. Assume $k\ge 2$ and define the $L_E(k-2;\mathbb{C})$-valued harmonic form by

$$\omega(F) := f(z,w)(X-zY)^n(X'-\overline{w}Y')^n dz\wedge d\overline{w}\quad (n = k-2)$$

on the Shimura subvariety Sh_δ as in (4.22). Then if $f(z,w)\neq 0$, for the mass factor $\mathfrak{m}_1$ as in (4.23),

$$\pi\int_{Sh_\delta}(n!)^{-2}\nabla^n(\omega(F))|_{Sh_\delta} = \mathfrak{m}_1E^{\pm}(1)\delta_{\pm}^{-k}(-\sqrt{-1})^{k+1}(2\pi)^{-k}\Gamma(k)L^{(C_s(\psi))}(1, Ad(F)\otimes\chi_E),$$

where $E^{\pm}(1) = \begin{cases}\prod_{p|C}(1-p^{-2})^{-1}\prod_{p|C_0}(1-\alpha_p^{-1}\overline{\alpha}_p p^{-1}) & \textit{if } V = D_\sigma^+,\\ a_{|\Delta_-|}\prod_{p|C}(1-p^{-2})^{-1}\prod_{p|\Delta_-, p\nmid C(\psi)}(1-p^{-1})\prod_{p|C_0}(1-\alpha_p^{-1}\overline{\alpha}_p p^{-1}) & \textit{if } V = D_\sigma^-.\end{cases}$

Here the constant in front of the L-value comes from the constants in Theorem 2.5, (4.17), (4.20) and (4.32). The source and the constant appearing these equations are summarized in the following table whose product in the second row gives rise to the constant:

(4.17)	(4.20)	(4.32)
$-2^{-1}\sqrt{-1}$	$(2\sqrt{-1})^k$	$(-1)^k 2(4\pi)^{-k}\delta_{\pm}^{-k} = (-1)^k 2^{1-k}(2\pi)^{-k}\delta_{\pm}^{-k}$

If $V = D_\sigma^-$, $a_{|\Delta_-|}$ can vanish killing the entire right-hand-side. The assumption $f(z,w) \neq 0$ implies the matching condition (M) just above Example 2.1. If the converse is true, by the non-vanishing of the adjoint L-values implies $f(z,w) \neq 0$.

Remark 4.8. We cannot choose ϕ_0 with the property that $\mathbf{r}(\mathrm{diag}[a, a^{-1}])\phi_0 = \boldsymbol{\chi}(a)\phi_0$ for a non-trivial character $\boldsymbol{\chi}$ as we need $\phi_0(0) \neq 0$ to have non-trivial $E(\phi_0^{D_0})$ (i.e., $(\mathbf{r}\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\phi_0^{D_0})(0) \neq 0$). Similarly, for the theta function to have Neben character ψ, we need to choose $\phi_Z^{(\infty)} = \psi$.

4.11. **Fourier expansion of theta descent.** In this subsection, the choice of the Bruhat function $\phi^{(\infty)}$ is arbitrary. As remarked in §3.1, we have a canonical surjection $\varrho_{D_\sigma} : G_{D_\sigma}^+ \twoheadrightarrow \mathrm{SO}_{D_\sigma}$ for $G_{D_\sigma}^+(\mathbb{Q}) := \{\alpha \in D_E^\times | N(\alpha)/N(\alpha^\sigma) = 1\}$ with $\alpha \in G_{D_\sigma}^+(\mathbb{Q})$ acting on D_σ by $v \mapsto \alpha^{-1} v \alpha^\sigma$. Then

$$\mathrm{O}_{D_\sigma} = \mathrm{SO}_{D_\sigma} \sqcup \mathrm{SO}_{D_\sigma}\iota = \mathrm{SO}_{D_\sigma} \sqcup \mathrm{SO}_{D_\sigma}\sigma \tag{4.35}$$

for the involution ι with $\mathrm{Tr}(x) = x + x^\iota$, and $\mathrm{Ker}(\varrho_{D_\sigma}) = \mathbb{G}_{m/\mathbb{Q}}$ is embedded into $G_{D_\sigma}^+$ to the center of $D^\times \subset D_E^\times$, which is the center Z_{G^+} of $G_{D_\sigma}^+$. Note that $D_E \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \times M_2(\mathbb{R})$ by the isomorphism sending $E \ni e \mapsto (e, e^\sigma) \in M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Thus

$$G_{D_\sigma}^+(\mathbb{R}) = \{(h_1, h_\sigma) \in \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) | \det(h_1)/\det(h_\sigma) = 1\}$$

with $\mathbb{R}^\times$ diagonally embedded onto the center of the product. Then $G_{D_\sigma}^+(\mathbb{R})$ has two connected components whose identity component $G_{D_\sigma}^+(\mathbb{R})^\circ$ modulo center is isomorphic to the target of

$$G_{D_\sigma}^+(\mathbb{R})^\circ \twoheadrightarrow \mathrm{SO}_{D_\sigma}^+(\mathbb{R}) = (\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{\pm 1\},$$

where $\{\pm 1\}$ is embedded into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ diagonally. We put $\mathrm{SO}_{D_\sigma}^+(\mathbb{A}) = \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})\mathrm{SO}_{D_\sigma}^+(\mathbb{R})$ and $\mathrm{SO}_{D_\sigma}^+(\mathbb{Q}) = \mathrm{SO}_{D_\sigma}(\mathbb{Q}) \cap \mathrm{SO}_{D_\sigma}^+(\mathbb{A})$ inside $\mathrm{SO}_{D_\sigma}(\mathbb{A})$. Writing $D_E \ni \gamma \mapsto \gamma \in M_2(\mathbb{R})$ for the left projection and $\gamma \mapsto \gamma^\sigma \in M_2(\mathbb{R})$ for the right projection, we let $\gamma \in D_E^\times$ with totally positive $N(\gamma)$ act on $\mathfrak{H}^2$ by $(z, w) \mapsto (\gamma(z), \gamma^\sigma(w))$.

Pick a Schwartz–Bruhat function $\phi : D_{\sigma,\mathbb{A}} \to \mathbb{C}$ and assume that $\phi_\infty = \overline{\Psi_k(\tau; z, w)}$ for $\Psi_k(\tau; z, w)$ as in (4.1). Since we compute the adjoint of the theta lift, we need to have complex conjugation applied to Ψ_k. In this subsection, the choice of the finite part $\phi^{(\infty)}$ is arbitrary. Consider Siegel's theta series and differential form for $n = k - 2$

$$\theta_k(\phi) = \theta_k(\phi)(\tau; z, w) = \sum_{v \in D_\sigma} \phi(v), \quad \Theta(\phi)(\tau; z, w; \mathbf{x}) = \theta_k(\phi)(\tau; z, w)(X - \overline{z}Y)^n (X' - wY')^n d\overline{z} \wedge dw,$$

where $\mathbf{x} = (X, Y; X', Y')$. This theta series depends on D_σ and hence on σ. However the groups O_{D_σ} and SO_{D_σ} do not depends on σ as seen above. Let

$$\Gamma_\phi := \{\gamma \in \mathrm{SO}_{D_\sigma}^+(\mathbb{Q}) = G_{D_\sigma}^+/Z_{G^+}(\mathbb{Q}) | \phi^{(\infty)}(\gamma^{-1} x \gamma^\sigma) = \phi^{(\infty)}(x) \text{ for all } x \in D_{\sigma,\mathbb{A}^{(\infty)}}\},$$

where $\mathrm{SO}_{D_\sigma}^+(\mathbb{Q}) = \mathrm{SO}_{D_\sigma}(\mathbb{Q}) \cap \mathrm{SO}_{D_\sigma}^+(\mathbb{R})$.

Let $S_{(k,k)}^{+-}(\widehat{\Gamma}_\phi)$ be the space of quaternionic modular forms $f : D^\times \backslash D_{E_\mathbb{A}}^\times \to \mathbb{C}$ of weight $k\infty + k\infty\sigma$ left invariant under $\widehat{\Gamma}_\phi$ (writing a fixed infinite place as ∞ and the other by $\infty\sigma$) holomorphic in z and anti-holomorphic in w. Pick $f \in S_{k,k}^{+-}(\widehat{\Gamma}_\phi)$ and restrict f to $G_{D_\sigma}^+(\mathbb{A})$. We compute the Fourier expansion of the theta descent $\int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{SO}_{D_\sigma}(\mathbb{A})} \theta(\phi)(g, h) f(h) d\mu_h$ and show that its Fourier coefficient for $\mathbf{e}(N(\alpha)\tau)$ is given by a finite sum of the period $P_\alpha = \int_{Sh_\alpha} (n!)^{-2} \nabla^n(\omega(f))$ for the harmonic differential 2-form $\omega(f)$ produced from f. Since $\sigma \in \mathrm{O}_{D_\sigma}(\mathbb{Q})$, we have $\mathrm{O}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{O}_{D_\sigma}(\mathbb{A}) = \mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{SO}_{D_\sigma}(\mathbb{A})$. We extend f originally defined on $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ to $\mathrm{O}_{D_\sigma}(\mathbb{A})$ by putting $f(\sigma x) = f(x)$ for $x \in \mathrm{SO}_{D_\sigma}(\mathbb{A})$. Then $f(x\sigma) = f(\sigma x \sigma) = f(x^\sigma)$, and by this extension, we have

$$\theta_*(f)(g) := \int_{\mathrm{SO}_{D_\sigma}^+(\mathbb{Q})\backslash \mathrm{SO}_{D_\sigma}^+(\mathbb{A})} \theta(\phi)(g, h) f(h) d\mu_h = \int_{\mathrm{O}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{O}_{D_\sigma}(\mathbb{A})} \theta(\phi)(g, h) f(h) d\mu_h. \tag{4.36}$$

The Haar measure $d\mu_h$ satisfies $\int_{\widehat{\Gamma}_\phi C_{\mathbf{i}}} d\mu_h = 1$ for the stabilizer $C_{\mathbf{i}}$ in $\mathrm{SO}^+_{D_\sigma}(\mathbb{R})$ of $\mathbf{i} := (\sqrt{-1}, \sqrt{-1}) \in \mathfrak{H}^2$, $d\mu_h = d\mu_{z,w} = y^{-2}dxdyv^{-2}dudv$ writing $z = x + y\sqrt{-1}$ and $w = u + v\sqrt{-1}$ and induces the Dirac measure at each point of $\mathrm{SO}^+_{D_\sigma}(\mathbb{Q})$. Plainly this integral vanishes if $f(x^\sigma) = -f(x)$. In other words, the descent factors through functions symmetric with respect to the action of σ.

Pick $\alpha \in D_\sigma$ with $N(\alpha) \neq 0$. Let us look into the action σ_α of $\mathrm{Gal}(E/\mathbb{Q})$ different from σ on D_E for $\alpha \in D_\sigma$. Recall $D_\alpha = H^0(\langle\sigma_\alpha\rangle, D_E)$ which is an indefinite quaternion algebra over $\mathbb{Q}$ with $D_\alpha \otimes_{\mathbb{Q}} E \cong D_E$. Then D_α is the even Clifford algebra of $D^\pm_{\alpha,0} = \{x \in \delta_\mp \cdot D_\alpha | x + x^\iota = 0\}$. Write

$$\mathrm{O}_\alpha := \mathrm{O}_{D_{\alpha,0}} \text{ and } \mathrm{SO}_\alpha(A) = \{x \in \mathrm{O}_\alpha(A) | \det_{D_{\alpha,0}}(x) = 1\}.$$

Then $\mathrm{SO}_\alpha(A) \cong (D_\alpha \otimes_{\mathbb{Q}} A)^\times/A^\times$. Since D_α is indefinite, we have $\mathrm{SO}_\alpha(\mathbb{R}) \cong \mathrm{PGL}_2(\mathbb{R})$, whose identity connected component is $\mathrm{SO}^+_\alpha(\mathbb{R}) = \mathrm{PGL}^+_2(\mathbb{R})$, and

$$\mathrm{SO}_\alpha(\mathbb{R}) = \mathrm{SO}^+_\alpha(\mathbb{R}) \sqcup \mathrm{SO}^-_\alpha(\mathbb{R}) \text{ with } \mathrm{SO}^-_\alpha(\mathbb{R}) = \mathrm{PGL}^+_2(\mathbb{R})\,\mathrm{diag}[1,-1].$$

Let $\mathrm{O}^+_\alpha(\mathbb{R}) = \mathrm{SO}^+_\alpha(\mathbb{R}) \sqcup \mathrm{SO}^+_\alpha(\mathbb{R})\iota$, $\mathrm{SO}^+_\alpha(\mathbb{Q}) = \mathrm{SO}_\alpha(\mathbb{Q}) \cap \mathrm{SO}^+_\alpha(\mathbb{R})$ and $\mathrm{O}^+_\alpha(\mathbb{Q}) = \mathrm{O}_\alpha(\mathbb{Q}) \cap \mathrm{O}^+_\alpha(\mathbb{R})$.

Note that the action of ι is scalar multiplication by -1 on the quadratic space $D_{\alpha,0}$, which acts on the symmetric space $\mathfrak{H}$ of $\mathrm{O}^+_\alpha(\mathbb{R})$ trivially. Thus, for any point $(z_0, w_0) \in \mathfrak{H}^2$, $\mathrm{O}^+_\alpha(\mathbb{R})(z_0, w_0) = \mathrm{SO}^+_\alpha(\mathbb{R})(z_0, w_0) \cong \mathfrak{H}$. Let $\Gamma_\alpha := \Gamma_\phi \cap \mathrm{SO}^+_\alpha(\mathbb{Q})$. Then $\Gamma_\alpha\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(z_0, w_0)$ is a Shimura sub-curve inside the Shimura surface associated to $\mathrm{SO}_{D_\sigma} \sim D^\times_E$, whose isomorphism class is independent of $(z_0, w_0) \in \mathfrak{H}^2$. Taking $(z_0, w_0) = \mathbf{i} := (\sqrt{-1}, \sqrt{-1})$, we write $Sh_\alpha := \Gamma_\alpha\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(\mathbf{i})$.

Pick a cusp form $f(z, w) \in S^{+-}_{(k,k)}(\widehat{\Gamma}_\phi)$. Consider the invariant form

$$\omega_{inv} := (z - \overline{z})^{-2}(w - \overline{w})^{-2} dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}. \tag{4.37}$$

The measure $d\omega_{inv}$ associated to ω_{inv} satisfies $d\omega_{inv} = (2\sqrt{-1})^{-2}d\mu_{z,w} = -4^{-1}d\mu_{z,w}$. Then we consider a differential 4-form given by

$$\Omega_\alpha(f) := [\alpha; z, \overline{w}]^k \exp(-\pi\frac{\eta|[\alpha; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})f(z, w)\omega_{inv}.$$

We pick $h_L, h_R \in \mathrm{SL}_2(\mathbb{R})$ so that $\alpha = h_L^{-1}\delta h_R$ for $\delta \in \mathbb{R}^\times$. Identify $(\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{\pm 1\} = \mathrm{SO}^+_{D_\sigma}(\mathbb{R})$ and put $h = (h_L, h_R) \in \mathrm{SO}^+_{D_\sigma}(\mathbb{R})$. Then

$$h\alpha h^{-\sigma} = (h_L, h_R)(\alpha, \alpha^\sigma)(h_R^{-1}, h_L^{-1}) = (h_L\alpha h_R^{-1}, *) = (\delta, *).$$

Since $(h\alpha h^{-\sigma})^\sigma = h^\sigma\alpha^\sigma h^{-1} = \pm h^\sigma\alpha^\iota h^{-1} = \pm(h\alpha h^{-\sigma})^{-1}$, we find that $* = \pm\delta = \delta^\sigma$ in the σ-component of $E_{\mathbb{R}}$. Thus in $D_{E_{\mathbb{R}}}$, we have $\alpha = h^{-1}\delta h^\sigma$ writing $(\delta, \pm\delta)$ as $\delta \in E_{\mathbb{R}}$.

Noting that ω_{inv} is invariant under the action of holomorphic automorphisms of $\mathfrak{H}^2$,

$$\begin{aligned}\Omega_\alpha(f) &= [h^{-1}\delta h^\sigma; z, \overline{w}]^k \exp(-\pi\frac{\eta|[h^{-1}\delta h^\sigma; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})f(z, w)\omega_{inv} \\ &\overset{h=(h_L,h_R)}{=} [\delta, g(z), h(\overline{w})]^k j(h_L, z)^k j(h_R, \overline{w})^k \exp(-\pi\frac{\eta|[\delta; h_L(z), h_R(\overline{w})]|^2}{|\operatorname{Im}(h_L(z))\operatorname{Im}(h_R(w))|})f(z, w)\omega_{inv} \\ &\overset{h_L(z)\mapsto z, h_R(w)\mapsto w}{=} [\delta; z, \overline{w}]^k \exp(-\pi\frac{\eta|[\delta; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})f|h^{-1}(z, w)\omega_{inv},\end{aligned} \tag{4.38}$$

where $f|h^{-1}(z, w) = j(h_L, z)^k j(h_R, \overline{w})^k f(h_L^{-1}(z), h_R^{-1}(w))$. The function $f|h^{-1}$ has invariance under $h\Gamma_\phi h^{-1}$. Recalling $\mathrm{SO}_\alpha = \mathrm{SO}_{D_{\alpha,0}}$, we have $Sh_\alpha = \Gamma_\alpha\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(\mathbf{i}) \cong h\Gamma_\alpha h^{-1}\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(\mathbf{i}) =: Sh^h_\alpha$ and $h\Gamma_\alpha h^{-1} \subset D^\times_{\mathbb{R}} = \mathrm{GL}_2(\mathbb{R}) \overset{\mathrm{diag}}{\hookrightarrow} (D_{E_{\mathbb{R}}})^\times = \mathrm{GL}_2(\mathbb{R})^2$ diagonally embedded. Then

$$\omega^h_\alpha(f)(z) := \delta^{-2}[\delta; z, \overline{z}]^k f|h^{-1}(z, z)(z - \overline{z})^{-2}dz \wedge d\overline{z} \overset{[\delta;z,\overline{z}]=\delta(z-\overline{z})}{=} [\delta; z, \overline{z}]^{k-2} f|h^{-1}(z, z)dz \wedge d\overline{z}$$

is a closed harmonic 2-form on Sh^h_α, and pulling $[\delta; z, \overline{z}]^{k-2}f|h^{-1}(z, z)dz \wedge d\overline{z}$ back to Sh_α by $h : Sh_\alpha \cong Sh^h_\alpha$, we define

$$\omega(f)(z, w; \mathbf{x}) := f(z, w)(X - zY)^{k-2}(X' - \overline{w}Y')^{k-2}dz \wedge d\overline{w} \text{ on } \mathfrak{H} \times \mathfrak{H}. \tag{4.39}$$

To introduce an SO_{D_σ}-invariant pairing $(\cdot, \cdot) : L_E(n; A) \otimes_A L_E(n; A) \to A$, we prepare another set of variables $\mathbf{s} := (S, T; S', T')$ as the variable of the right factor $L_E(n; A)_r = L_{\mathrm{id}}(n; A)_r \otimes_A L_\sigma(n; A)_r$:

$$L_{\mathrm{id}}(n; A)_r := AS^n + AS^{n-1}T + \cdots + AT^n \text{ and } L_\sigma(n; A)_r := AS'^n + AS'^{n-1}T' + \cdots + AT'^n.$$

The variable of the left factor $L_E(n;A)_l$ are denoted by $(X,Y;X',Y')$ as before. We pair $L_{\mathrm{id}}(n;A)_l = AX^n + AX^{n-1}Y + \cdots + AY^n$ on the left with $L_{\mathrm{id}}(n;A)_r = AS^n + AS^{n-1}T + \cdots + AT^n$ on the right to have the pairing $(\cdot,\cdot)_{\mathrm{id}} : L_{\mathrm{id}}(n;A)_l \otimes_A L_{\mathrm{id}}(n;A)_r \to A$ given by $P(X,Y;S,T) \mapsto (n!)^{-2}\nabla^n_{\mathrm{id}}(P(X,Y;S,T))$, where $\nabla_{\mathrm{id}} = \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial Y \partial S}$. Similarly, we define $\nabla_\sigma = \frac{\partial^2}{\partial X' \partial T'} - \frac{\partial^2}{\partial Y' \partial S'} : L_\sigma(n;A)_l \otimes_A L_\sigma(n;A)_r \to A$. and put $(\cdot,\cdot)_\sigma = (n!)^{-2}\nabla^n_\sigma$. Finally we define the desired pairing

$$(\cdot,\cdot) = (\cdot,\cdot)_n = (\cdot,\cdot)_{\mathrm{id}} \otimes (\cdot,\cdot)_\sigma : L_E(n;A) \otimes_A L_E(n;A) \to A \tag{4.40}$$

by $(P,Q)_n = n!^{-2}\nabla^n_{\mathrm{id}} n!^{-2}\nabla^n_\sigma(PQ)$. The pairing $(\cdot,\cdot)$ is SO_{D_σ}-invariant [H94, (11.2a,b)].

Since $\pi_\alpha := (n!)^{-2}\nabla^n : L_E(n;A)|_{\mathrm{SO}_\alpha} \to A$ and $\pi_{D_\sigma} := (n!)^{-2}\nabla^n_{\mathrm{id}}(n!)^{-2}\nabla^n_\sigma : L_E(n;A) \otimes_A L_E(n;A) \to A$ is SO_α-equivariant, we have a commutative diagram up to constants

$$\begin{array}{ccc} L_E(n;A)|_{\mathrm{SO}_\alpha} \otimes_A L_E(n;A)|_{\mathrm{SO}_\alpha} & \longrightarrow & L_E(n;A) \otimes_A L_E(n;A) \\ {\scriptstyle \pi_\alpha \otimes \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \pi_{D_\sigma}} \\ A = A \otimes_A A & = & A. \end{array} \tag{4.41}$$

Writing the variables of the left (resp. right) factor of $L_E(n;A)|_{\mathrm{SO}_\alpha}$ as X,Y,X',Y' (resp. S,T,S',T'), we find from [H99, page 141] $\pi_\alpha(X^{n-j}Y^jX'^jY'^{n-j}) = (-1)^j\binom{n}{j}^{-1} = \pi_\alpha(S^{n-j}T^jS'^jT'^{n-j})$ and $\pi_{D_\sigma}(X^{n-j}Y^jX'^jY'^{n-j}S^{n-j}T^jS'^jT'^{n-j}) = \binom{n}{j}^{-2}$; so, the above diagram commutes.

Note that $(n!)^{-2}\nabla^n(\omega(f))$ is given by $\delta^{2-k}[\alpha;z,\overline{w}]^{k-2}f|(z,w)dz \wedge d\overline{w}$ ($\alpha = h^{-1}\delta h^\sigma$ with $h \in \mathrm{SO}_{D_\sigma}(\mathbb{R})$; so, $\delta^2 = N(\alpha)$), and

$$(n!)^{-2}\nabla^n(\omega(f))|_{Sh_\alpha} = \delta^{2-k}h^*(\omega^h_\alpha(f)(z)|_{Sh^h_\alpha}). \tag{4.42}$$

Then

$$(\Theta(\phi)(\tau;z,w;\mathbf{x}),\omega(f)(z,w;\mathbf{s})) = \sum_{\alpha \in D_\sigma} \phi^{(\infty)}(\alpha)\Omega_\alpha(f)\mathbf{e}(\pm N(\alpha)\tau).$$

We are going to compute $\sum_{\gamma \in \Gamma_\phi/\Gamma_\alpha} \int_{Sh} \Omega_{\gamma^{-1}\alpha\gamma^\sigma}(f)$ converting it into an integral over $\Gamma_\alpha \backslash \mathfrak{H}^2$ by averaging.

By the diagonally embedded $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SL}^\Delta_2(\mathbb{R}) \subset \mathrm{SL}_2(E_\mathbb{R})$, $\exp(-\pi\frac{\eta|[\delta;z,\overline{w}]|^2}{|\mathrm{Im}(z)\,\mathrm{Im}(w)|})$ is invariant under $\mathrm{SL}^\Delta_2(\mathbb{R})$. Note $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H}$ whose coordinate on the right is given by (z,v) ($w = u + v\sqrt{-1}$). Consider the exact sequence of analytic manifolds:

$$1 \to \mathrm{SL}^\Delta_2(\mathbb{R}) \hookrightarrow \mathrm{SL}_2(\mathbb{R})^2 \xrightarrow[\twoheadrightarrow]{(h_L,h_R)\mapsto h_L^{-1}h_R} \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2 \cong \mathrm{SL}_2(\mathbb{R}) \to 1. \tag{4.43}$$

Writing $\mathrm{SO}^\Delta_2(\mathbb{R})$ for the image of $\mathrm{SO}_2(\mathbb{R})$ in $\mathrm{SL}_2(\mathbb{R})^2$, we make the quotient of $\mathrm{SL}^\Delta_2(\mathbb{R})$ by $\mathrm{SO}^\Delta_2(\mathbb{R})$ from the right. Since $(h_Lg, h_Rg) \mapsto g^{-1}h_L^{-1}h_Rg$ under the quotient map, this induces a conjugation by $\mathrm{SO}_2(\mathbb{R})$ on $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2 \cong \mathrm{SL}_2(\mathbb{R})$.

We make the right quotient by $\mathrm{SO}_2(\mathbb{R})^2$ of $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2 \cong \mathrm{SL}_2(\mathbb{R})$, which produces

$$\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2 = \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2/\mathrm{SO}_2(\mathbb{R})^2 \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong T^+(\mathbb{R}) \cong \mathbb{R}^\times_+, \tag{4.44}$$

where T is the diagonal torus of SL(2) and $T^+(\mathbb{R})$ is the identity connected component. This is because of the Cartan decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R})T^+(\mathbb{R})\mathrm{SO}_2(\mathbb{R})$ for the diagonal torus T of SL(2) [SL2, VII.2], and the isomorphism is induced by $w \mapsto v = \mathrm{Im}(w) \in \mathbb{R}^\times_+$. Thus we obtain

$$\begin{aligned} &(\mathrm{SL}^\Delta_2(\mathbb{R})/\mathrm{SO}^\Delta_2(\mathbb{R})) \times \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2 \cong \mathfrak{H} \times \mathrm{SL}_2(\mathbb{R}) \ \text{ given by } ((g_z,g_z),g) \leftrightarrow (z,g) \\ &(\mathrm{SL}^\Delta_2(\mathbb{R})/\mathrm{SO}^\Delta_2(\mathbb{R})) \times \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathfrak{H} \times \mathbb{R}^\times_+ \ \text{ given by } ((g_z,g_z),g_v) \leftrightarrow (z,v). \end{aligned} \tag{4.45}$$

Then we have $h\Gamma_\alpha h^{-1}\backslash\mathfrak{H}^2 \cong Sh^h_\alpha \times (\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2) \cong Sh^h_\alpha \times \mathbb{R}^\times_+$ by sending $(z,w) \in h\Gamma_\alpha h^{-1}\backslash\mathfrak{H}^2$ to the pair $(z,z) \in Sh^h_\alpha$ and $v = \mathrm{Im}(w) \in \mathbb{R}^\times_+$.

Fix the differential form $d\theta$ inducing the Haar measure on $\mathrm{SO}_2(\mathbb{R}) \cong S^1$ of measure 1. We have the factor $\mathbb{R}^\times_+ = \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H}$ in the above argument and hence $\mathfrak{H}^2 = \mathfrak{H}^\Delta \times \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^\times_+$ by $\mathfrak{H}^2 \ni (z,w) \mapsto ((z,z),\theta,v) \in \mathfrak{H}^\Delta \times \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R}^\times_+$, where $\mathfrak{H}^\Delta$ is $\mathfrak{H}$ diagonally embedded into $\mathfrak{H}^2$. Then we can write $\omega_{inv} = (z-\overline{z})^{-2} dz \wedge d\overline{z} \wedge d\theta \wedge d\varphi_v$ for a differential form $d\varphi_v$ on $\mathbb{R}_+^\times$. We will compute $d\varphi_v$ later in the proof of Theorem 4.10. Thus

$$h^{-*}\Omega_\alpha(f) = h^{-*}\omega_\alpha^h(f) \wedge \exp(-\pi \frac{\eta|[\delta; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|}) d\theta \wedge d\varphi_v. \tag{4.46}$$

Define the period of $\omega(f)$ over Sh_α as

$$P_\alpha(f) = P(f;\alpha,\Gamma_\phi) := \delta^n \int_{Sh_\alpha} (n!)^{-2}\nabla^n(\omega(f))|_{Sh_\alpha} \overset{(4.42)}{=} \int_{Sh_\alpha^h} [\delta; z, \overline{z}]^{k-2} f|h^{-1}(z,z) dz \wedge d\overline{z}, \tag{4.47}$$

where $\alpha = h^{-1}\delta h^\sigma = \delta h^{-1} h^\sigma$ for $h \in \mathrm{SO}_{D_\sigma}(\mathbb{R})$. Comparing this definition with (4.15), we find

$$P_\delta(\theta^*(F)) = \delta^n P'_\delta(\theta^*(F)). \tag{4.48}$$

Since $Sh_\alpha \cong Sh_{\gamma^{-1}\alpha\gamma^\sigma}$ for $\gamma \in \Gamma_\phi$ and this isomorphism brings $\omega_\alpha(f)$ to $\omega_{\gamma^{-1}\alpha\gamma^\sigma}(f)$, $P_\alpha(f)$ only depends on the class of α in D_σ/Γ_ϕ.

Remark 4.9. To have the non-vanishing period, the locally constant sheaf in which $\omega(f)$ has values needs to have trivial constant sheaf as a quotient (i.e., $\nabla^n\omega(f) \neq 0$), since the projection to any non-constant simple direct factor has vanishing integral. In other words, the Neben character of the theta lift has to be unramified everywhere. This follows from the fact that $\mathrm{SO}_{D_\sigma} = G^+_{D_\sigma}/Z_{G^+}$ which is embedded into $D_E^\times/E^\times$, and hence the center acts trivially on the theta lift. Thus the theta lift differs from the usual base-change lift (and actually it is the base-change lift twisted by a character to have trivial central character; i.e., this is the central character identity imposed in [H99, §2.4]).

We have the following explicit q-expansion of the theta descent for any indefinit quaternion algebra $D_{/\mathbb{Q}}$ including $M_2(F)$:

Theorem 4.10. *Suppose $\phi_\infty = \overline{\Psi_k(\tau; z, w)}$ for $\Psi_k(\tau; z, w)$ as in (4.1) and that $f \in S^{+-}_{(k,k)}(\widehat{\Gamma}_\phi)$ is a cusp form on $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ of weight $k > 0$ anti-holomorphic in w and holomorphic in z as above. Then*

$$\int_{\Gamma_\phi\backslash\mathfrak{H}^2} \theta_k(\phi)(\tau; z, w) f(z,w)\omega_{inv} = (8\sqrt{-1})^{-1} \sum_{\alpha \in D_\sigma/\Gamma_\phi; N(\alpha)>0} \phi^{(\infty)}(\alpha) P_\alpha(f) \mathbf{e}(N(\alpha)\tau_\mp),$$

where $\tau_+ = \tau$ and $\tau_- = -\overline{\tau}$.

By applying complex conjugation to the formula in Theorem 4.10, we get the result for ϕ with $\phi_\infty = \Psi_k(\tau; z, w)$ for $\Psi_k(\tau; z, w)$ as in (4.1); so, this assumption is harmless.

This theorem is far more explicit than [O77, Theorem 1] and covers general D_σ not treated in [Sh81] II Proposition 5.1 and [H06] Theorem 3.2. Another paper of Shimura [Sh82, Theorem 2.2] gives a similar result for D_0 in place of D_σ.

Remark 4.11.
1. Assume that f is the theta lift of an elliptic Hecke eigenform F via the quadratic space $(D^\pm_{\sigma_\alpha}, \pm N)$ for $\alpha \in D^\pm_\sigma$ with $D \cong D_\alpha$. Then $\alpha = \xi\delta\xi^{\sigma\iota}$ for $\xi \in D_E^\times$ and a scalar $\delta \in D^\pm_{\sigma_\alpha}$ by Lemma 3.2, and $D_\sigma = \xi D^\pm_{\sigma_\alpha}\xi^{-1}$ by Corollary 3.3. Thus Sh_α with respect to $\alpha \in D^\pm_\sigma$ is isomorphic to $Sh_{\delta,\sigma_\alpha}$ for $Sh_{\delta,\sigma_\alpha} = Sh_\delta$ with respect to $D^\pm_{\sigma_\alpha}$, and if we compute the period with respect to the theta lift for D_{σ_α}, $P_\alpha(f)$ is the L-value $L(1, Ad(F) \otimes \chi_E)$ times a constant depending on α as ξ induces a correspondence between $Sh_{\delta,\sigma_\alpha}$ and Sh_α. The constant is an adjoint generalization of the mass of Siegel–Shimura. In particular, the vanishing of the eigenvalue for $T(l)$ for a prime l only comes from the vanishing of the mass factor, which is an interesting fact.
2. Since the Howe conjecture is solved for general dual reductive pairs in [W90] and [GT16], the theta lift map is (essentially) Hecke equivariant for almost all SL(2)-Hecke operators, and hence the theta descent is also essentially Hecke equivariant. The Howe conjecture concerns only for SL or Mp, and hence it does not guarantee fully Hecke equivariance for GL-Hecke operators. Therefore we expect that the period $P_\alpha(f)$ vanishes if f is not the theta lift from F via the quadratic space $(D^\pm_{\sigma_\beta}, \pm N)$ for any choice of $\beta \in D^\pm_\sigma$. We study this point in our subsequent paper.

(3) If $\beta \in D_\sigma$ cannot be written as $\xi\delta\xi^{\sigma\iota}$, it is not clear that the theta lift $\theta^*_\beta(F)$ of F with respect to D_{σ_β} coincides with the one $\theta^*(F)$ with respect to D_σ? Since we expect that the two theta lifts are Hecke equivariant, by a representation theoretic multiplicity one, if $\theta^*(F)\theta^*_\beta(F) \neq 0$, $\theta^*(F)$ and $\theta^*_\beta(F)$ span the same automorphic representation of $D^\times_{E_\mathbb{A}}$, and this would imply the periods with respect to Shimura subvarieties in Sh_E coming from $D^\times$ and $D^\times_\beta$ have the almost equal integrality (i.e., a consequence of the Tate conjecture). We hope to study this point in our subsequence paper.

We prove the result for D^+_σ as the proof of the case of D^-_σ is almost identical.

Proof. Fix $\alpha \in D^+_\sigma$. To shorten the formula, we remove $\mathbf{e}(-N(\alpha)\overline{\tau})$ from each term in the sum and put it back at the end. First suppose that $N(\alpha) > 0$. Let Φ be a fundamental domain in $\mathfrak{H}^2$ of $\Gamma = \Gamma_\phi$. Then, writing $\Gamma_\alpha := \Gamma \cap \mathrm{SO}_\alpha(\mathbb{Q})$,

$$\begin{aligned}(*) &:= \eta \sum_{\gamma\in\Gamma_\alpha\backslash\Gamma} \int_\Phi \exp(-\pi\frac{\eta|[\gamma^{-1}\alpha\gamma^\sigma; z, \overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})[\gamma^{-1}\alpha\gamma^\sigma; z, \overline{w}]^k f(z,w)\omega_{inv}\\ &= \eta \sum_\gamma \int_\Phi \exp(-\pi\frac{\eta|[\alpha;\gamma(z),\gamma^\sigma(\overline{w})]|^2}{|\operatorname{Im}(\gamma(z))\operatorname{Im}(\gamma^\sigma(w))|}) j(\gamma;z)^k j(\gamma^\sigma,\overline{w})^k[\alpha;\gamma(z),\gamma^\sigma(\overline{w})]^k f(z,w)\omega_{inv}\\ &\overset{(1)}{=} \eta \sum_\gamma \int_{\gamma(\Phi)} \exp(-\pi\frac{\eta|[\alpha;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|}) j(\gamma;\gamma^{-1}(z))^k j(\gamma^\sigma,\gamma^{-\sigma}(\overline{w}))^k[\alpha;z,\overline{w}]^k f(\gamma^{-1}(z),\gamma^{-\sigma}(w))\omega_{inv}\\ &= \eta \int_{\Gamma_\alpha\backslash\mathfrak{H}^2} \exp(-\pi\frac{\eta|[\alpha;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})[\alpha;z,\overline{w}]^k f(z,w)\omega_{inv}\end{aligned}$$

is the coefficient of $\mathbf{e}(-N(\alpha)\overline{\tau})$. Here at (1), we made variable change: $z \mapsto \gamma^{-1}(z)$ and $w \mapsto \gamma^{-\sigma}(w)$.

Write $\mathrm{SO}_{D_\sigma}(\mathbb{R}) = \mathrm{SO}^+_{D_\sigma}(\mathbb{R}) \sqcup \mathrm{SO}^-_{D_\sigma}(\mathbb{R})$ with $\mathrm{SO}^-_{D_\sigma}(\mathbb{R}) = ((\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{\pm1\})(\epsilon,\epsilon)$ for $\epsilon = \mathrm{diag}[-1,1]$. Then $\mathrm{SO}_2(\mathbb{R})^2/\{\pm1\} \sqcup (\mathrm{SO}_2(\mathbb{R})^2/\{\pm1\})(\epsilon,\epsilon)$ is a maximal compact subgroup of $\mathrm{SO}_{D_\sigma}(\mathbb{R})$. Let $C = \mathrm{SO}_2(\mathbb{R})^2/\{\pm1\} \sqcup (\mathrm{SO}_2(\mathbb{R})^2/\{\pm1\})(\epsilon,\epsilon) \subset \mathrm{SO}_{D_\sigma}(\mathbb{R})$ and $C^+ = \mathrm{SO}_2(\mathbb{R})^2/\{\pm1\} \subset \mathrm{SO}^+_{D_\sigma}(\mathbb{R})$. Then

$$\mathrm{SO}_{D_\sigma}(\mathbb{R})/C \cong \mathrm{SO}^+_{D_\sigma}(\mathbb{R})/C^+ \cong \mathrm{PSL}_2(\mathbb{R})^2/\mathrm{PSO}_2(\mathbb{R})^2 \cong \mathfrak{H}^2.$$

The component $\mathrm{SO}^+(\mathbb{R})$ acts on $\mathfrak{H}^2$ holomorphically, and $\mathrm{SO}^-(\mathbb{R})$ acts anti-holomorphically. In each case, C, C^+ and $\mathrm{PSO}_2(\mathbb{R})^2$ are the stabilizer of $\mathbf{i} = (\sqrt{-1},\sqrt{-1}) \in \mathfrak{H}^2$.

Now we identify $\mathfrak{H}^2$ with $\mathrm{SO}^+_{D_\sigma}(\mathbb{R})(\mathbf{i})$. Choose a fundamental domain Φ_α of $\mathrm{SO}^+_\alpha(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2$ (by j) and write the image of $(z,w) \in \Phi_\alpha$ in $\mathrm{SO}^+_\alpha(\mathbb{R})\backslash\mathfrak{H}^2$ as (z_0,w_0). Then for $g \in \mathrm{SO}^+_\alpha(\mathbb{R})$, by (4.4), we have $\frac{|[\alpha;g(z),g^\sigma(\overline{w})]|^2}{|\operatorname{Im}(g(z))\operatorname{Im}(g^\sigma(\overline{w}))|} = \frac{|[g^{-1}\alpha g^{-\sigma\iota};z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|} = \frac{|[\alpha;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|}$ and

$$(*) = \eta\int_{\Phi_\alpha}\exp(-\pi\eta\frac{|[\alpha;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})\int_{\Gamma_\alpha\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(z_0,w_0)}[\alpha;g(z_0),g^\sigma(\overline{w}_0)]^k f(g(z_0,w_0))d\varphi_\alpha d\varphi', \tag{4.49}$$

where $d\varphi_\alpha$ is the differential form on $\Gamma_\alpha\backslash\mathrm{SO}^+_\alpha(\mathbb{R})(z_0,w_0)$ given by $h^*((z-\overline{z})^{-2}dz \wedge d\overline{z})$ for h as in (4.38) and $d\varphi'$ is a measure associated to an invariant 1-form φ' on $\mathrm{SO}^+_\alpha(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathbb{R}^\times_+$ so that $\omega_{inv} = \varphi_\alpha \wedge \varphi' \wedge d\theta$. Note that $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})^2/\mathrm{SO}_2(\mathbb{R})^2 \xrightarrow[i]{\sim} [0,1)$. We will make explicit the isomorphism i and the differential form φ' later.

As explained in (4.38), replacing $D_{\alpha,\mathbb{R}} \subset D_{E_\mathbb{R}}$ by $D_\mathbb{R} = hD_{\alpha,\mathbb{R}}h^{-1} \subset D_{E_\mathbb{R}}$, we may assume that $\alpha = \delta \in E_\mathbb{R}$ (but $\delta \in E_\mathbb{R}$ not necessarily in E); so, we pretend $\sigma = \sigma_\alpha$ and $D_{\alpha,\mathbb{R}} = D_\mathbb{R}$. Then we just as notation replace Sh_α by Sh^h_α. Our $\delta \in E^\times_\mathbb{R}$ satisfies $\delta^2 = N(\delta) = N(\alpha)$. Since writing $h\Gamma h^{-1}$ for Γ and $h\Gamma_\alpha h^{-1}$ for Γ_α all the time is cumbersome, we hereafter assume that $\delta \in E^\times$. By doing this, we do not lose the details and we can simplify a lot the notation. However we do need to conjugate back at the end to α by h and remember that $N(\alpha) = N(\delta)$.

If we choose different $(z'_0,w'_0) \in \Phi_\delta$, taking a path $\gamma := [(z_0,w_0),(z'_0,w'_0)]$ in Φ_δ diffeomorphic to the real interval $[0,1]$, we find $\Delta := \{\Gamma_\delta\backslash(\mathrm{SO}^+_\delta(\mathbb{R})(z,w))|(z,w) \in \gamma\}$ is isomorphic to $\Gamma_\delta\backslash\mathrm{SO}^+_\delta(\mathbb{R})\times\gamma$, and hence its boundary $\partial\Delta = \Gamma_\delta\backslash\mathrm{SO}^+_\delta(\mathbb{R})(z_0,w_0) - \Gamma_\delta\backslash\mathrm{SO}^+_\delta(\mathbb{R})(z_0,w_0)$. Since $\mathrm{SO}_\delta(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$, we identify $\mathrm{SO}^+_\delta(\mathbb{R})(z,w) \cong \mathfrak{H}$, $\Gamma_\delta\backslash(\mathrm{SO}^+_\delta(\mathbb{R})(z,w)) = \Gamma_\delta\backslash\mathfrak{H}$ is a Shimura curve whose isomorphism class is independent of (z,w). Thus we may take $(z_0,w_0) = \mathbf{i}$. Then we have

$$(*) = \delta^2 P_\delta(f)\eta \times \int_{\mathrm{SO}^+_\delta(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\varphi'$$
$$= \delta^2 P_\delta(f)\eta \times \int_{\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\varphi',$$

where $\mathrm{SL}^\Delta_2(\mathbb{R})$ is the image of $\mathrm{SL}_2(\mathbb{R})$ embedded diagonally into $\mathrm{SL}_2(\mathbb{R})^2$.

We need to compute

$$\int_{\Phi_\delta} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\varphi' = \int_{\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\varphi'.$$

For $g_z = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $g_z(\sqrt{-1}) = z$, we have $(g_z,g_z)^{-1}(z,w) = (\sqrt{-1}, g_z^{-1}(w))$. If $(\sqrt{-1},w)$ and $(\sqrt{-1},w')$ is equivalent in $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2$, then $(\sqrt{-1},w') = (g(\sqrt{-1}), g(w))$; so, $g \in \mathrm{SO}_2(\mathbb{R})$ and $w' = g(w)$. Thus $\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2 \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H}$. As before, write the variable of $\mathfrak{H}$ on the right-hand-side as w. Consider $i : w \mapsto \frac{w-\sqrt{-1}}{w+\sqrt{-1}}$ which induces an isomorphism $\mathfrak{H} \cong \mathfrak{D} := \{\mathfrak{z} \in \mathbb{C} \big| |\mathfrak{z}| < 1\}$ whose inverse is $\mathfrak{z} \mapsto \sqrt{-1}\frac{1+\mathfrak{z}}{1-\mathfrak{z}}$. Since $\mathrm{SO}_2(\mathbb{R})$ acts on $\mathfrak{D}$ by rotation, we find $\mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{H} \cong \mathrm{SO}_2(\mathbb{R})\backslash\mathfrak{D} \cong [0,1)$. We see $|\sqrt{-1}-\overline{w}|^2 = 4|1-\mathfrak{z}|^{-2}$ and $w - \overline{w} = 2\sqrt{-1}\frac{1-\mathfrak{z}\overline{\mathfrak{z}}}{|1-\mathfrak{z}|^2}$. Writing $\mathfrak{z} = re^{2\pi\sqrt{-1}\theta}$ for $r,\theta \in [0,1)$, we have $1 - \mathfrak{z}\overline{\mathfrak{z}} = 1 - r^2$. Thus

$$\frac{|i-\overline{w}|^2}{\operatorname{Im}(w)} = \frac{4|1-\mathfrak{z}|^{-2}}{\frac{(1-\mathfrak{z}\overline{\mathfrak{z}})}{|1-\mathfrak{z}|^2}} = 4(1-\mathfrak{z}\overline{\mathfrak{z}})^{-1} = 4(1-r^2)^{-1}.$$

Since $i_*((w-\overline{w})^{-2}dw \wedge d\overline{w}) = (1-\mathfrak{z}\overline{\mathfrak{z}})^{-2}d\mathfrak{z} \wedge d\overline{\mathfrak{z}} = -2\pi\sqrt{-1}(1-r^2)^{-2}rdr \wedge d\theta$, we find

$$d\varphi' = -2\pi\sqrt{-1}(1-r^2)^{-2}rdr$$

and

$$\int_{\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\varphi' = -2\pi\sqrt{-1}\int_0^1 \exp(-8\pi\eta|\delta|^2(1-r^2)^{-1})(1-r^2)^{-2}rdr.$$

By the isomorphism $i : \mathfrak{H} \cong \mathfrak{D}$, $[\sqrt{-1}, \infty\sqrt{-1}) \cong [0,1)$ by $\sqrt{-1}v \mapsto r$ with $r = \frac{v-1}{v+1}$. Thus $dr = 2(v+1)^{-2}dv$ and $1-r^2 = 4v(v+1)^{-2}$. Therefore, we can rewrite

$$-2\pi\sqrt{-1}\int_0^1 \exp(-8\pi\eta|\delta|^2(1-r^2)^{-1})(1-r^2)^{-2}rdr$$
$$= -4^{-1}\pi\sqrt{-1}\exp(-4\pi\eta|\delta|^2)\int_1^\infty \exp(-2\pi\eta|\delta|^2(v^{-1}+v))\frac{v^2-1}{v^2}dv.$$

Writing $a = 2\pi\eta|\delta|^2$ and $f(v) = \exp(-a(v^{-1}+v))$, we have $f'(v) = -a\frac{v^2-1}{v^2}f(v)$. Thus we have

$$\int_1^\infty \exp(-a(v^{-1}+v))\frac{v^2-1}{v^2}dv = -a^{-1}(f(\infty)-f(1)) = a^{-1}\exp(2a).$$

This shows

$$(*) = \delta^2 P_\delta(f)\eta \times \int_{\mathrm{SL}^\Delta_2(\mathbb{R})\backslash\mathfrak{H}^2} \exp(-\pi\eta\frac{|[\delta;z,\overline{w}]|^2}{|\operatorname{Im}(z)\operatorname{Im}(w)|})d\mu_{z,w} = (8\sqrt{-1})^{-1}P_\delta(f).$$

It is well known that if f is a cusp form, then the image of the theta correspondence is also a cusp form; so, the term of α with $N(\alpha) \leq 0$ vanishes (e.g., [Sh82, Lemma 2.1] or [O77, page 108]). $\square$

5. Definite D with E real

In this section, we assume $E_{\mathbb{R}} = \mathbb{R}\times\mathbb{R}$ and that $N : D \to \mathbb{Q}$ is positive definite; so, $D\otimes_{\mathbb{Q}}\mathbb{R} \cong \mathbb{H}$. We follow the notation and the assumption introduced in (D$^{\pm}$) in §3.1. In particular, K is an imaginary quadratic field K so that $D\otimes_{\mathbb{Q}} K \cong M_2(K)$. We choose a maximal order R of D so that $R\otimes_{\mathbb{Z}} O_K \subset M_2(O_K)$ for the integer ring O_K of K. We identify $\mathrm{SO}_{D_\sigma} = G^+_{D_\sigma}/Z_{G^+_{D_\sigma}}$.

Let $L = KE$ and $\langle\varsigma\rangle = \mathrm{Gal}(L/E)$ and $\langle\sigma\rangle = \mathrm{Gal}(L/K)$. Then $D_L = D\otimes_{\mathbb{Q}} L = M_2(L)\times M_2(L)$ on which σ interchanges two components, and ς acts each component $M_2(L)$ by ς. To distinguish two components, we write $M_2(L)_l$ for the left component and $M_2(L)_r$ for the right component. Let (X,Y) be the variable vector on which D_L acts through the left component $M_2(L)$ by $(X,Y)(\ell,\ell') = (X,Y)\ell^\iota$ for $(\ell,\ell')\in M_2(L)_l\times M_2(L)_r$. Let (X',Y') be the variable vector on which $(\ell,\ell')\in M_2(L)_l\times M_2(L)_r$ acts by $(X',Y')\ell'^\iota$. We write simply $\mathbf{x} = (X,Y;X',Y')$. Set $[\mathbf{x}] = [\mathbf{x}]_I := J\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right][X',Y']$. We will later define $[\mathbf{x}]_D$ in §6.2 when E is imaginary and D is definite, and if we need to distinguish $[\mathbf{x}]_I$ and $[\mathbf{x}]_D$, we add subscripts I and D. Then $(\ell,\ell')\in M_2(L)_r\times M_2(L)_r$ acts on $[\mathbf{x}]$ by

$$[\mathbf{x}] \mapsto \ell J\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right][X',Y']\ell'^\iota = J^t\ell^\iota\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right][X',Y']\ell'^\iota. \tag{5.1}$$

We embed $D_\sigma\subset D_E$ into $M_2(L)_l\times M_2(L)_r$. Since $s|_{D_\sigma}$ is definite, a spherical homogeneous polynomial of degree d on $D_\sigma\otimes_{\mathbb{Q}} L$ is a linear combination of $v\mapsto s(v,w)^m$ for $w\in D_{\sigma,\mathbb{C}}$ with $N(w) = \pm s(w,w) = 0$ [HMI, §2.5.2]. Note that

$$D^{\pm}_{\sigma,L} = \{(\ell,\ell')\in M_2(L)_l\times M_2(L)_r \mid (\ell',\ell) = \pm(\ell^\iota,\ell'^\iota)\}\cong M_2(L)_l,$$

where the last isomorphism is the projection to the left factor. Similarly to (4.2), define, for $v\in D^{\pm}_{\sigma,\mathbb{C}}$,

$$[v;\mathbf{x}] := \mathrm{Tr}_{D_E/E}(v^\iota[\mathbf{x}]) = \mathrm{Tr}_{D_E/E}(v[\mathbf{x}]^\iota) = dYX' + bXX' - cYY' - aXY' \quad \left(v = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right). \tag{5.2}$$

Then we consider its power

$$[v;\mathbf{x}]^n = s_+(v^\iota,[\mathbf{x}])^n = \mathrm{Tr}_{D_E/E}(v^\iota[\mathbf{x}])^n = \mathrm{Tr}_{D_E/E}(v[\mathbf{x}]^\iota)^n \quad (0<n\in\mathbb{Z}). \tag{5.3}$$

Since $N([\mathbf{x}]) = \det[\mathbf{x}] = 0$, the function $v\mapsto[v;\mathbf{x}]^n$ (for each $\mathbf{x}$) is a spherical polynomial homogeneous in X,Y and in X',Y' of degree n [HMI, §2.5.2]. We simply put $[v;\mathbf{x}]^0 = 1$ for all v.

5.1. **Definite theta series.** Fix an infinite place ∞ of E and write its conjugate as $\infty\sigma$. Identify $E_{\mathbb{C}} = E\otimes_{\mathbb{Q}}\mathbb{C} = \mathbb{C}\times\mathbb{C}$ by $e\otimes z\mapsto(ez,e^\sigma z)$. For a subring $A\subset\mathbb{C}$, consider polynomials $P(\mathbf{x})\in A[X,Y,X',Y']$ in $(X,Y)\in E\otimes_{E,\mathrm{id}}\mathbb{C}^2$ and $(X',Y')\in E\otimes_{E,\sigma}\mathbb{C}^2$. We let $\gamma\in D_E$ act on $P(\mathbf{x})$ from the left by $P|\gamma(X,Y;X',Y') = P((X,Y)\gamma^\iota;(X',Y')\gamma^{\iota\sigma})$; $\gamma\in D_E$ acts on $[\mathbf{x}]$ as above; so, $[\mathbf{x}]|\gamma = [\mathbf{x}\gamma^\iota] = J^t\gamma^\iota\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right][X',Y']\gamma^{\iota\sigma}$.

Note for $g\in\mathrm{GL}_2(E_{\mathbb{C}})$,

$$[g^\iota v g^\sigma;\mathbf{x}] = \mathrm{Tr}_{D_E/E}((g^\iota v g^\sigma)^\iota[\mathbf{x}]^\iota) = \mathrm{Tr}_{D_E/E}(g^{\sigma\iota}v^\iota g[\mathbf{x}]) = [v;g[\mathbf{x}]g^{\sigma\iota}] = [v;\mathbf{x}g^\iota] \tag{5.4}$$

with $\mathbf{x}g^\iota = ((X,Y)g^\iota,(X',Y')g^{\iota\sigma})$. Define a Schwartz function Ψ_k (of weight $2\le k\in\mathbb{Z}$) by

$$\Psi_k(v) = [v;\mathbf{x}]^n\mathbf{e}(-N(v)\overline{\tau}_\pm)\in\mathcal{S}(D^{\pm}_{\sigma,\mathbb{R}}) \quad (n = k-2), \tag{5.5}$$

where $\tau_+ = \tau$ and $\tau_- = -\overline{\tau}$ (the notation $\tau^{\pm} = \overline{\tau}_\pm$ is used in §4.3 which is different from $\tau_\pm$). For a Bruhat function $\phi^{(\infty)}\in\mathcal{S}(D^{\pm}_{\sigma,\mathbb{A}^{(\infty)}})$, putting $\phi = \phi^{(\infty)}\Psi_k$ we consider, for $\tau\in\mathfrak{H}$ and $h\in G^+_{D_\sigma}(\mathbb{A})$

(5.6)
$$\theta(\phi)(\tau;h) = \theta_{D^{\pm}_\sigma}(\tau;h;\mathbf{x}) = \sum_{\alpha\in D^{\pm}_\sigma}\phi(h^{-1}\alpha h^\sigma) = \sum_{\alpha\in D^{\pm}_\sigma}\phi^{(\infty)}(h^{-1}\alpha h^\sigma)[h_\infty^{-1}\alpha h^\sigma_\infty;\mathbf{x}]^n\mathbf{e}(-N(\alpha)\overline{\tau}_\pm).$$

Since $[\alpha;\mathbf{x}]\in L_E(n;\mathbb{C})$, we may regard $\theta(\phi)$ as having values in $L_E(n;\mathbb{C})$.

Lemma 5.1. *Write $\widehat{\Gamma}_\phi := \{u\in G^+_{D_\sigma}(\mathbb{A}^{(\infty)}) \mid \phi^{(\infty)}(u^{-1}vu^\sigma) = \phi^{(\infty)}(v)$ for all $v\in D_{\sigma,\mathbb{A}^{(\infty)}}\}$. Then*

1. *$\theta(\tau;\gamma h) = \theta(\tau;h)$ for $\gamma\in G^+_{D_\sigma}(\mathbb{Q})$,*
2. *$\theta(\tau;hu) = \theta(\tau;h)$ for all $u\in\widehat{\Gamma}_\phi$,*
3. *For z in the center $Z_{G^+_{D_\sigma}}(\mathbb{A})$ of $G^+_{D_\sigma}(\mathbb{A})$, $\theta(\tau;zh) = \theta(\tau;h)$,*
4. *For $u_\infty\in G^+_{D_\sigma}(\mathbb{R})$ with $N(u_\infty) = 1$, $\theta(\tau;hu_\infty;\mathbf{x}) = \theta(\tau;h;\mathbf{x}u_\infty^{-1})$,*
5. *As a function of τ, $\theta_{D^{\pm}_\sigma}(\tau;h;\mathbf{x})$ is in $S^{\mp}_k(\Gamma_\tau)$ for a suitable congruence subgroup Γ_τ of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. Since θ is the sum of $\phi(h^{-1}\alpha h^{\sigma})$, we have (1–3). The assertion (4) follows from (5.4), and the last assertion is a restatement of [HMI, Theorem 2.65]. $\square$

As before, we split $v = \mathfrak{z} + \mathfrak{x}$ for $\mathfrak{z} \in Z^{\pm}$ and $\mathfrak{x} \in D_0^{\pm}$; so,

$$[v; \mathbf{x}]^n = ([\mathfrak{z}; \mathbf{x}] + [\mathfrak{x}; \mathbf{x}])^n = \sum_{j=0}^{n} \binom{n}{j} [\mathfrak{x}; \mathbf{x}]^j [\mathfrak{z}; \mathbf{x}]^{n-j}.$$

To evaluate the integral over $D^{\times}\backslash D^{\times}_{\mathbb{A}}/D^{\times}_{\mathbb{R}}$, we need to project down to the $D^{\times}_{\mathbb{R}}$-invariant quotient of $L_E(n;\mathbb{C})$. Thus we need to compute $(n!)^{-2}\nabla^n[\alpha;\mathbf{x}]^n$. By (4.9), we find, for a scalar $\mathfrak{z} \in Z^{\pm}$,

$$(n!)^{-2}\nabla^n[\mathfrak{z};\mathbf{x}]^n = (n!)^{-2}\nabla^n \mathfrak{z}^n(YX' - XY')^n = \mathfrak{z}^n \sum_{j=0}^{n}(-1)^j\binom{n}{j}(-1)^j\binom{n}{j}^{-1} = \mathfrak{z}^n(n+1). \tag{5.7}$$

For $\mathfrak{x} \in D_0^{\pm}$, we can choose g in $\mathrm{SL}_2(\mathbb{C})$ diagonally embedded in $\mathrm{SL}_2(E_{\mathbb{C}})$ by $g \mapsto (g, g^{\sigma})$ so that

$$[\mathfrak{x};\mathbf{x}]|g = [g^{-1}\mathfrak{x}g;\mathbf{x}] = [\mathrm{diag}[z,-z];\mathbf{x}] = z(YX' + XY')$$

for $z \in \mathbb{C}$ and remark that $[\mathfrak{z};\mathbf{x}]|g = [\mathfrak{z};\mathbf{x}g^{-1}]$. Assuming $\mathfrak{z} \neq 0$ and $\mathfrak{x} \neq 0$, by Clebsch–Gordan decomposition [H94, (11.2a,b)], the action of $\mathrm{SL}_2(\mathbb{C})$ on the space spanned by the right translation of $[\mathfrak{z};\mathbf{x}]^{n-j}[\mathfrak{x};\mathbf{x}]^j$ by $\mathrm{SL}_2(\mathbb{C})$ is isomorphic to the $2j$-th symmetric tensor representation of $\mathrm{SL}_2(\mathbb{C})$. Since $(n!)^{-2}\nabla^n P|g = (n!)^{-2}\nabla^n P$ for $g \in \mathrm{SL}_2(\mathbb{C}) \hookrightarrow D^{\times}_{\mathbb{C}}$ [H99, page 141], we have for any $0 < j \leq n$

$$(n!)^{-2}\nabla^n[\mathfrak{z};\mathbf{x}]^{n-j}[\mathfrak{x};\mathbf{x}]^j = (n!)^{-2}\nabla^n[\mathfrak{z}^{n-j}z^j(YX' + XY')^j(YX' - XY')^{n-j}] = 0. \tag{5.8}$$

5.2. **Factoring theta series in the definite case with E real.** Recall $\Delta_+ = 1$ and we assume $\Delta_- > 0$. Then $\delta_{\pm} = \sqrt{\Delta_{\pm}}$. As described in §3.1, we decompose $D^{\pm}_{\sigma} = Z^{\pm} \oplus D_0^{\pm}$ so that $Z^{\pm} = \delta_{\pm}\mathbb{Q} \subset D^{\pm}_{\sigma}$ with $Q(x) = x^2$ and $L_Z = N\delta_{\pm}\mathbb{Z}$. We take the Bruhat function $\phi_Z^{(\infty)}$ on $L_Z^*/L_Z = N^{-1}\mathbb{Z}/N\mathbb{Z}$ defined in §4.6 for a Dirichlet character $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. We take an Eichler order $R(N_0)$ in D for N_0 prime to the discriminant ∂ of D. Then let $\phi_0^{(\infty)} \in \mathcal{S}(D_{0,\mathbb{A}^{(\infty)}})$ be as in (4.25) for the characteristic function $\phi_{\widehat{L}}$ of $\widehat{L} := \delta_{\mp}\widehat{R}(N_0) \cap D^{\pm}_{0,\mathbb{A}^{(\infty)}}$. Remark 4.5 applies; so, often in computation, we pretend $\phi_0^{(\infty)} = \phi_{\widehat{L}}$. We put $\phi^{(\infty)} = \phi_Z^{(\infty)} \otimes \phi_0^{(\infty)}$ and

$$\phi(x) := \phi^{(\infty)}(x^{(\infty)})[x_{\infty};\mathbf{x}]^n\mathbf{e}(N(x_{\infty})\tau_{\pm}). \tag{5.9}$$

Decompose $\mathbf{x} = \mathfrak{z} + \mathfrak{x}$ with scalar $\mathfrak{z} \in Z^{\pm}_{\mathbb{R}}$ and $\mathfrak{x} \in D^{\pm}_{0,\mathbb{R}}$. Then $[\mathfrak{z};\mathbf{x}] = \mathfrak{z}(YX' - XY')$ and $[\mathfrak{z}+\mathfrak{x};\mathbf{x}] = [\mathfrak{z};\mathbf{x}] + [\mathfrak{x};\mathbf{x}]$. Thus we find

$$[\mathfrak{z}+\mathfrak{x};\mathbf{x}]^n = \sum_{j=0}^{n}\binom{n}{j}\mathfrak{z}_{\infty}^j(YX' - XY')^j[\mathfrak{x};\mathbf{x}]^{n-j}.$$

Defining $\phi_j^Z(\mathfrak{z}) = \phi_Z^{(\infty)}(\mathfrak{z}^{(\infty)})\mathfrak{z}^j\mathbf{e}(\Delta_{\pm}\mathfrak{z}_{\infty}^2\tau_{\pm})$ and $\phi_j^{D_0} = \phi_0^{(\infty)}(\mathfrak{x}^{(\infty)})[\mathfrak{x}_{\infty};\mathbf{x}]^j\mathbf{e}(N(\mathfrak{x}_{\infty})\tau_{\pm})$, we have

$$\theta(\phi)|_{\mathrm{O}_{\delta}(\mathbb{A})} = \sum_{j=0}^{n}(YX' - XY')^j\binom{n}{j}\theta(\phi_j^Z)\theta(\phi_{n-j}^{D_0}). \tag{5.10}$$

and by (5.7) and (5.8)

$$(n!)^{-2}\nabla^n\theta(\phi)|_{\mathrm{O}_{\delta}(\mathbb{A})} = \theta((n!)^{-2}\nabla^n\phi_n^Z)\theta(\phi_0^{D_0}). \tag{5.11}$$

5.3. **Siegel–Weil formula in the real definite case.** Note

$$\mathbf{r}_Z(g_{\tau})L_Z(g)(\mathfrak{z}_{\infty}^j\mathbf{e}(\mathfrak{z}_{\infty}^2\sqrt{-1})) = \eta^{(1+2j)/4}\mathfrak{z}_{\infty}^j\mathbf{e}(\mathfrak{z}_{\infty}^2\tau_{\pm}) \quad (\tau = \xi + \eta\sqrt{-1} \in \mathfrak{H}) \tag{5.12}$$

and for $\tau_{\pm}$ as in (5.5),

$$\mathbf{r}_{D_0}(g_{\tau})L_{D_0}(g)([\mathfrak{x};\mathbf{x}]^j\mathbf{e}(N(\mathfrak{x}_{\infty})\sqrt{-1})) = \eta^{(3+2j)/4}[\mathfrak{x};\mathbf{x}]^j\mathbf{e}(N(\mathfrak{x}_{\infty})\tau_{\pm}). \tag{5.13}$$

Recall $\boldsymbol{\theta}(\phi)(\tau) = \sum_{\alpha\in V}(\mathbf{w}(g_{\tau})\phi)(\alpha)$ for $\phi \in \mathcal{S}(V_{\mathbb{A}})$ with $V = Z, D_0$ and D_{σ}.

Recall the Haar measure $d\mu_g$ defined above (4.20) and

$$Sh_{\delta} = D^{\times}\backslash D^{\times}_{\mathbb{A}}/\varrho_0^{-1}(\widehat{\Gamma}_{\delta})D^{\times}_{\mathbb{R}} = \mathrm{SO}_{\delta}(\mathbb{Q})\backslash\mathrm{SO}_{\delta}(\mathbb{A})/\widehat{\Gamma}_{\delta}C_{\delta} = \mathrm{O}_{\delta}(\mathbb{Q})\backslash\mathrm{O}_{\delta}(\mathbb{A})/\widehat{\Gamma}_{\delta}C_{\delta}$$

for $C_\delta = C_\infty(\mathrm{O}_\delta)$ as $\mathrm{O}_\delta = \mathrm{SO}_\delta \sqcup \mathrm{SO}_\delta\iota$. We now study

$$\int_{Sh_\delta} \boldsymbol{\theta}(\phi)(g)d\mu_g = \int_{Sh_\delta}\sum_{j=0}^{n}\boldsymbol{\theta}((n!)^{-2}\nabla^n\phi_j^Z)\boldsymbol{\theta}(\phi_0^{D_0})d\mu_g \overset{(*)}{=} \mathfrak{m}\boldsymbol{\theta}((n!)^{-2}\nabla^n\phi_n^Z)E(\phi_0^{D_0}), \tag{5.14}$$

where $\mathfrak{m}$ is as in (4.23) and for $\Phi \in \mathcal{S}(D_{0,\mathbb{A}})$ and for $g \in \mathrm{Mp}(\mathbb{A})$,

$$E(\Phi) = \sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})} |a(g)|^{s-(1/2)}(\mathbf{w}(\gamma g)\Phi)(0)|_{s=\frac{1}{2}}.$$

At the identity $(*)$, the sum $\sum_{j=0}^{n}$ in (5.10) reduces to the term $j = n$ because of (5.8) and Lemma 4.4 (see also Remark 4.5).

Take $F(\tau) = \sum_{m=1}^{\infty} a_m\mathbf{e}(-m\tau_\mp) \in S_k^\mp(M, \varphi\chi_{D_\sigma})$ for M as in (4.26). Again in the same manner as getting (4.32) from (4.29) and (4.31): for $\overline{B} := B(\mathbb{Q})\backslash B(\mathbb{A})/B(\widehat{\mathbb{Z}})$,

$$\begin{aligned}\int_{\overline{B}} F(\tau)\eta^{k/2}\boldsymbol{\theta}((n!)^{-2}\nabla^n\phi_n^Z)(g_\tau)\mathbf{r}(g_\tau)(\phi_0^{D_0})(0)d\mu_\tau &= \int_{\overline{B}} F(\tau)\theta((n!)^{-2}\nabla^n\phi_n^Z)(g_\tau)\eta^{k-2}d\xi d\eta \\ &= \int_0^\infty\int_0^1 F(\tau)\theta((n!)^{-2}\nabla^n\phi_n^Z)(g_\tau)d\xi\eta^{k-2}d\eta \\ &= 2\delta_\pm^{k-2}\int_0^\infty \sum_{0<m\in\mathbb{Z}}\psi(m)m^{k-2}\exp(-4\pi|\Delta_\pm|m^2\eta)\eta^{k-2}d\eta \\ &= 2\delta_\pm^{k-2}|\Delta_\pm|^{1-k}(4\pi)^{-k+1}(n+1)\Gamma(k-1)\sum_{0<m\in\mathbb{Z}}\psi(n)a_{m^2}m^{-k}.\end{aligned} \tag{5.15}$$

The factor $(n+1)$ shows up at the end by (5.7), and $\delta_\pm^{k-2}|\Delta_\pm|^{1-k} = \delta_\pm^{-k}$. Noting $(n+1)\Gamma(k-1) = \Gamma(k)$ as $n+1 = k-1$, we get in the same manner as in Theorem 4.7

Theorem 5.2. *Suppose $E_\mathbb{R} \cong \mathbb{R}$ and $D_\mathbb{R} \cong \mathbb{H}$. Let F be a primitive form in $S_k^\mp(C, \psi^{-1}\chi_{D_\sigma})$ with the conductor $C|M$ for the level M as in (4.26) and $f := \theta^*(F)$ be the theta lift:*

$$f(g) = \int_{\Gamma_\tau\backslash\mathfrak{H}} \theta(\phi)(\tau; g)F(\tau)\eta^{k-2}d\xi d\eta.$$

Choose $\phi_Z^{(\infty)}$ associated to Dirichlet character ψ of conductor $C(\psi)$ as specified above and $\phi_0^{(\infty)}$ as in (4.25)*. Let ϕ be a Schwartz-Bruhat function of $D_{\sigma,\mathbb{A}}^\pm$ as in* (5.9)*, and choose the measure $d\mu_g$ on* $\mathrm{O}_\delta(\mathbb{A})$ *as in* Theorem 4.2*. Then if $f(g) \neq 0$, for the mass factor $\mathfrak{m}_1$ as in* (4.23) *and $E^\pm(1)$ as in* Theorem 4.7,

$$\pi^2\int_{Sh_\delta}(n!)^{-2}\nabla^n f(g)d\mu_g = \mathfrak{m}_1E^\pm(1)\delta_\pm^{-k}2(4\pi)^{-k+1}\Gamma(k)L^{(C_s(\psi))}(1, Ad(F)\otimes\chi_E)$$

for the compatible system ρ_F attached to F and the finite set of points Sh_δ as in (4.22)*.*

5.4. **Mass formula and the adjoint L-value formula.** Recall from (4.22) that

$$Sh_\delta = \mathrm{O}_\delta(\mathbb{Q})\backslash\mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta\mathrm{O}_\delta(\mathbb{R}) \cong D^\times\backslash D_\mathbb{A}^\times/\mathbb{A}^\times\widehat{R}(N_0)^\times D_\mathbb{R}^\times.$$

Since D is definite, Sh_δ is a finite set, and $d\mu_g$ is the measure given by

$$\int_{Sh_\delta}\varphi d\mu_g = \sum_{x\in Sh_\delta} e_x^{-1}\varphi(x),$$

where $e_x = |x\widehat{R}(N_0)^\times x^{-1}\cap D^\times|$. Thus the measure computes the mass of the quotient Sh_δ in the sense of [Sh99, page 1]. Then Theorem 5.2 tells us

Corollary 5.3. *Let the notation and the assumption be as in* Theorem 5.2 *in addition to $C_0 = 1$. Then if $V = D_\sigma^+$,*

$$\frac{\mathfrak{m}_1}{\pi^2}2(4\pi)^{-k+1}\Gamma(k)L^{(C_s(\psi))}(1, Ad(F)\otimes\chi_E) = \sum_{x\in Sh_\delta} e_x^{-1}(n!)^{-2}(\nabla^n f)(x).$$

A slightly more complicated formula holds when $C_0 \neq 1$ or $V = D_\sigma^-$ whose explicit form is left to the reader. The above formula is an adjoint generalization of a mass formula of Siegel–Shimura in (4.23):

$$\frac{\mathfrak{m}_1}{\pi^2}\zeta(2) = \sum_{x \in Sh_\delta} e_x^{-1},$$

and also a generalization of Dirichlet class number formula.

5.5. **Fourier expansion of theta descent for $E = \mathbb{Q} \times \mathbb{Q}$ and definite D.** Since the Fourier expansion in the definite case with $E = \mathbb{Q}\times\mathbb{Q}$ is particularly simple, we insert here its description in the simplest case of the weight $k = 2$ (and hence $n = 0$). The general case for a real quadratic field E will be dealt with in §5.8. Note in this case, $D_\sigma^\pm \cong D^\pm = (D, \pm N)$ by $D_\sigma^\pm \ni (v, \pm v^\iota) \mapsto v \in D$. For simplicity, we assume that the weight is 2 on SL_2 and $(2,2)$ on $D_E^\times$.

Let $Sh = Sh_\delta := \mathrm{SO}_{D_0}(\mathbb{Q})\backslash \mathrm{SO}_{D_0}(\mathbb{A}^{(\infty)})/\widehat{R}^\times \mathrm{SO}_{D_0}(\mathbb{R}) = D^\times\backslash D_{\mathbb{A}}^\times/\mathbb{A}^\times\widehat{R}^\times D_{\mathbb{R}}^\times$ and $Sh_E = Sh \times Sh$. Since $N(\widehat{R}^\times) = \widehat{\mathbb{Z}}^\times$ by [BNT, Proposition X.3.6] and $\mathbb{A}^\times = \mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\mathbb{R}_+^\times$, we may assume that $N(a) = 1$ for all representatives of Sh. For the class $[a] \in Sh$ represented by $a \in D_{\mathbb{A}^{(\infty)}}^\times$, write $\Gamma_a := a\widehat{R}^\times a^{-1} \cap D^\times$ and define $\overline{\Gamma}_a$ for the image of Γ_a in $D^\times/\mathbb{Q}^\times$, which are finite groups with $|\Gamma_a| = 2|\overline{\Gamma}_a|$. Put $e_a := |\Gamma_a|$.

As described in §3.1, ϱ_0 is an isomorphism of $\mathrm{SO}_\delta = \mathrm{SO}_{D_0}$ onto $D^\times/\mathbb{G}_m$ as algebraic groups, where $\mathbb{G}_m$ is identified with the center of $D^\times$. Also we know $\mathrm{O}_{D_0} = \mathrm{SO}_{D_0} \sqcup \mathrm{SO}_{D_0}\sigma$ (regarding the Galois action as an element of O_{D_0}). Thus the stabilizer $\Gamma'_a \subset \mathrm{O}_{D_0}(\mathbb{Q})$ of the lattice $aRa^{-1} \cap D_0^-$ fits into the following exact sequence:

$$1 \to \mathrm{Gal}(E/\mathbb{Q}) \to \Gamma'_a \to \overline{\Gamma}_a \to 1, \tag{5.16}$$

as $\sigma \in \Gamma'_a$. Thus $|\Gamma'_a| = |\Gamma_a| = e_a$ and the number $|\Gamma'_a|$ appears in [Sh99, Introduction] as $[\Gamma_i : 1]$; so, in order to resort to the results in [Sh99], we need to use the alternative definition $e_a = |\Gamma'_a|$.

Recalling some results in [H05, §4], we study the Doi–Naganuma lift in this simplest case. Set for a subring A of $\mathbb{C}$

$$\begin{aligned} &S(A) := \{\mathcal{F} : Sh \to A \mid \sum_{[a] \in Sh} e_a^{-1}\mathcal{F}([a]) = 0\} \\ &S_E(A) = \{f : Sh_E \to A \mid \sum_{[a],[b] \in Sh} e_a^{-1}e_b^{-1}f([a],[b]) = 0\} = S(A) \otimes_A S(A). \end{aligned} \tag{5.17}$$

We take $D_\sigma^+ \cong D$ and $\phi^{(\infty)}$ to be the characteristic function of $\widehat{\mathbb{Z}} \oplus \widehat{R}_0$ for $R_0 = D_0 \cap R$ with $\phi_\infty(\tau; v) = \mathbf{e}(N(v)\tau) = \eta^{-1}\mathbf{r}_D(g_\tau)\phi(\sqrt{-1}; v)$. Thus writing $\phi_Z^{(\infty)}$ (resp. $\phi_0^{(\infty)}$) for the characteristic function of $\widehat{\mathbb{Z}}$ (resp. $\widehat{R}_0$) and $\phi_{Z,\infty}$ (resp. $\phi_{0,\infty}$) for $\mathbf{e}(x^2\tau) = \eta^{-1/4}\mathbf{r}_Z(g_\tau)\mathbf{e}(\sqrt{-1}x^2)$ (resp. $\mathbf{e}(N(v)\tau) = \eta^{-3/4}\mathbf{r}_{D_0}(g_\tau)\mathbf{e}(\sqrt{-1}N(v))$),

$$\theta(\phi)(\tau; g) = \theta(\phi_Z)(\tau)\theta(\phi_0)(\tau; g),$$

where $\theta(\phi_Z)(\tau) = \sum_{n \in \mathbb{Z}} \mathbf{e}(n^2\tau)$ and $\theta(\phi_0)(\tau; g) = \sum_{\alpha \in g\widehat{R}_0 g^{-1} \cap D_0} \mathbf{e}(N(\alpha)\tau)$. They are holomorphic modular form of level $[4, \partial]$ (see (4.26)) and of weight $\frac{1}{2}$ and $\frac{3}{2}$, respectively. As a function on $\mathrm{O}_\delta(\mathbb{A})$, $\theta(\phi_0)(\tau; g)$ is invariant under $\widehat{R}^\times = \mathrm{SO}_\delta(\widehat{\mathbb{Z}})$ and σ ($\mathrm{O}_\delta = \mathrm{SO}_\delta \sqcup \mathrm{SO}_\delta\sigma$). Set, for $\widetilde{F}(z) = F(-\overline{z})$,

$$f(h) = \int_{X_0(4\partial)} \theta(\phi)(\tau; h)\widetilde{F}(\tau)\eta^{-2}d\xi d\eta$$

and for $e_a = |a\widehat{R}^\times a^{-1} \cap D^\times|$,

$$\begin{aligned} \int_{Sh} f(g)dg &= \sum_{a \in Sh} e_a^{-1} \int_{X_0(4\partial)} \widetilde{F}(\tau)\theta(\phi_Z)(\tau)\theta(\phi_0(\tau; a))\eta^{-2}d\xi d\eta \\ &= \int_{X_0(4\partial)} \widetilde{F}(\tau)\theta(\phi_Z)(\tau) \sum_{a \in Sh} e_a^{-1}\theta(\phi_0(\tau; a))\eta^{-2}d\xi d\eta. \end{aligned} \tag{5.18}$$

By Siegel–Weil formula, we have $E(\phi_0)(\tau) = \sum_{a\in Sh} e_a^{-1}\theta(\phi_0(\tau;a))$ is a Siegel–Weil Eisenstein series on $\Gamma_0(4\partial)$ of weight $\frac{3}{2}$. Note that $\theta(\phi)(\tau;a,b) = \sum_{\xi\in D}\phi(a^{-1}\xi b)\mathbf{e}(N(\xi)\tau) = \theta_{a,b}$. Since $(a^{-1}\xi b)^\iota = b^{-1}\xi^\iota a$ as $N(a) = N(b) = 1$, we have $\theta_{a,b} = \theta_{b,a}$. Thus

$$f(a,b) = f(b,a). \tag{5.19}$$

Hereafter in this section, we assume

(I) $\frac{1}{6}\in A$,

and take the Schwartz-Bruhat function $\boldsymbol{\phi} = \boldsymbol{\phi}^{(\infty)}\phi_\infty$ slightly different from ϕ; i.e., we take the characteristic function $\boldsymbol{\phi}^{(\infty)}$ of $\widehat{R}$ in place of that of $\widehat{\mathbb{Z}}\oplus R_0$. The condition (I) assures us that $e_a^{-1}\in A$ for all $a\in Sh$. We have a perfect A-linear pairings

$$\langle\cdot,\cdot\rangle : S(A)\times S(A)\to A$$

given by $\langle\varphi,\varphi'\rangle = \sum_{a\in Sh} e_a^{-1}\varphi(a)\varphi'(a)$.

We have Hecke operators $T(n)$ acting on $S(A)$ as follows: Let

$$\mathbf{T}(n) := \{b\in\widehat{R}|N(b)\widehat{\mathbb{Z}} = n\widehat{\mathbb{Z}}\}$$

for a positive integer n. Decompose $\mathbf{T}(n) = \bigsqcup_{c\in C(n)} c\widehat{R}^\times$. Then $\mathcal{F}|T(n)(a) = \sum_{c\in C(n)}\mathcal{F}(ac)$. The algebra $H(A)$ is defined to be the A-subalgebra of $\mathrm{End}_A(S(A))$ generated by $T(n)$ for all positive integers n. For $\mathcal{F},\mathcal{G}\in S(A)$, we define $\mathcal{F}\otimes\mathcal{G} : Sh_E\to A$ by $\mathcal{F}\otimes\mathcal{G}(a,b) = \mathcal{F}(a)\mathcal{G}(b)$. In this way, $S_E(A)\cong S(A)\otimes_A S(A)$. Writing the variable $h\in D^\times_{E_{\mathbb{A}}} = D^\times_{\mathbb{A}}\times D^\times_{\mathbb{A}}$ as $h = (h_L, h_R)$ and $\widehat{\Gamma}_E := \widehat{R}^\times\times\widehat{R}^\times$ and taking the Schwartz-Bruhat function $\boldsymbol{\phi} = \boldsymbol{\phi}^{(\infty)}\phi_\infty$ for the characteristic function $\boldsymbol{\phi}^{(\infty)}$ of $\widehat{R}$, define the theta descent $\theta_*(\mathcal{F}\otimes\mathcal{G})$ of $\mathcal{F}\otimes\mathcal{G}$ by

$$\theta_*(\mathcal{F}\otimes\mathcal{G})(\tau) = \int_{Sh_E}\theta(\boldsymbol{\phi})(\tau;h)\mathcal{F}(h_L)\mathcal{G}(h_R)d\mu_h = \sum_{(a,b)\in Sh^2}\mathcal{F}(a)\mathcal{G}(b)e_a^{-1}e_b^{-1}\sum_{\xi\in D}\phi(a^{-1}\xi b)\mathbf{e}(N(\xi)\tau).$$

Note that $\mathrm{Supp}(\phi^{(\infty)}) = \widehat{\mathbb{Z}}\oplus\widehat{R}_0\subset\widehat{R} = \mathrm{Supp}(\boldsymbol{\phi}^{(\infty)})$; so, the functions $\phi^{(\infty)}$ and $\boldsymbol{\phi}^{(\infty)}$ are slightly different. We have $[\widehat{R},\widehat{\mathbb{Z}}\oplus\widehat{R}_0] = 2$ as $\mathbb{Z}\oplus R_0$ is the kernel of $R\ni x\mapsto\mathrm{Tr}(x)\bmod 2\in\mathbb{F}_2$. The theta function $\theta(\boldsymbol{\phi})$ has level group $\Gamma_\tau = \Gamma_0(\partial)$.

Theorem 5.4. *Assume $\frac{1}{6}\in A$. Then we have $\theta_*(\mathcal{F}\otimes\mathcal{G}) = \sum_{n=1}^\infty\langle\mathcal{F}|T(n),\mathcal{G}\rangle q^n\in S_2(\Gamma_0(\partial);A)$.*

We get an A-linear map $\theta_* : S_E(A)\to S_2(\Gamma_0(\partial);A)$ given by $\theta_*(\mathcal{F}\otimes\mathcal{G}) = \sum_{n=1}^\infty\langle\mathcal{F}|T(n),\mathcal{G}\rangle q^n$. Since $\langle\mathcal{F}|T(n),\mathcal{G}\rangle = \langle\mathcal{F},\mathcal{G}|T(n)\rangle$, the map θ_* factors through $S(A)\otimes_{H(A)}S(A)$. A main result in [H05, Theorem 4.1] assuming that ∂ is a prime p is that this induces an A-linear isomorphism

$$\theta_* : S(A)\otimes_{H(A)}S(A)\cong S_2(\Gamma_0(p);A), \tag{5.20}$$

if $(p-1)(\zeta(2)/\pi^2)^{-1} = 6(p-1)$ is a unit in A. Since the image of θ_* lands in the space of new forms, for a general composite ∂, the cokernel of θ_* is large.

Proof. By definition,

$$\begin{aligned}\theta_*(\mathcal{F}\otimes\mathcal{G}) &= \int_{Sh_E}\theta(\boldsymbol{\phi})(\tau;h)\mathcal{F}(h_L)\mathcal{G}(h_R)d\mu_h\\ &= \sum_{(a,b)\in Sh\times Sh}\mathcal{F}(a)\mathcal{G}(b)e_a^{-1}e_b^{-1}\sum_{\xi\in D}\phi(a^{-1}\xi b)\mathbf{e}(N(\xi)\tau) = \sum_{(a,b)\in Sh\times Sh}e_a^{-1}e_b^{-1}\mathcal{F}(a)\mathcal{G}(b)\theta_{a,b}(\tau),\end{aligned}$$

where $\theta_{a,b} = \theta(\phi_{a,b})$ with the characteristic function $\phi_{a,b}^{(\infty)}$ of $a\widehat{R}b^{-1}$ and the measure $d\mu_h$ is chosen so that $\int_{D_E^\times\backslash D_E^\times(a,b)\widehat{\Gamma}_E D^\times_{E_{\mathbb{R}}}/\widehat{\Gamma}_E D^\times_{E_{\mathbb{R}}}} d\mu_h = e_a^{-1}e_b^{-1}$. Since $\theta_{a,b}\in S_2(\Gamma_0(\partial),\mathbb{Z})$ as is well known, we have $\theta_*(\mathcal{F}\otimes\mathcal{G})\in M_2(\Gamma_0(\partial);A)$.

For a decomposition $\mathbf{T}(n) = \bigcup_{c\in C(n)} c\widehat{R}^\times$, we have $\mathcal{F}|T(n)(x) = \sum_{c\in C(n)}\mathcal{F}(xc)$. Writing $ac = \xi b_c u$ for $\xi\in\Gamma_a\backslash D^\times/\Gamma_b$, $b_a\in Sh$ and $u\in\widehat{R}^\times$, $\mathcal{F}|T(n)(a) = \sum_c\mathcal{F}(b_c)$ and $\xi\in D^\times\cap a\widehat{R}b_c^{-1}$. This shows $\theta_*(\mathcal{F}\otimes\mathcal{G})(\tau) = \sum_{n=1}^\infty\langle\mathcal{F}|T(n),\mathcal{G}\rangle\mathbf{e}(n\tau)$ as the constant term is given by $\sum_{a,b}e_a^{-1}e_b^{-1}\mathcal{F}(a)\mathcal{G}(b) = \sum_a e_a^{-1}\mathcal{F}(a)\sum_b e_b^{-1}\mathcal{G}(b) = 0$. $\square$

5.6. **Hecke equivariance of the theta descent.** We prove Hecke equivariance of the theta descent keeping the assumption of §5.5. We first compute

$$a(n, \theta_*(\mathcal{F} \otimes \mathcal{G})|T(p)) = a(\frac{n}{p}, \theta_*(\mathcal{F} \otimes \mathcal{G})) + p^{k-1} a(pn, \theta_*(\mathcal{F} \otimes \mathcal{G})),$$

while

$$\begin{aligned}\theta_{a,b}|T(p) &= a(\frac{n}{p}, \theta_{a,b}) + pa(pn, \theta_{a,b}) \\ &= e_a^{-1} e_b^{-1} (|\{\xi \in a\widehat{R}b^{-1} \cap D, N(\xi) = n/p\}| + p|\{\xi \in a\widehat{R}b^{-1} \cap D, N(\xi) = np\}|).\end{aligned}$$

Proposition 5.5. *On $S_k(\Gamma_0(N), \psi)$, we have $T(n)T(m) = \sum_{0<d|(m,n),(d,N)=1} \psi(d) d^{k-1} T(\frac{mn}{d^2})$, the same formula is valid on $S(A)$ for $k = 2$ and $N = \partial$, and writing $a(n, f)$ for the n-th q-expansion coefficient of $f \in S_k(\Gamma_0(N), \psi)$, we have $a(n, f|T(m)) = \sum_{0<d|(m,n), d\nmid N} \psi(d) d^{k-1} a(\frac{mn}{d^2}, f)$.*

Proof. We have a general formula [IAT, Theorem 3.24, (iv)]

$$T(m)T(n) = \sum_{0<d|(m,n),(d,N)=1} \psi(d) d^{k-1} T(\frac{mn}{d^2}).$$

This specializes to the formula for $S_k(\Gamma_0(N), \psi)$ as $f|T(d,d) = \psi(d) d^{k-1} f$ if $f \in S_k(\Gamma_0(N), \psi)$. On $S(A)$, we limit the formula to $T(l)$ with $R_l \cong M_2(\mathbb{Z}_l)$, there is no change. If $l|\partial$, as R_l has a sequence of two sided ideal $\varpi^n R_l$ with $\varpi^n R_l / \varpi^{n+1} R_l \cong \mathbb{F}_l$ and $N(\varpi) = l$, the formula becomes as indicated. □

By Proposition 5.5, we get

$$\begin{aligned}\theta_*(\mathcal{F} \otimes (\mathcal{G}|T(m))) &= \sum_{n>0} \langle \mathcal{F}|T(n), \mathcal{G}|T(m) \rangle q^n = \sum_{n>0} \langle \mathcal{F}|T(n)T(m), \mathcal{G} \rangle q^n \\ &= \sum_{n>0} \sum_{d|(n,m), d\nmid\partial} \langle \mathcal{F}|T(\frac{nm}{d^2}), \mathcal{G} \rangle q^n = \theta_*(\mathcal{F} \otimes \mathcal{G})|T(m).\end{aligned}$$

Thus we get

Corollary 5.6. *Letting $h \in H(A)$ act on $S(A) \otimes_{H(A)} S(A)$ by $(\mathcal{F} \otimes \mathcal{G})|h := \mathcal{F} \otimes (\mathcal{G}|h)$, the morphism $\theta_* : S(A) \otimes_{H(A)} S(A) \to S_2(\Gamma_0(\partial); A)$ becomes Hecke equivariant.*

Since $\theta_*(\mathcal{F} \otimes \mathcal{G})(\tau) = \int_{Sh_E} \theta(\phi)(\tau; h_L, h_R) \mathcal{F}(h_L) \mathcal{G}(h_R) d\mu_h$, by (5.19), we get

$$\theta(\tau; h_L, h_R)|T_\tau(n) = \theta(\tau; h_L, h_R)|T_R(n) = \theta(\tau; h_L, h_R)|T_L(n),$$

where $T_\tau(n)$ (resp. $T_L(n), T_R(n)$) is the elliptic Hecke operator (resp. the left and right quaternionic Hecke operator).

5.7. **Congruence number formula.** We find in [EMI, §9.3.1] a congruence number formula via an adjoint L-value for $D = M_2(\mathbb{Q})$. We generalize this to a definite D under the assumption of §5.5.

The $H(\mathbb{C})$-module $S(\mathbb{C})$ is semi-simple with multiplicity one for each algebra homomorphism $\lambda : H(\mathbb{C}) \to \mathbb{C}$. Write $\mathbb{Z}[\lambda]$ (resp. $\mathbb{Q}[\lambda]$) for the subring of $\mathbb{C}$ generated by $\lambda(T(n))$ for all $0 < n \in \mathbb{Z}$ over $\mathbb{Z}$ (resp. $\mathbb{Q}$). Since we have Hecke equivariant isomorphism $\theta_* : S(\mathbb{C}) \otimes_{H(\mathbb{C})} S(\mathbb{C}) \cong S_2^{new}(\Gamma_0(\partial))$ by Eichler and Jacquet–Langlands, choosing a Hecke eigenvector $\mathcal{F}_\lambda \in S(\mathbb{Z}[\lambda])$, the Hecke equivariance of Corollary 5.6 tells us that for $F_\lambda := \sum_{n=1}^\infty \lambda(T(n)) q^n \in S_2^{new}(\Gamma_0(\partial))$, $\theta^*(F_\lambda) = \Omega_\lambda(\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda)$ for a constant $\Omega_\lambda \neq 0$. Since θ_* induces an injection $S(A) \otimes_{H(A)} S(A) \hookrightarrow S_2^{new}(\Gamma_0(\partial); A)$ with finite cokernel C by the argument of [H05] applied to D, assuming $A \supset \mathbb{Z}[\lambda]$, the Hecke equivariance of Corollary 5.6 tells us that for $F_\lambda := \sum_{n=1}^\infty \lambda(T(n)) q^n \in S_2^{new}(\Gamma_0(\partial))$, $\theta_*^{-1}(F_\lambda)) = \varepsilon_\lambda(\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda)$ for a constant $\varepsilon_\lambda \in A[\frac{1}{C}]^\times$. We want to study Ω_λ and ε_λ.

Let for a $\mathbb{Z}[\lambda]$-subalgebra A of $\mathbb{C}$,

$$\begin{aligned}S(A)_\lambda &= \{\mathcal{F} \in S(A) | \mathcal{F}|T(n) = \lambda(T(n))\mathcal{F} \text{ for all } 0 < n \in \mathbb{Z}\}, \\ S(A)^\lambda &= \{\mathcal{G} \in S(A)_\lambda \otimes_A \mathrm{Frac}(A) | \langle \mathcal{G}, S(A)_\lambda \rangle \subset A\}.\end{aligned} \tag{5.21}$$

Define D-congruence module

$$C_0^D(\lambda; A) := S(A)^\lambda / S(A)_\lambda. \tag{5.22}$$

Theorem 5.7. *We have* $|\langle\mathcal{F}_\lambda,\mathcal{F}_\lambda\rangle|_{\mathfrak{p}} = \big||C_0^D(\lambda;\mathbb{Z}[\lambda])|\big|_{\mathfrak{p}}$ *for all primes* $\mathfrak{p}$ *such that* $S(\mathbb{Z}[\lambda]_{\mathfrak{p}})^\lambda$ *and* $S(\mathbb{Z}[\lambda]_{\mathfrak{p}})_\lambda$ *are both* $\mathbb{Z}[\lambda]_{\mathfrak{p}}$*-free, where* $\mathbb{Z}[\lambda]_{\mathfrak{p}}$ *is the subring of* $\overline{\mathbb{Q}}_p$ *generated by* $\lambda(T(n))$ *for all* n *over* $\mathbb{Z}_p$ *under the embedding of* $\mathbb{Z}[\lambda]$ *into* $\overline{\mathbb{Q}}_p$ *by the place* $\mathfrak{p}|p$ *of* $\mathbb{Q}[\lambda]$ *and* $|\cdot|_{\mathfrak{p}}$ *is the* $\mathfrak{p}$*-adic absolute value with* $|\varpi|_{\mathfrak{p}} = N_{\mathbb{Q}[\lambda]/\mathbb{Q}}(\mathfrak{p})^{-1}$ *for a prime element* ϖ *of the valuation ring of* $\mathfrak{p}$ *in* $\mathbb{Q}[\lambda]$.

The right-hand-side of the formula of Corollary 5.3 applied to $\theta^*(F_\lambda)$ is exactly $\langle\mathcal{F}_\lambda,\mathcal{F}_\lambda\rangle$, and therefore $\big||C_0^D(\lambda;\mathbb{Z}[\lambda])|\big|_{\mathfrak{p}} = |\langle\mathcal{F}_\lambda,\mathcal{F}_\lambda\rangle|_{\mathfrak{p}}$ gives an expression of the congruence number $|C_0^D(\lambda;\mathbb{Z}[\lambda])|$ by the adjoint L-value, generalizing the formula of [EMI, Theorem 9.3.2] for $D = M_2(\mathbb{Q})$ to the definite D.

Proof. By integration over $S = \mathrm{SO}_{D_0}(\mathbb{Q})\backslash\mathrm{SO}_{D_0}(\mathbb{A})/\widehat{R}^\times$ identifying $\mathrm{SO}_{D_0} = D^\times/Z(D^\times)$ for the center $Z(D^\times)$ of the algebraic group $D^\times$, we get $\int_S \theta(F_\lambda)d\mu_h = \Omega_\lambda\sum_{a\in S} e_a^{-1}\mathcal{F}_\lambda(a)^2 = \Omega_\lambda\langle\mathcal{F}_\lambda,\mathcal{F}_\lambda\rangle$. Let $A = \mathbb{Z}[\lambda]_{\mathfrak{p}}$. If $S(A)_\lambda$ is A-free, it is generated by one element over A, and hence $S(A)_\lambda = A\mathcal{F}_\lambda$. If $S(A)^\lambda$ is A-free, again it is generated by one element, say $\mathcal{F}^\lambda$. Then we may assume that $\langle\mathcal{F}^\lambda,\mathcal{F}_\lambda\rangle = 1$, and hence $C_0^D(\lambda;A) = A/aA$ for a given by $a\mathcal{F}^\lambda = \mathcal{F}_\lambda$. Thus $\langle\mathcal{F}_\lambda,\mathcal{F}_\lambda\rangle = \langle a\mathcal{F}^\lambda,\mathcal{F}_\lambda\rangle = a$, and $\big||C_0^D(\lambda;A)|\big|_{\mathfrak{p}} = |a|_{\mathfrak{p}}$. Since $C_0^D(\lambda;\mathbb{Z}[\lambda]_{\mathfrak{p}}) \cong C_0^D(\lambda;\mathbb{Z}[\lambda])\otimes_{\mathbb{Z}[\lambda]}\mathbb{Z}[\lambda]_{\mathfrak{p}}$, the desired assertion follows. $\square$

Identify $S(A)\otimes_A S(A) = \mathrm{End}_A(S(A))$ by sending $\mathcal{F}\otimes\mathcal{G}$ to $\Phi_{\mathcal{F}\otimes\mathcal{G}} := \mathcal{H}\mapsto\langle\mathcal{H},\mathcal{G}\rangle\mathcal{F}$. Write $S_2^{new}(\Gamma_0(\partial))$ for the space spanned by new forms inside $S_2(\Gamma_0(\partial))$ and put $S_2^{new}(\Gamma_0(\partial);A) = S_2^{new}(\Gamma_0(\partial))\cap A[[q]]$. Then defining $\mathbf{h}(A) = A[T(n)|n=1,2,\dots]\subset\mathrm{End}(S_2^{new}(\Gamma_0(\partial);A))$, we have a morphism $H:\mathbf{h}(A)\to\mathrm{End}_A(S(A))$ given by $H(T(n)) = T(n)|_{S(A)}$. Since $S(A)\otimes_{H(A)}S(A)$ is sent to $\mathrm{End}_{H(A)}(S(A))$, we may regard $H:\mathbf{h}(A)\to\mathrm{End}_{H(A)}(S(A))$. Is this a surjective isomorphism? By the solution of Eichler's basis problem, H is an isomorphism if $A\subset\mathbb{C}$ is a field, and in general $\mathrm{Coker}(H)$ is A-torsion module.

Assume that $\partial = p$ and $V = D_\sigma^+$. Take $\delta = 1$. By Theorem 5.2 (and Theorem 0.1) specialized to our case, we have

$$\int_{Sh_1}\theta(\phi)^*(F)(g)d\mu_g = \frac{p^2}{4\pi^3(p+1)}L(1,Ad(F)), \tag{5.23}$$

which is the Fourier coefficient of $\theta(\phi)_*(\theta(\phi)^*(F))$ in $\mathbf{e}(\tau)$ by Theorem 5.9, taking $\phi^{(\infty)}$ to be the characteristic function of $\widehat{\mathbb{Z}}\oplus\widehat{R}_0$. Since $\mathbb{Z}\oplus R_0$ is the kernel of $R\ni x\mapsto\mathrm{Tr}(x) \bmod 2\in\mathbb{F}_2$, we find $\mathcal{R} := \{\alpha(\mathbb{Z}\oplus R_0)\subset\mathbb{Z}\oplus R_0\}$ is equal to $\mathbb{Z}+\mathfrak{m}_2R$, where $\mathfrak{m}_2$ is the maximal two-sided ideal of R above (2).

Recall $\boldsymbol{\phi}$ defined in §5.5 below (I) and ϕ defined below (5.17). Since $\Gamma_\delta = \mathrm{SO}_D(\mathbb{Q})\cap\Gamma_\phi = \mathrm{SO}_D(\mathbb{Q})\cap\Gamma_{\boldsymbol{\phi}}$, we find that Sh_δ for $\boldsymbol{\phi}$ and ϕ are equal. Thus by Theorem 5.9, the value (5.23) is also the Fourier coefficient of $\theta(\boldsymbol{\phi})_*(\theta(\phi)^*(F))$ in $\mathbf{e}(\tau)$ taking $\boldsymbol{\phi}^{(\infty)}$ to be the characteristic function of $\widehat{R}$. Since $\theta(\boldsymbol{\phi})$ is $\widehat{R}^\times$-invariant, we find $\theta(\boldsymbol{\phi})_*(\theta(\phi)^*(F)) = \theta(\boldsymbol{\phi})_*(\mathrm{Tr}(\theta(\phi)^*(F)))$, where $\mathrm{Tr}(f)(h) = \sum_{\gamma\in\widehat{R}^\times/\widehat{\mathcal{R}}^\times} f(h\gamma)$. By Hecke equivariance of $\theta(\boldsymbol{\phi})_*$ and $\boldsymbol{\phi}_v = \phi_v$ for all places v outside 2, $\theta(\phi)^*(F)$ and $\theta(\boldsymbol{\phi})^*(F)$ has the same Hecke eigenvalues for $T(n)$ for n prime to 2, we have $\mathrm{Tr}(\theta(\phi)^*(F)) = c\theta(\boldsymbol{\phi})^*(F)$ for a constant c by the strong multiplicity one theorem. Since their theta descents have the equal non-zero Fourier coefficients in $q = \mathbf{e}(\tau)$, we find $c = 1$. Thus

$$\theta(\boldsymbol{\phi})^*(F) = \mathrm{Tr}(\theta(\phi)^*(F)) = \frac{p^2}{4\pi^3(p+1)}L(1,Ad(F))(\mathcal{F}_\lambda\otimes\mathcal{F}_\lambda)\in S_E(\mathbb{C}). \tag{5.24}$$

This combined with Theorems 5.4, 5.7 and [EMI, Theorem 9.3.2] shows

Corollary 5.8. *Assume* $\partial = p$*, and let the notation and assumption be as in* Theorem 5.7 *for a prime* $\mathfrak{p}$ *of* $\mathbb{Z}[\lambda]$*. Then for a valuation ring* A *with* $\mathbb{Z}[\lambda]\subset A\subset\mathbb{Z}[\lambda]_{\mathfrak{p}}\cap\overline{\mathbb{Q}}$,

$$\theta(\boldsymbol{\phi})_*(\theta(\boldsymbol{\phi})^*(F)) = \frac{p^2}{4\pi^3(p+1)}L(1,Ad(F))F \quad\text{and}\quad C_0(\lambda;A)\cong C_0^D(\lambda;A), \tag{5.25}$$

and $A\frac{4\pi^3(p+1)}{p^2}\frac{\theta(\boldsymbol{\phi})^*(F)}{\Omega_+\Omega_-} = S(A)_\lambda\otimes_A S(A)_\lambda$ *inside* $S_E(A)$ *for* $\Omega_\pm := \Omega(\pm,\lambda;A)$ *as in* [EMI, (9.18)].

The assumption of this corollary holds if $\mathfrak{p}$ is prime to $6(p-1)$ as verified in [H05, Theorem 4.1].

5.8. **Fourier expansion of theta descent for definite D and real E.** In this subsection, the choice of $\phi^{(\infty)}$ is arbitrary. Recall $\mathrm{O}_{D_\sigma} = \mathrm{SO}_{D_\sigma} \sqcup \mathrm{SO}_{D_\sigma}\iota$ and $\mathrm{SO}_{D_\sigma} = G^+_{D_\sigma}/Z_{G^+_{D_\sigma}}$. Note that $D_{E_\mathbb{R}} = D_E \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{H} \times \mathbb{H}$ by an isomorphism sending $E \ni e \mapsto (e, e^\sigma) \in \mathbb{R} \times \mathbb{R}$. Thus

$$G^+_{D_\sigma}(\mathbb{R}) = \{(h_1, h_\sigma) \in \mathbb{H}^\times \times \mathbb{H}^\times \mid \det(h_1)/\det(h_\sigma) = 1\} \text{ and } \mathrm{SO}_{D_\sigma}(\mathbb{R}) = G^+_{D_\sigma}(\mathbb{R})/\mathbb{R}^\times$$

with $\mathbb{R}^\times$ diagonally embedded into the center of the product, which is compact and connected. Note that $D^\pm_{\sigma,\mathbb{R}} = \{(h, \pm h^\iota) | h \in \mathbb{H}\} \subset D_{E_\mathbb{R}}$. We identify $D_{\sigma,\mathbb{R}}$ with $\mathbb{H}$ by the left projection. Writing $D_E \ni \gamma \mapsto \gamma \in \mathbb{H}$ for the left projection and $\gamma \mapsto \gamma^\sigma \in \mathbb{H}$ for the right projection, we let $\gamma \in D_E^\times$ act on $D^\pm_{\sigma,\mathbb{R}} = \mathbb{H}$ by $h \mapsto \gamma^{-1} h \gamma^\sigma$.

Pick a Schwartz-Bruhat function $\phi = \phi^{(\infty)} \otimes \phi_\infty : D_{\sigma,\mathbb{A}} \to \mathbb{C}$ and assume that

$$\phi_\infty(v) = \Psi_k(v) = [v; \mathbf{x}]^n \mathbf{e}(N(v)\tau_\pm) \in \mathcal{S}(D^\pm_{\sigma,\mathbb{R}})$$

as in (5.5). Consider the associated classical theta series $\theta_k(\phi) = \theta_k(\phi)(\tau; z, w) = \sum_{v \in D_\sigma} \phi(v)$. Let

$$\widehat{\Gamma}_\phi := \{\gamma \in \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)}) = G^+_{D_\sigma}/Z_{G^+}(\mathbb{A}^{(\infty)}) | \phi^{(\infty)}(\gamma^{-1} x \gamma^\sigma) = \phi^{(\infty)}(x) \text{ for all } x \in D_{\sigma,\mathbb{A}^{(\infty)}}\}.$$

An automorphic form $f(h; \mathbf{x}) : D^\times \backslash D^\times_{E_\mathbb{A}} \to L_E(n; \mathbb{C})$ $(k = n+2)$ of weight $k\infty + k\infty\sigma$ satisfies

$$f(\gamma z x u; \mathbf{x}) = f(x; \mathbf{x} u_\infty^{-1}) \text{ for } u \in \widehat{\Gamma}_\phi D^1_E(\mathbb{R}), z \in E^\times_\mathbb{A} \text{ and } \gamma \in D^\times_E, \tag{5.26}$$

where $D^1_E(A) = \{\gamma \in (D_E \otimes_\mathbb{Q} A)^\times | N(\gamma) = 1\}$ is an algebraic group over $\mathbb{Q}$.

To define the theta descent, for any E-algebra such that $D_{E_A} \cong M_2(A) \times M_2(A)$, we describe $\mathrm{SL}_2(E_\mathbb{R})$-invariant self duality of $L_E(n; A)$. As before $L(n, 0; A)$ (resp. $L(0, n; A)$) is the space of homogeneous polynomials of degree n in (X, Y) (resp. (X', Y')). Let $(h, h_\sigma) \in D_E$ act on $P(X, Y) \in L(n, 0; A)$ (resp. $Q(X, Y) \in L(0, n; A)$) by $P(X, Y) \mapsto P((X, Y)h^\iota) = P|h(X, Y)$ (resp. $P(X, Y) \mapsto Q((X, Y)h^\iota_\sigma) = Q|h_\sigma(X, Y)$). Then $P(X, Y) \otimes Q(X', Y') \mapsto P(X, Y)Q(X', Y')$ gives a D_E-equivariant isomorphism $L(n, 0; A) \otimes_A L(0, n; A) \cong L_E(n; A)$.

We prepare another set of variables $\mathbf{s} := (S, T; S', T')$ for $L(n, 0; A) = AS^n + AS^{n-1}T + \cdots + AT^n$ and $L(0, n; A) = AS'^n + AS'^{n-1}T' + \cdots + AT'^n$. Regarding that $L(n, 0; A) \otimes_A L(n, 0; A)$ is made of polynomials in X, Y, S, T homogeneous of degree n in X, Y and also in S, T, we consider the pairing $(\cdot, \cdot)_L : L(n, 0; A) \otimes_A L(n, 0; A) \to A$ given by $P(X, Y; S, T) \mapsto (n!)^{-2} \nabla^n_L(P(X, Y; S, T))$, where $\nabla_L = \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial Y \partial S}$. Similarly, we define $\nabla_R = \frac{\partial^2}{\partial X' \partial T'} - \frac{\partial^2}{\partial Y' \partial S'} : L(0, n; A) \otimes_A L(0, n; A) \to A$, and put $(\cdot, \cdot)_R = (n!)^{-2} \nabla^n_R$. Finally we define $(\cdot, \cdot) : L_E(n; A) \otimes_A L_E(n; A) \to A$ by $(\cdot, \cdot)_L \otimes (\cdot, \cdot)_R$. We have $(P|(h, h_\sigma), Q|(h, h_\sigma)) = (N(h)N(h_\sigma))^n (P, Q)$.

Taking the measure $d\mu_h$ with $\int_{\widehat{\Gamma}_\phi} d\mu_h = 1$ on $\mathrm{SO}_{D_\sigma}(\mathbb{Q}) \backslash \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$ and restricting f in (5.26) to $G^+_{D_\sigma}(\mathbb{A}) \subset D^\times_{E_\mathbb{A}}$, we define the theta descent $\theta_*(f)(\tau)$ by

$$\theta_*(f)(\tau) := \int_{Sh_E} (\theta(\phi)(\tau; h; \mathbf{x}), f(h; \mathbf{s})) d\mu_h \tag{5.27}$$

for $Sh_E := \mathrm{SO}_{D_\sigma}(\mathbb{Q}) \backslash \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$. We now like to show that its Fourier coefficient for $\mathbf{e}(N(\alpha)\tau)$ is given by a finite sum of the period

$$P_\alpha(f) = \int_{Sh_\alpha} (\theta(\phi)(\tau; h; \mathbf{x}), f(h; \mathbf{s})) d\mu_h$$

for $Sh_\alpha = \mathrm{SO}_\alpha(\mathbb{Q}) \backslash \mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$ with $\widehat{\Gamma}_\alpha = \widehat{\Gamma}_\phi \cap \mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})$.

By approximation theorem, we can choose a finite set $\mathcal{A} \subset \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$ such that

$$\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)}) = \bigsqcup_{a \in \mathcal{A}} \mathrm{SO}_{D_\sigma}(\mathbb{Q}) a \widehat{\Gamma}_\phi \text{ so, } Sh_E \cong \mathcal{A}.$$

We have

$$\int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q}) \backslash \mathrm{SO}_{D_\sigma}(\mathbb{Q}) a \widehat{\Gamma}_\phi a^{-1}} = 1/|\mathrm{SO}_{D_\sigma}(\mathbb{Q}) \cap a \widehat{\Gamma}_\phi a^{-1}| = e_a^{-1}$$

for $e_a = |\Gamma^a_\phi|$ with $\Gamma^a_\phi := \mathrm{SO}_{D_\sigma}(\mathbb{Q}) \cap a \widehat{\Gamma}_\phi a^{-1}$, and for $\tau_\pm$ as in (5.5)

$$\theta_*(f)(\tau) = \int_{Sh_E} (\theta(\phi)(\tau; h; \mathbf{x}), f(h; \mathbf{s})) d\mu_h = \sum_{a \in \mathcal{A}} e_a^{-1} \sum_{\alpha \in D_\sigma} (\phi(a^{-1} \alpha a^\sigma; \mathbf{s})(\tau), f(a; \mathbf{x})) \mathbf{e}(N(\alpha)\tau_\pm).$$

Writing $\phi_a(v;\mathbf{s})=\phi(a^{-1}va^\sigma;\mathbf{s})$, we have for $k=n+2$

$$\begin{aligned}(5.28)\quad \theta_*(f)(\tau)&=\sum_{a\in\mathcal{A}}e_a^{-1}\sum_{\alpha\in D_\sigma}(\phi_a(\alpha;\mathbf{x}),f(a;\mathbf{s}))\mathbf{e}(N(\alpha)\tau_\pm)\\ &=\sum_{a\in\mathcal{A}}\sum_{\alpha\in D_\sigma/\Gamma^a_\phi}\phi^{(\infty)}(a^{-1}\alpha a^\sigma)([\alpha;\mathbf{x}]^n,f(a;\mathbf{s}))\mathbf{e}(N(\alpha)\tau_\pm).\end{aligned}$$

As before $\mathrm{O}_{D_\sigma}=\mathrm{SO}_{D_\sigma}\sqcup\mathrm{SO}_{D_\sigma}\iota=\mathrm{SO}_{D_\sigma}\sqcup\mathrm{SO}_{D_\sigma}\sigma$ and $\mathrm{O}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{O}_{D_\sigma}(\mathbb{A})=\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})$. We extend f originally defined on $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ to $\mathrm{O}_{D_\sigma}(\mathbb{A})$ by putting $f(\sigma x)=f(x)$ for $x\in\mathrm{SO}_{D_\sigma}(\mathbb{A})$. Then $f(x\sigma)=f(\sigma x\sigma)=f(x^\sigma)$, and by this extension, we have

$$(5.29)\qquad \int_{Sh_E}\theta(\phi)(\tau;h)f(h)d\mu_h=\int_{\mathrm{O}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{O}_{D_\sigma}(\mathbb{A})}\theta(\phi)(\tau;h)f(h)d\mu_h.$$

Consider the embedding $\mathrm{SO}_\alpha\hookrightarrow\mathrm{SO}_{D_\sigma}$ given by $\mathrm{SO}_\alpha=\mathrm{Aut}(D^\pm_{\alpha,0},\pm N)\ni h_\alpha\mapsto\mathrm{diag}[\mathrm{id}_{Z^\pm},h_\alpha]\in\mathrm{Aut}(D_\sigma,N)=\mathrm{SO}_{D_\sigma}$, which is compatible with the natural embedding $D_\alpha^\times=G^+_{D_{\alpha,0}}\hookrightarrow G^+_{D_\sigma}\subset D_E^\times$. If the image of $s,s'\in\mathrm{SO}_\alpha(\mathbb{A})$ coincides in $\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})$, we have $\gamma s=\gamma's'$ for $s,s'\in\mathrm{SO}_\alpha(\mathbb{A})$ with $\gamma,\gamma'\in\mathrm{SO}_{D_\sigma}(\mathbb{Q})$. Thus for $\delta=\gamma^{-1}\gamma'$, we have $s=\delta s'$. Applying σ_α, we find $s=s^{\sigma_\alpha}=\delta^{\sigma_\alpha}s'^{\sigma_\alpha}=\delta^{\sigma_\alpha}s'$. Thus $\delta^{\sigma_\alpha}\delta^{-1}=\delta^{\sigma_\alpha}s'(\delta s')^{-1}=1$, which implies $\delta\in\mathrm{SO}_\alpha(\mathbb{Q})$. Thus the natural map $\mathrm{SO}_\alpha(\mathbb{Q})\backslash\mathrm{SO}_\alpha(\mathbb{A})\to\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})$ is injective. More easily, we can show $\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\alpha$ injects into $\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$. We let $\mathrm{SO}_{D_\sigma}(\mathbb{Q})$ act on $\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$ by left multiplication. If $\xi\in\mathrm{SO}_{D_\sigma}(\mathbb{Q})$ fixes the image of $s\in\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$ in $\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$, we find $\xi s=s\gamma$ for $\gamma\in\widehat{\Gamma}_\phi$. Thus $\xi\in\mathrm{SO}_{D_\sigma}(\mathbb{Q})\cap s\widehat{\Gamma}_\phi s^{-1}$ which is a finite group and is trivial for all $s\in\mathrm{SO}_{D_\sigma}(\mathbb{A})$ if $\widehat{\Gamma}_\phi$ is sufficiently small. Indeed, if $\phi\circ\zeta\neq\phi$ for any root $1\neq\zeta\in\mathrm{SO}_{D_\sigma}(\mathbb{Q})$ of unity, $\mathrm{SO}_{D_\sigma}(\mathbb{Q})\cap s\widehat{\Gamma}_\phi s^{-1}$ is trivial for all $s\in\mathrm{SO}_{D_\sigma}(\mathbb{A})$. Thus

$$(5.30)\qquad \mathrm{SO}_\alpha(\mathbb{Q})\backslash\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\alpha\text{ injects into }\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\phi$$

if $\widehat{\Gamma}_\phi$ is small enough. We say that $\widehat{\Gamma}_\phi$ is neat if (5.30) holds.

Choose a complete representative set $S_\alpha\subset\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})$ for

$$Sh_\alpha:=\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{Q})\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})\widehat{\Gamma}_\phi/\widehat{\Gamma}_\phi.$$

If $\widehat{\Gamma}_\phi$ is neat, $S_\alpha\cong\mathrm{SO}_\alpha(\mathbb{Q})\backslash\mathrm{SO}_\alpha(\mathbb{A}^{(\infty)})/\widehat{\Gamma}_\alpha$. By adjusting the representative set $\mathcal{A}$, we may assume that $S_\alpha\subset\mathcal{A}$. Then the period of f over Sh_α is given by

$$(5.31)\qquad P_\alpha(f):=\int_{Sh_\alpha}(\theta(\tau;h;\mathbf{x}),f(h;\mathbf{s}))=\sum_{s\in S_\alpha}\phi^{(\infty)}(s^{-1}\alpha s^\sigma)([\alpha,\mathbf{x}]^n,f(s;\mathbf{s})),$$

which only depends on the class $\alpha\in D_\sigma/\Gamma_\phi$ and the support of the function $\alpha\mapsto P_\alpha(f)$ is contained in a lattice of D_σ.

We thus obtain, combining (5.28) and (5.31)

Theorem 5.9. *Suppose that f is an automorphic form on $D^\times_{E_\mathbb{A}}$ satisfying* (5.26). *If $n=0$, we further assume that $\int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})}f(h)d\mu_h=0$. Then we have, for $\tau_\pm$ as in* (5.5)

$$\theta_*(f)(\tau)=\sum_{\alpha\in D_\sigma/\Gamma_\phi;N(\alpha)>0}P_\alpha(f)\mathbf{e}(N(\alpha)\tau_\pm)\quad\textit{for an arbitrary }\phi^{(\infty)}.$$

We now compare $f(s;\mathbf{s})|_{Sh_\alpha}$ and $([\alpha;\mathbf{x}]^n,f(s;\mathbf{s}))$. Since $\pi_\alpha:=(n!)^{-2}\nabla^n:L_E(n;A)|_{\mathrm{SO}_\alpha}\to A$ and $\pi_{D_\sigma}:=(n!)^{-2}\nabla^n_L(n!)^{-2}\nabla^n_R:L_E(n;A)\otimes_A L_E(n;A)\to A$ are SO_α-equivariant, we have a commutative diagram up to constants

$$(5.32)\qquad \begin{array}{ccc} L_E(n;A)|_{\mathrm{SO}_\alpha}\otimes_A L_E(n;A)|_{\mathrm{SO}_\alpha} & \longrightarrow & L_E(n;A)\otimes_A L_E(n;A)\\ {\scriptstyle\pi_\alpha\otimes\pi_\alpha}\downarrow & & \downarrow{\scriptstyle\pi_{D_\sigma}}\\ A=A\otimes_A A & = & A.\end{array}$$

Writing the variables of the left (resp. right) factor of $L_E(n;A)|_{\mathrm{SO}_\alpha}$ as X,Y,X',Y' (resp. S,T,S',T'), we find from [H99, page 141] $\pi_\alpha(X^{n-j}Y^jX'^jY'^{n-j})=(-1)^j\binom{n}{j}^{-1}=\pi_\alpha(S^{n-j}T^jS'^jT'^{n-j})$ and

$\pi_{D_\sigma}(X^{n-j}Y^jX'^jY'^{n-j}S^{n-j}T^jS'^jT'^{n-j}) = \binom{n}{j}^{-2}$, and the above diagram commutes without ambiguity. For $\delta \in Z^\pm$, we have $([\delta,\mathbf{x}]^n, f(s;\mathbf{s})) = \delta^n((YX'-XY')^n, f(s;\mathbf{s}))$. Since $YX'-XY' = (X',Y')\wedge(X,Y)$, $(YX'-XY')^n$ generates a unique sub-factor invariant under SO_δ. Thus $((YX'-XY')^n,(TS'-T'S)^n) = c$ for a constant c. Since $\pi_\delta((YX'-XY')^n) = n+1$ by (5.7), we find $c=(n+1)^2$ from (5.32). Define

$$\pi'_\delta(P) = \pi_\delta(P)(YX'-XY')^n/(n+1).$$

By the above commutative diagram (5.32), we find $(\pi'_\alpha(P),Q) = (P,\pi'_\alpha(Q))$. Thus we conclude

$$\begin{aligned}([\delta,\mathbf{x}]^n, f(s;\mathbf{s})) &= \delta^n((YX'-XY')^n, f(s;\mathbf{s})) = \delta^n(\pi'_\delta((YX'-XY')^n), f(s;\mathbf{s}))\\ &= \delta^n((YX'-XY')^n, \pi'_\delta(f(s;\mathbf{s}))) = \delta^n(n+1)!^{-1}n!^{-1}\nabla^n(f(s;\mathbf{s})).\end{aligned}$$

For general $\alpha \in D_\sigma^\pm$ outside $Z^\pm$, since $D_{\alpha,\mathbb{R}} \cong \mathbb{H} \cong D_\mathbb{R}$, by Lemma 3.2, we find $h \in D_{E_\mathbb{R}}^\times$ with $N(h)=1$ such that $\alpha = h^{-1}\delta h^\sigma$ for $\delta \in \mathbb{R}$, and

$$\begin{aligned}(5.33)\quad ([\alpha,\mathbf{x}]^n, f(s;\mathbf{s})) &= ([h^{-1}\delta h^\sigma,\mathbf{x}]^n, f(s;\mathbf{s})) = ([\delta,\mathbf{x}h]^n, f(s;\mathbf{s})) = ([\delta,\mathbf{x}]^n, f(s;\mathbf{s}h^{-1}))\\ &= \delta^n(n+1)!^{-1}n!^{-1}\nabla^n(f(s;\mathbf{s}h^{-1})) = N(\alpha)^{n/2}(n+1)!^{-1}n!^{-1}\nabla^n(f(s;\mathbf{s}h^{-1})),\end{aligned}$$

where the last identity follows from $N(\alpha) = N(h^{-1}\delta h^\sigma) = \delta^2$. Thus we find

Corollary 5.10. *We have the following alternative expression of $P_\alpha(f)$:*

$$P_\alpha(f) = \sum_{s\in S_\alpha} e_s^{-1} \sum_{\beta\in\alpha\cdot{}^s\Gamma_\alpha} N(\beta)^{n/2}\phi^{(\infty)}(s^{-1}\beta s^\sigma)(n+1)!^{-1}n!^{-1}\nabla^n(f(s;\mathbf{s}h^{-1})),$$

where ${}^s\Gamma_\alpha = \mathrm{SO}_\alpha(\mathbb{Q}) \cap s\widehat{\Gamma}_\phi s^{-1}$ and $h \in D_{E_\mathbb{R}}^\times$ with $N(h)=1$ such that $h\alpha h^{-\sigma} \in Z_\mathbb{R}^\pm$.

6. General theory for imaginary E

Hereafter, the field E is imaginary quadratic. We assume K as in ($\mathrm{D}^\pm$) in §3.1 to be also imaginary quadratic so that $D_K \cong M_2(K)$. We have identified $D_K = M_2(K)$ so that $R\cdot O_K \subset M_2(K)$ for the O_K-linear span $R\cdot O_K \cong R\otimes_\mathbb{Z} O_K$. Note that $D_E\otimes_\mathbb{Q}\mathbb{R} \cong M_2(\mathbb{C})$ by an isomorphism sending $D_E \ni \left(\begin{smallmatrix} a & b\\ \partial b^\varsigma & a^\varsigma\end{smallmatrix}\right)\otimes e \mapsto \left(\begin{smallmatrix} ea & eb\\ \partial eb^\varsigma & ea^\varsigma\end{smallmatrix}\right) \in M_2(\mathbb{C})$. Thus $G^+_{D_\sigma}(\mathbb{R}) = \{h \in \mathrm{GL}_2(\mathbb{C}) | \det(h) \in \mathbb{R}^\times\}$ and $\mathrm{SO}_{D_\sigma}(\mathbb{R}) = G^+_{D_\sigma}(\mathbb{R})/\mathbb{R}^\times$ with $\mathbb{R}^\times$ embedded into the center of the product. Let $\mathrm{GL}_2^+(\mathbb{C}) := \{g \in \mathrm{GL}_2(\mathbb{C}) | 0 < \det(g) \in \mathbb{R}\}$. Then $\mathrm{SO}^+_{D_\sigma}(\mathbb{R}) = \mathrm{GL}_2^+(\mathbb{C})/\mathbb{R}^\times$ is the identity connected component of $\mathrm{SO}_{D_\sigma}(\mathbb{R})$. We identify SO_{D_σ} with $G^+_{D_\sigma}/Z_{G^+_{D_\sigma}}$ for the center $Z_{G^+_{D_\sigma}}$ of $G^+_{D_\sigma}$, and we let $\gamma \in G^+_{D_\sigma}$ act on $D_\sigma^\pm$ by $v \mapsto \gamma^{-1}v\gamma^\sigma$. Then $\mathrm{SO}_{D_\sigma}(\mathbb{R})$ has the identity connected component $\mathrm{SO}^+_{D_\sigma}(\mathbb{R})$ isomorphic to $\mathrm{PSL}_2(\mathbb{C})$.

Writing $x \mapsto \overline{x}$ for complex conjugation on $\mathbb{C}$, the diagonalized regular representation $\rho : \mathbb{C} \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ is given by $\rho(a) = \mathrm{diag}[a,\overline{a}]$. If $a \in E$, we have $\rho(a) = \mathrm{diag}[a,a^\sigma]$. Take the real 3–dimensional upper half space

$$\mathcal{H} := \left\{z = \left(\begin{smallmatrix} x & -y\\ y & \overline{x}\end{smallmatrix}\right) \middle| 0 < y \in \mathbb{R}, x \in \mathbb{C}\right\}.$$

As in [H94, (2.2)], we let $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{C})$ act on $\mathcal{H}$ by

$$\gamma(z) = (\rho(a)z+\rho(b))(\rho(c)z+\rho(d))^{-1}.$$

Since $\mathrm{PGL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R}) \cong \mathrm{GL}_2^+(\mathbb{C})/\mathbb{R}^\times\mathrm{SU}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R})$, we can extend the action of $\mathrm{GL}_2^+(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})$ making the center to act trivially; in other words, for $g \in \mathrm{GL}_2(\mathbb{C})$, taking $g' = \sqrt{\det(g)}^{-1}g$ and define $g(z) := g'(z)$. This action of $g \in \mathrm{GL}_2(\mathbb{C})$ outside $\mathrm{GL}_2^+(\mathbb{C})$ cannot be written as $(\rho(a)z+\rho(b))(\rho(c)z+\rho(d))^{-1}$ for $g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. For $\varepsilon = -J \in \mathcal{H}$, the stabilizer of ε in $\mathrm{SL}_2(\mathbb{C})$ is $\mathrm{SU}_2(\mathbb{R})$.

6.1. **Cohomological modular forms on** $\mathrm{SL}_2(\mathbb{C})$**.** In [H94, §3], we defined the notion of cohomological modular form on $\mathrm{SL}_2(\mathbb{C})$ for an arithmetic group $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. We translate and adelize the notion to $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ as $\mathrm{SO}^+_{D_\sigma}(\mathbb{R})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ via the isomorphism $G^+_{D_\sigma}(\mathbb{R})/\mathbb{R}^\times \cong \mathrm{SO}_{D_\sigma}(\mathbb{R})$. The arithmetic subgroup Γ is replaced by an open compact subgroup $\widehat{\Gamma} \subset \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$. A function $f : \mathrm{SO}_{D_\sigma}(\mathbb{A}) \to L(n^*;\mathbb{C})$ written as $f(h;\mathbf{s})$ for the variable $\mathbf{s} = (S,T)$ of $L(n^*;\mathbb{C})$ is called automorphic form of weight $k\infty + k\infty\sigma$ for $k = n+2$ if f satisfies

(M1) $f(\gamma h u;\mathbf{s}) = f(h;\mathbf{s}^t u^\sigma_\infty)$ for $h \in \mathrm{SO}_{D_\sigma}(\mathbb{A})$, $\gamma \in \mathrm{SO}_{D_\sigma}(\mathbb{Q})$ and $u \in \widehat{\Gamma}\cdot \mathrm{SO}_P(\mathbb{R})$, where P is the standard positive majorant of $s_\pm$ in Lemma 6.1 and $\mathrm{SO}_P(\mathbb{R})$ is as in (6.5);

(M2) $D_v f = \left(\frac{n^2}{2}+n\right) f$ for the Casimir operator D_v at each archimedean place $v = \infty, \infty\sigma$;

(M3) $f|_{\mathrm{SO}_{D_\sigma}(\mathbb{R})}$ is slowly increasing towards cusps of Γ if $D_E \cong M_2(E)$.

The theta series $\mathrm{SO}_{D_\sigma}(\mathbb{A}) \ni h \mapsto \theta(\tau;h;\mathbf{s}) \in \mathbb{C}[\mathbf{s}]$ for each τ satisfies (M1) and (M3). For an elliptic cusp form $F \in S^\pm_k(\Gamma_\tau)$, $\theta^*(\phi)(F)(h;\mathbf{s}) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{Mp}(\mathbb{A})} F(g)\theta(\phi)(\tau;h;\mathbf{s})d\mu_g$ satisfies (M2) (see [S75, §5-§7]).

6.2. **Realization of** $D^\pm_{\sigma,\mathbb{C}}$**.** Note that $D_{E_\mathbb{C}} \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. We use the notation introduced in §3.1 in Cases II and ID. In Case II, identifying $D_\mathbb{R} = M_2(\mathbb{R})$ and $D_\mathbb{C} = M_2(\mathbb{C})$, we have $D_\mathbb{R} = H^0(\langle\sigma_1\rangle, D_\mathbb{C})$ for σ_1 as in (II) in §3.1. In Case ID, identifying $D_\mathbb{R} = \mathbb{H}$ and $D_\mathbb{C} = M_2(\mathbb{C})$, we have $D_\mathbb{R} = H^0(\langle\sigma_J\rangle, D_\mathbb{C})$ for σ_J as in (ID) in §3.1.

Note

$$D^\pm_{\sigma,\mathbb{C}} = \begin{cases} \{(x, \pm x^\iota) | x \in M_2(\mathbb{C})\} & \text{if } \sigma = \sigma_1 \text{ (i.e., in case II)},\\ \{(x, \pm {}^t x) | x \in M_2(\mathbb{C})\} & \text{if } \sigma = \sigma_J \text{ (i.e., in case ID)},\end{cases}$$

since $Jx^\iota J^{-1} = {}^t x$. We have an embedding $D_{\sigma,\mathbb{R}} \hookrightarrow D_{\sigma,\mathbb{C}}$ given by

$$\begin{cases} x \mapsto (x, \overline{x}) & \text{in Case II with } \sigma = \sigma_1,\\ x \mapsto (x, J\overline{x}J^{-1}) & \text{in Case ID with } \sigma = \sigma_J.\end{cases}$$

We can identify $D^\pm_{\sigma,\mathbb{C}} = M_2(\mathbb{C})$ by the projection to the left factor. The action of $G^+_{D_\sigma}(\mathbb{C}) = \{(g,h) \in \mathrm{GL}_2(\mathbb{C})^2 | \det(g) = \det(h)\}$ on $M_2(\mathbb{C})$ is different in the following way:

$$\begin{cases} x \mapsto (g,h)x(g,h)^{\sigma_1\iota} = gxh^\iota & \text{in Case II with } \sigma = \sigma_1,\\ x \mapsto (g,h)x(g,h)^{\sigma_J\iota} = gx^t h & \text{in Case ID with } \sigma = \sigma_J.\end{cases}$$

Consider $\mathbf{x} := (X,Y;X',Y') \in (E\otimes_\mathbb{Q}\mathbb{C})^2 = \mathbb{C}^2 \oplus \mathbb{C}^2$, and define

$$[\mathbf{x}]_D := \left[\begin{smallmatrix} X\\ Y\end{smallmatrix}\right][X',Y'] = \left(\begin{smallmatrix} XX' & XY'\\ YX' & YY'\end{smallmatrix}\right) \quad\text{and}\quad [\mathbf{x}]_I = J\left[\begin{smallmatrix} X\\ Y\end{smallmatrix}\right][X',Y'] = \left(\begin{smallmatrix} X'Y & YY'\\ -XX' & -XY'\end{smallmatrix}\right).$$

When the case we are working is clear in the context, we just write $[\mathbf{x}]$ for $[\mathbf{x}]_I$ or $[\mathbf{x}]_D$. Then the action of $G^+_{D_\sigma}(\mathbb{C}) = \{(g,h) \in \mathrm{GL}_2(\mathbb{C})^2 | \det(g) = \det(h)\}$ on $M_2(\mathbb{C})$ is as follows:

$$\begin{cases} [\mathbf{x}]_I \mapsto (g,h)[\mathbf{x}]_I(g,h)^{\sigma_1\iota} = g[\mathbf{x}]_I h^\iota = [(X,Y)g^\iota, (X',Y')h^\iota] & \text{in Case II with } \sigma = \sigma_1,\\ [\mathbf{x}]_D \mapsto (g,h)[\mathbf{x}]_D(g,h)^{\sigma_J\iota} = g[\mathbf{x}]_D{}^t h = [(X,Y)^t g, (X',Y')^t h] & \text{in Case ID with } \sigma = \sigma_J.\end{cases}$$

The case $\sigma = \sigma_1$ is verified in (5.1), and the case $\sigma = \sigma_J$ can be verified by a computation. Define for $v = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in D^\pm_{\sigma,\mathbb{C}}$

$$\begin{cases} [v;\mathbf{x}] = \mathrm{Tr}_{D_E/E}(v^\iota[\mathbf{x}]_I) = dYX' + bXX' - cYY' - aXY' & \text{in Case II with } \sigma = \sigma_1,\\ [v;\mathbf{x}] = \mathrm{Tr}_{D_E/E}(v^\iota[\mathbf{x}]_D) = dXX' - bYX' - cXY' + aYY' & \text{in Case ID with } \sigma = \sigma_J.\end{cases} \tag{6.1}$$

We then have for $g \in G^+_{D_\sigma}(\mathbb{R})$

$$\begin{cases} [g^\iota v g^\sigma;\mathbf{x}] = [v; g[\mathbf{x}]_I g^{\sigma\iota}] = [v;\mathbf{x}g^\iota] & \text{in Case II with } \sigma = \sigma_1,\\ [g^\iota v g^\sigma;\mathbf{x}] = [v; g[\mathbf{x}]_D g^{\sigma_J\iota}] = [v; g[\mathbf{x}]_D{}^t g^{\sigma_1}] = [v;\mathbf{x}^t g] & \text{in Case ID with } \sigma = \sigma_J.\end{cases} \tag{6.2}$$

The first formula for $\sigma = \sigma_1$ is (5.4). In the second formula, $\mathbf{x}^t g = ((X,Y)^t g, (X',Y')^t g^{\sigma_J})$.

Define a standard positive majorant of $s_\pm$ in Case ID by

$$P(x,y) = P_D(x,y) = \mathrm{Tr}_{D_E/E}(xJ\overline{y}^\iota J^{-1}) = \mathrm{Tr}_{D_E/E}(xy^*) \tag{6.3}$$

for $y^* = {}^t\overline{y}$. If $x \in D^{\pm}_{\sigma_J,A}$ and $J \in D_{E_A} \cap D_{E_{\mathbb{C}}}$ for a finite extension $A_{/\mathbb{Q}} \subset \mathbb{C}$, then

$$(Jx)^{\iota} = -x^{\iota}J = \mp Jx^{\sigma_1}J^{-1}J = \mp Jx^{\sigma_1} = \mp J^{\sigma_1}x^{\sigma_1} = \mp(Jx)^{\sigma_1}.$$

Thus $x \mapsto Jx$ induces an isomorphism $D^{\pm}_{\sigma_J,A} \cong D^{\mp}_{\sigma_1,A}$. In particular, this isomorphism sends $[\mathbf{x}]_D \in D_{\sigma_J,\mathbb{C}}$ to $[\mathbf{x}]_I \in D_{\sigma_1,\mathbb{C}}$. Define the standard positive majorant in Case II by

$$P(x,y) = P_I(x,y) = P_D(Jx,Jy) = \mathrm{Tr}_{D_E/E}(xy^*) \text{ in Case II.} \tag{6.4}$$

Lemma 6.1. *Let the notation be as above. Then P defined as above is a positive majorant of $s_{\pm}$ and $v \mapsto [v;\mathbf{x}]^{n+1}$ is a spherical harmonic polynomial for the standard positive majorant P. Moreover the stabilizer of P in $\mathrm{SL}_2(\mathbb{C})$ is $\mathrm{SU}_2(\mathbb{R})$,*

Proof. Recall $s(x,y) = s_{\pm}(x,y) = \mathrm{Tr}_{D_E/E}(xx^{\sigma}) = \pm\mathrm{Tr}_{D_E/E}(xx^{\iota})$. As for P being a positive majorant, we only need to prove this for $s = s_+$ and $\sigma = \sigma_J$ (i.e., in Case ID). We have $s(x,y) = \mathrm{Tr}_{D_E/E}(xJ\overline{y}J^{-1})$ over $D_{\sigma,\mathbb{R}}$. On $Z_{\mathbb{R}}$, $s = P$ and on $D_{0,\mathbb{R}}$, $0 \leq P(x,y) = \mathrm{Tr}_{D_E/E}(xJ\overline{y}^{\iota}J^{-1}) = -\mathrm{Tr}_{D_E/E}(xJ\overline{y}J^{-1}) = -s(x,y)$, which shows that P is a positive majorant in Case ID. Since $P(x,y) = \mathrm{Tr}_{D_E/E}(xy^*)$, its stabilizer in $\mathrm{SL}_2(\mathbb{C})$ is $\mathrm{SU}_2(\mathbb{R})$ by definition. Since $J \in \mathrm{SU}_2(\mathbb{R})$, $\mathrm{SU}_2(\mathbb{R})$ is also the stabilizer of P_I.

Now we prove that $[v;\mathbf{x}]^{n+1}$ is a spherical harmonic polynomial. Since $N([\mathbf{x}]) = \det([\mathbf{x}]) = 0$, we need to show $s_{\pm}(v;[\mathbf{x}]) = \pm P(v,[\mathbf{x}])$ [HMI, page 143]. This follows from the above computation in Case ID. By the isomorphism $J : D^{\pm}_{\sigma_J,A} \cong D^{\mp}_{\sigma_1,A}$, this also shows the result in Case II. □

Define $g_z = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ for $z = \left(\begin{smallmatrix} x & -y \\ y & \overline{x}\end{smallmatrix}\right) \in \mathcal{H}$. Then $g_z(\varepsilon) = z$.

Corollary 6.2. *Let $P_z(x,y) = P(g_z^{-1}xg_z^{\sigma}, g_z^{-1}yg_z^{\sigma})$, which is a positive majorant associated to $z \in \mathcal{H}$. Then if $g(\varepsilon) = z$, the polynomial $v \mapsto [g^{-1}vg^{\sigma};\mathbf{x}]^{n+1}$ is spherical harmonic with respect to P_z.*

Let

$$\mathrm{SO}_P(\mathbb{R}) := \{u \in \mathrm{SO}_{D^{\pm}_{\sigma}}(\mathbb{R}) | P[u^{-1}vu^{\sigma}] = P[v] \text{ for all } v \in D^{\pm}_{\sigma,\mathbb{R}}\}. \tag{6.5}$$

If $u \in \mathrm{SU}_2(\mathbb{R})$ (i.e., $u^tu^{\sigma} = 1 \Leftrightarrow {}^tu = u^{-\sigma} = u^{\iota\sigma}$),

$$P(u^{-1}vu^{\sigma}, u^{-1}vu^{\sigma}) = \mathrm{Tr}_{D_E/E}(u^{-1}vu^{\sigma t}(u^{-1}vu^{\sigma})^{\sigma}) = \mathrm{Tr}_{D_E/E}(v^tv^{\sigma}) = P(v,v).$$

Thus $\mathrm{SO}_{P_D}(\mathbb{R}) = \mathrm{SU}_2(\mathbb{R})/\{\pm 1\}$ and $\mathrm{SO}_{P_I}(\mathbb{R}) = J \cdot \mathrm{SO}_{P_D}(\mathbb{R})J^{-1} = \mathrm{SU}_2(\mathbb{R})/\{\pm 1\}$ as $J \in \mathrm{SU}_2(\mathbb{R})$.

By Lemma 6.1, the function $v \mapsto [v;\mathbf{x}]^{n+1}$ is a spherical harmonic polynomial on $D_{\sigma,\mathbb{C}}$ of homogeneous degree $n+1$. Write H_n for the space of spherical harmonics of homogeneous degree $n+1$ on D_{σ} and homogeneous of degree $n+1$ in (X,Y) and (X',Y'). We have $\dim_{\mathbb{C}} H_n = 2n+3$ (in [A78, Lemma 2], this space is denoted H_{n+1}).

By (6.2), as a function of $v \in D_{\sigma}$ and $\mathbf{x} = (X,Y;X',Y')$, $[v;\mathbf{x}]^{n+1}$ intertwines the representation of $\mathrm{SO}_{D_{\sigma}}$ on the space of spherical functions of degree $n+1$ with the symmetric $(n+1)$-th power representation of $g \mapsto g \otimes g^{\sigma}$. On the maximal compact subgroup $\mathrm{SU}_2(\mathbb{R})/\{\pm 1\} = \mathrm{SO}_P(\mathbb{R}) \subset \mathrm{SO}_{D_{\sigma}}(\mathbb{R})$, $g \mapsto g^{\sigma}$ is equivalent to the standard representation. Thus, on $L_E(n+1;\mathbb{C})$, the action of $\mathrm{SU}_2(\mathbb{R})$ is equivalent to the symmetric $(n+1)$ power representation of $g \mapsto g \otimes g$ which contains the symmetric n^*-power $g \mapsto g^{\otimes n^*}$ (for $n^* = 2n+2$) with multiplicity one [H94, (11.2a)]. On the other hand, as seen in [A78, Lemma 2], on the space H_n of spherical functions of degree $n+1$, $\mathrm{SU}_2(\mathbb{R})$ acts by the symmetric n^*-th tensor representation irreducibly.

6.3. **Locally constant sheaves on $\Gamma_{\phi}\backslash\mathcal{H}$.** Recall $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SO}_{D_{\sigma}}(\mathbb{R}) = G^+_{D_{\sigma}}(\mathbb{R})/Z(G^+_{D_{\sigma}}(\mathbb{R}))$. Let M be a discrete left $\mathrm{PSL}_2(\mathbb{C})$-module. Regard M as a right $\mathrm{PSL}_2(\mathbb{C})$-module by $mg = g^{-1}m$, and write ${}^tM$ for the right $\mathrm{PSL}_2(\mathbb{C})$-module M. Write $\mathrm{PSU}_2(\mathbb{R})$ for the image of $\mathrm{SU}_2(\mathbb{R})$ in $\mathrm{PSL}_2(\mathbb{C})$. We construct on the automorphic manifold $\mathcal{S} := \Gamma_{\phi}\backslash\mathcal{H}$ a covering space in the following two ways:

(A) $\widetilde{M} := \Gamma_{\phi}\backslash(\mathcal{H} \times M)$ via the action $\gamma(z,m) = (\gamma(z),\gamma m)$ for $\gamma \in \Gamma_{\phi}$ and $z \in \mathcal{H}$;

(B) $M^* := ((\Gamma_{\phi}\backslash\mathrm{PSL}_2(\mathbb{C})) \times {}^tM)/\mathrm{PSU}_2(\mathbb{R})$ via the action $\gamma(g,m)u = (\gamma gu, mu)$ ($u \in \mathrm{PSU}_2(\mathbb{R})$, $\gamma \in \Gamma_{\phi}$) regarding ${}^tM$ as a right $\mathrm{PSU}_2(\mathbb{R})$-module.

We use the symbol $\mathcal{S}$ for $\Gamma_{\phi}\backslash\mathcal{H}$ (not Sh) as $\mathcal{S}$ is not an algebraic variety (so, not a Shimura variety). The covering spaces are étale over $\mathcal{S}$ if $\Gamma_{\phi} \cap g\mathrm{PSU}_2(\mathbb{R})g^{-1} = \{1\}$ for all $g \in \mathrm{PSL}_2(\mathbb{C})$. The definition (B) as above works well for any right $\mathrm{PSU}_2(\mathbb{R})$-module X. However X^* may not have a matching

$\widetilde{X}$ (i.e., X may not have compatible left action of Γ_ϕ without enlarging X). In this way, we can construct the sheaf $L^*(n^*;\mathbb{C})$ on $\mathcal{S}$ for the $\mathrm{PSU}_2(\mathbb{R})$-module $L(n^*;\mathbb{C})$. Here is an archimedean version of [H88, Proposition 6.1]:

Proposition 6.3. *Let M be a discrete left $\mathrm{PSL}_2(\mathbb{C})$-module. We have a canonical isomorphism of covering spaces $M^* \cong \widetilde{M}$ induced by $(g,m) \mapsto (g(\varepsilon), gm)$ for $g \in \mathrm{PSL}_2(\mathbb{C})$, whose converse is given by $(z,m) \mapsto (g_z, mg_z) = (g_z, g_z^{-1}m)$.*

Proof. Define a map $i : \mathrm{SL}_2(\mathbb{C}) \times {}^tM \to \mathcal{H} \times M$ by $(g,m) \mapsto (g(\varepsilon), gm)$ as above. Then for $\gamma \in \Gamma_\phi$ and $u \in \mathrm{SU}_2(\mathbb{R})$.

$$i(gu, mu) = (g(\varepsilon), guu^{-1}m) = (g(\varepsilon), gm) \text{ and } i(\gamma g, m) = (\gamma(g(\varepsilon)), \gamma gm) \sim (g(\varepsilon), gm) \text{ (in } \widetilde{M}).$$

Thus i indices a morphism of covering spaces $i : M^* \to \widetilde{M}$. Since this is an isomorphism on the fiber at g if $\gamma(g(\varepsilon)) = g(\varepsilon)$ for $\gamma \in \Gamma_\phi$ implies $\gamma = 1$, we conclude the isomorphism of the fiber $M^*_{g(\varepsilon)} \cong \widetilde{M}_{g(\varepsilon)}$ at $g(\varepsilon) \in \mathcal{H}$. Suppose the stabilizer $\Gamma_{g(\varepsilon)}$ in Γ_ϕ of $g(\varepsilon)$ is non-trivial. For $1 \neq \gamma \in \Gamma_{g(\varepsilon)}$, $(g(\varepsilon), gm) = (\gamma(g(\varepsilon)), \gamma gm) = (g(\varepsilon), \gamma gm)$ in $\widetilde{M}_{g(\varepsilon)}$. This implies $m \in H^0(g^{-1}\Gamma_{g(\varepsilon)}g, M)$. Thus $\widetilde{M}_{g(\varepsilon)} = H^0(g^{-1}\Gamma_{g(\varepsilon)}g, M)$. Note $g^{-1}\Gamma_{g(\varepsilon)}g \subset \mathrm{PSU}_2(\mathbb{R})$. Then in $M^*_{g(\varepsilon)}$, we have $(g,m) = (gg^{-1}\gamma g, mg^{-1}\gamma g) = (\gamma g, mg^{-1}\gamma g) = (g, mg^{-1}\gamma g)$. Thus again $M^*_{g(\varepsilon)} = H^0(g^{-1}\Gamma_{g(\varepsilon)}g, M)$, and hence i induces an isomorphism fiber by fiber, as desired. $\square$

We adelize the construction as follows: Write $\mathcal{S}_{\mathbb{A}} = \mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{SO}_{D_\sigma}(\mathbb{A})/\widehat{\Gamma}_\phi \mathrm{SO}_P(\mathbb{R})$ adelically for the closure $\widehat{\Gamma}_\phi$ of Γ_ϕ in $\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$. Then we define an adelized covering space:

($\mathrm{B}_{\mathbb{A}}$) $M^*_{\mathbb{A}} := (\mathcal{S}_{\mathbb{A}} \times {}^tM)/\widehat{\Gamma}_\phi \mathrm{SO}_P(\mathbb{R})$ through the action $(g,m)u = (gu, mu_\infty)$ for $u \in \widehat{\Gamma}_\phi \mathrm{SO}_P(\mathbb{R})$.

By the strong approximation theorem for $\mathrm{SO}_{D_\sigma}(\mathbb{A})$, we have $\mathcal{S}_{\mathbb{A}} = \mathcal{S}$, and by [H88, Proposition 6.1],

Corollary 6.4. *We have a canonical isomorphism $M^*_{\mathbb{A}} \cong M^* \cong \widetilde{M}$ induced by the projection to the ∞-component.*

Remark 6.5. If the above covering space is étale, it defines a locally constant sheaf which we denote by the same symbol $X_{/\mathcal{S}} = M^*_{\mathbb{A}}, M^*, \widetilde{M}$; so, we have a well defined sheaf cohomology $H^\bullet(\mathcal{S}, X)$. Even if the covering space is not étale, we have a normal subgroup of finite index $\Gamma \subset \Gamma_\phi$ such that the covering is étale over $\mathcal{S}' = \Gamma\backslash\mathcal{H}$. Thus we can define the cohomology group $H^\bullet(\mathcal{S}, X_{/\mathcal{S}}) := H^\bullet(\mathcal{S}', X_{/\mathcal{S}'})^{\Gamma_\phi}$, which is well defined as long as the multiplication by $|\Gamma_\phi/\Gamma|$ is invertible on X. In this sense, we pretend that X is étale over $\mathcal{S}$ (assuming to have a choice of Γ with $|\Gamma_\phi/\Gamma|$ invertible on coefficients).

6.4. **Vector valued theta series for imaginary** E**.** Define for the majorant P in Lemma 6.1

$$\Psi_k(v;\tau) = \eta^{1/2}[v;\mathbf{x}]^{n+1}\mathbf{e}(\pm N(v)\xi + \frac{\eta P[v]}{2}\sqrt{-1}) \in \mathcal{S}(D^\pm_{\sigma,\mathbb{R}}) \quad (n = k-2, \tau \in \mathfrak{H}, v \in D^\pm_\sigma). \tag{6.6}$$

We have $\eta^{1/2}$ in front to adjust the metaplectic weight to be k (e.g., [HMI, Theorem 2.65]). Here, for $u \in \mathrm{SU}_2(\mathbb{R})/\{\pm1\} = \mathrm{SO}_P(\mathbb{R}) \subset \mathrm{SO}_{D_\sigma}(\mathbb{R})$ and $g \in \mathrm{SO}_{D_\sigma}(\mathbb{R})$, the coefficient polynomial satisfies

$$gu \mapsto [u^{-1}g^{-1}vg^\sigma u^\sigma;\mathbf{x}]^{n+1} = [g^{-1}vg^\sigma;\mathbf{x}{}^tu^\sigma]^{n+1}, \tag{6.7}$$

where ${}^tu^\sigma = u^{-1}$ or ${}^tu$ according to whether in Case II ($\sigma = \sigma_1$) or in Case ID ($\sigma = \sigma_J$).

For $\phi^{(\infty)} \in \mathcal{S}(D^\pm_{\sigma,\mathbb{A}^{(\infty)}})$, putting $\phi = \phi^{(\infty)}\Psi_k$ we consider, for $\tau \in \mathfrak{H}$ and $g \in \mathrm{SO}_{D_\sigma}(\mathbb{A})$

$$\begin{aligned}\theta(\phi)(\tau;g) = \theta(\phi)(\tau;g;\mathbf{x}) &= \sum_{\alpha \in D^\pm_\sigma} \phi(g^{-1}\alpha g^\sigma) \\ &= \eta^{1/2} \sum_{\alpha \in D^\pm_\sigma} \phi^{(\infty)}(g^{-1}\alpha g^\sigma)[g_\infty^{-1}\alpha g_\infty^\sigma;\mathbf{x}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_\infty^{-1}\alpha g_\infty^\sigma]}{2}\sqrt{-1}).\end{aligned} \tag{6.8}$$

As before, let $\Gamma_\phi := \{\gamma \in \mathrm{SO}_{D_\sigma}(\mathbb{Q}) | \phi^{(\infty)} \circ \gamma = \phi^{(\infty)}\}$ and write $\widehat{\Gamma}_\phi$ for its closure in $\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$. Then for $u \in \widehat{\Gamma}_\phi \mathrm{SU}_2(\mathbb{R}) = \mathrm{SO}_P(\mathbb{R})$ as in (6.7) and $g \in \mathrm{SO}_{D_\sigma}(\mathbb{A})$,

$$\begin{aligned}(6.9)\quad \theta(\phi)(\tau; gu; \mathbf{x}) &= \sum_{\alpha \in D_\sigma^\pm} \phi(u^{-1}g^{-1}\alpha g^\sigma u^\sigma) \\ &= \eta^{1/2} \sum_\alpha \phi^{(\infty)}(u^{-1}g_\infty^{-1}\alpha g_\infty^\sigma u^\sigma)[u_\infty^{-1}g^{-1}\alpha g^\sigma u_\infty^\sigma; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[u_\infty^{-1}g_\infty^{-1}\alpha g_\infty^\sigma u_\infty^\sigma]}{2}\sqrt{-1}) \\ &= \eta^{1/2} \sum_\alpha \phi^{(\infty)}(g^{-1}\alpha g^\sigma)[g_\infty^{-1}\alpha g_\infty^\sigma; \mathbf{x}^t u_\infty^\sigma]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_\infty^{-1}\alpha g_\infty^\sigma]}{2}\sqrt{-1}) = \theta(\phi)(\tau; g; \mathbf{x}^t u_\infty^\sigma).\end{aligned}$$

Since $[\alpha; \widetilde{\mathbf{x}}] \in L(n^*; \mathbb{C})$ $(n^* = 2n+2)$ for the $\mathrm{SU}_2(\mathbb{R})$-module $L(n^*; \mathbb{C})$, we may regard $\theta(\phi)(\tau; h; \widetilde{\mathbf{x}})$ has values in $L(n^*; \mathbb{C})$ with $\theta(\phi)(gu_\infty) = \theta(\phi)(g)\rho_{n^*}(u_\infty)$ writing the action of $\mathrm{SU}_2(\mathbb{R})$ on $L(n^*; \mathbb{C})$ as ρ_{n^*}. As seen in [A78, §2], base change image using $\theta(\phi)$ as the kernel function is an eigenfunction of the Casimir operator with eigenvalue equal to that of $L_E(n; \mathbb{C})$ (cf. [H94, §2.3]).

For $\gamma \in G_{D_\sigma}^+(\mathbb{Q})/Z_{G_{D_\sigma}^+}(\mathbb{Q}) = \mathrm{SO}_{D_\sigma}(\mathbb{Q})$, we have

$$\begin{aligned}(6.10)\quad \theta(\phi)(\tau; \gamma g; \mathbf{x}) &= \sum_{\alpha \in D_\sigma^\pm} \phi(g^{-1}\gamma^{-1}\alpha\gamma^\sigma g^\sigma) \\ &= \eta^{1/2} \sum_\alpha \phi^{(\infty)}(g_\infty^{-1}\gamma^{-1}\alpha\gamma^\sigma g_\infty^\sigma)[g_\infty^{-1}\gamma^{-1}\alpha\gamma^\sigma g_\infty^\sigma; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_\infty^{-1}\gamma^{-1}\alpha\gamma^\sigma g_\infty^\sigma]}{2}\sqrt{-1}) \\ &\overset{\gamma^{-1}\alpha\gamma^\sigma \mapsto \alpha}{=} \eta^{1/2} \sum_\alpha \phi^{(\infty)}(g^{-1}\alpha g^\sigma)[g^{-1}\alpha g^\sigma; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_\infty^{-1}\alpha g_\infty^\sigma]}{2}\sqrt{-1}) = \theta(\phi)(\tau; g; \mathbf{x}).\end{aligned}$$

Let $h^? = h^\iota$ in Case II and ${}^t h$ in Case ID for $h \in G_{D_{E_\sigma}}^+(\mathbb{A})$. Define

$$(6.11)\quad \theta(\tau; z; \mathbf{x}) = \theta(\tau; g_z; \mathbf{x}g_z^{-?}) = \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm} \phi^{(\infty)}(\alpha)[\alpha; \mathbf{x}g_z^{-1}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_z^{-1}\alpha g_z^\sigma]}{2}\sqrt{-1}),$$

This definition could be given by choosing any g_∞ with $g_\infty(\varepsilon) = z$ in place of g_z, as $P[g_z^{-1}\alpha g_z^\sigma] = P[u^{-1}g_z^{-1}\alpha g_z^\sigma u^\sigma]$ for any $u \in \mathrm{SU}_2(\mathbb{R})$. We can check for any $\gamma \in \Gamma_\phi$,

$$\begin{aligned}(6.12)\quad \theta(\tau; \gamma(z); \mathbf{x}) &= \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm} \phi^{(\infty)}(\alpha)[\alpha; \mathbf{x}g_{\gamma(z)}^{-1}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_{\gamma(z)}^{-1}\alpha g_{\gamma(z)}^\sigma]}{2}\sqrt{-1}) \\ &= \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm} \phi^{(\infty)}(\alpha)[\alpha; \mathbf{x}g_{\gamma(z)}^{-1}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_z^{-1}\gamma^{-1}\alpha\gamma^\sigma g_z^\sigma]}{2}\sqrt{-1}) \\ &\overset{\alpha \mapsto \gamma^{-\iota}\alpha\gamma^{-\sigma}}{=} \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm} \phi^{(\infty)}(\alpha)[\alpha; \mathbf{x}g_z^{-1}]^{n+1}\mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_z^{-1}\alpha g_z^\sigma]}{2}\sqrt{-1}).\end{aligned}$$

The last equality of (6.12) follows from (7.9) and (7.10). By Corollary 6.4 and the invariance in (6.12), $\theta(\phi)(\tau; z; \mathbf{x})$ is a global section of $\widetilde{L}_E(n+1; \mathbb{C})$ over $\mathcal{S}$ and by the invariance (6.9), $\theta(\phi)(\tau; g; \mathbf{x})$ is the equivalent global section of $L_E^*(n+1; \mathbb{C})$ over $\mathcal{S}$.

6.5. **Explicit form down to earth of Schwartz functions.** By Lemma 3.2, there are two isomorphism classes of the action on $D_{E_\mathbb{R}}$ of $\mathrm{Gal}(E/\mathbb{Q})$ over $\mathbb{R}$ and hence on $D_{\sigma,\mathbb{R}}$ such that $H^0(\langle\sigma_1\rangle, D_{E_\mathbb{R}}) \cong M_2(\mathbb{R})$ and $H^0(\langle\sigma_J\rangle, D_{E_\mathbb{R}}) = \mathbb{H}$. For $z = \left(\begin{smallmatrix} x & -y \\ y & \overline{x} \end{smallmatrix}\right) \in \mathcal{H}$, we put $g_z = y^{-1/2}\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{H}$; so, $g_z(\varepsilon) = z$. We first deal with Case ID where $\sigma = \sigma_J$; so, $D_\mathbb{R} \cong \mathbb{H}$.

As a subspace of $M_2(\mathbb{C})$, we have

$$(6.13)\qquad D_{\sigma_J}^\pm = D_{\sigma_J,\mathbb{R}}^\pm = \left\{ \left(\begin{smallmatrix} u\sqrt{\pm 1} & w \\ \overline{w} & t\sqrt{\pm 1} \end{smallmatrix}\right) \middle| u, t \in \mathbb{R} \text{ and } w \in \mathbb{C} \right\}.$$

Let $p_+(z) := yg_z^\iota g_z^{\sigma_J} = yg_z^{-1}g_z^{\sigma_J} = \left(\begin{smallmatrix} 1+x\overline{x} & -xy \\ -y\overline{x} & y^2 \end{smallmatrix}\right)$. Then $p_+(z) \in D_{\sigma_J}^+$ and $p_-(z) := \sqrt{-1}p(z) \in D_{\sigma_J}^-$ with $\pm s_\pm[p_\pm(z)] > 0$. Let $D_{\sigma_J}^\pm = \mathbb{R}p_\pm(z) \oplus (\mathbb{R}p_\pm(z))^\perp$, and write $V_z := (\mathbb{R}p_\pm(z))^\perp$ on which $\mp s_\pm$

is positive definite. For $h \in G^+_{D_{\sigma_J}}(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{C})$, $hg_z = g_{h(z)}u$ for $u \in \mathbb{H}^\times$ with $N(u) = N(h) \in \mathbb{R}^\times$. Thus

$$h^{-1}p_\pm(z)h^{\sigma_J} = h^{-1}yg_z^{-1}g_z^{\sigma_J}h^{\sigma_J} = u^{-1}y(h(z))g^\iota_{h(z)}g^{\sigma_J}_{h(z)} = u^{-1}p_\pm(h(z))u^{\sigma_J},$$

where $y(h(z))$ is the y-entry of $h(z) \in \mathcal{H}$. Hence we have verified the following fact for $\sigma = \sigma_J$

$$P[h^{-1}p_\pm(z)h^\sigma] = P[p_\pm(h(z))]. \tag{6.14}$$

Writing $v = ap_\pm(z) + w$ and $v' = a'p_\pm(z) + w'$ with $s_\pm(\mathbb{R}p_\pm(z), \mathbb{R}w + \mathbb{R}w') = 0$ and $a, a' \in \mathbb{R}$, we define

$$P_z(v, v') = aa's_\pm[p_\pm(z)] - s_\pm(w, w'),$$

which is a positive majorant of $s_\pm$. Note, for $\sigma = \sigma_J$

$$P_z(h^{-1}vh^\sigma, h^{-1}v'h^\sigma) = aa's_\pm(h^{-1}p_\pm(z)h^\sigma) - s_\pm(h^{-1}wh^\sigma, h^{-1}w'h^\sigma) = P_{h(z)}(v, v'). \tag{6.15}$$

Then we find

$$P_z[v] + s_\pm[v] = a^2s_\pm[p_\pm(z)] - s_\pm[w] + (a^2s_\pm[p_\pm(z)] + s_\pm[w]) = 2a^2s_\pm[p_\pm(z)] = 2\frac{s_\pm(p_\pm(z), v)^2}{s_\pm[p_\pm(z)]},$$

since $s_\pm(p_\pm(z), v) = as_\pm[p_\pm(z)]$. Note

$$s_\pm[p_\pm(z)] = \mathrm{Tr}_{D_\mathbb{C}/\mathbb{C}}(p_\pm(z)p_\pm(z)^{\sigma_J}) = \pm\mathrm{Tr}_{D_\mathbb{C}/\mathbb{C}}(p_\pm(z)p_\pm(z)^\iota) = \pm 2\det(p_\pm(z)) = 2y(z)^2,$$

where $y(z) = y$ for $z = \left(\begin{smallmatrix} x & -y \\ y & \overline{x}\end{smallmatrix}\right) \in \mathcal{H}$. Write $[z; v] = \mathrm{Tr}_{D_\mathbb{C}/\mathbb{C}}(p_-(z)v^\iota)$. Combining these formulas,

$$\frac{P_z[v]}{2} = \frac{1}{2}\left\{-s_\pm[v] \pm \frac{[z; v]^2}{y(z)^2}\right\} = \mp N(v) \pm \frac{[z; v]^2}{2y(z)^2}. \tag{6.16}$$

Therefore, the exponential factor of the standard Schwartz function is

$$\mathbf{e}(\pm N(v)\xi + \frac{P_z[v]\eta\sqrt{-1}}{2}) = \mathbf{e}(\pm N(v)\overline{\tau} \pm \frac{[z; v]^2\eta\sqrt{-1}}{2y(z)^2}).$$

By (6.15), $P_z[h^{-1}vh^\sigma] = P_{h(z)}[v]$ for $\sigma = \sigma_J$, and hence

$$\frac{[z; h^{-1}vh^\sigma]^2}{2y(z)^2} = \frac{[h(z); v]^2}{2y(h(z))^2}, \tag{6.17}$$

which is also valid in Case II for $[z; v]$ defined below. Therefore when $\sigma = \sigma_J$, we choose a standard Schwartz function $\phi_\infty : D_{\sigma,\mathbb{A}} \to \mathbb{C}$ of weight $k = n + 2$ as follows

$$\phi_\infty(v) = \Psi_k(v, \mathbf{x}; z, \tau) = \Psi_{k,\sigma_J}(v, \mathbf{x}; z, \tau) = \eta^{1/2}[g_z^{-1}vg_z^{\sigma_J}; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(v)\overline{\tau} \pm \frac{[z; v]^2\eta\sqrt{-1}}{2y(z)^2}) \tag{6.18}$$

as in (6.6).

We show a formula similar to (6.18) when $\sigma = \sigma_1$ choosing $p_\pm(z)$ differently. We are in Case II, $D_\mathbb{R} \cong M_2(\mathbb{R})$, and

$$D^\pm_{\sigma_1} = D^\pm_{\sigma_1,\mathbb{R}} = \left\{\begin{pmatrix} x & \sqrt{\mp 1}b \\ \sqrt{\mp 1}c & \pm\overline{x}\end{pmatrix} \middle| x \in \mathbb{C} \text{ and } b, c \in \mathbb{R}\right\}. \tag{6.19}$$

This $D^\pm_{\sigma_1}$ has signature $(3, 1)$ and the one dimensional negative definite space is generated by $\sqrt{\mp 1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$. Let $p_-(z) = yg_z^\iota\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)g_z^{\sigma_1} = \left(\begin{smallmatrix} -xy & 1-x\overline{x} \\ y^2 & \overline{x}y\end{smallmatrix}\right) \in D^-_{\sigma_1}$ for $z = \left(\begin{smallmatrix} x & -y \\ y & \overline{x}\end{smallmatrix}\right) \in \mathcal{H}$ and $p_+(z) = \sqrt{-1}p_-(z) \in D^+_{\sigma_1}$. Again (6.14) is valid for $\sigma = \sigma_1$ for this choice of $p_\pm(z)$, and similar to the case of $D_\mathbb{R} = \mathbb{H}$, defining $P_z(v, v') = -aa's_\pm[p_\pm(z)] + s_\pm(w, w')$ for $v = ap_\pm(z) + w$ and $v' = a'p_\pm(z) + w'$ with $s_\pm(\mathbb{R}p_\pm(z), \mathbb{R}w + \mathbb{R}w') = 0$ and $a, a' \in \mathbb{R}$ and we find

$$P_z[v] - s_\pm[v] = -2\frac{s_\pm(p_\pm(z), v)^2}{s_\pm[p_\pm(z)]},$$

and writing $[z; v] = \mathrm{Tr}_{D_\mathbb{C}/\mathbb{C}}(p_+(z)v^\iota)$,

$$\frac{P_z[v]}{2} = \frac{1}{2}\left\{s_\pm[v] \mp \frac{[z; v]^2}{y(z)^2}\right\} = \pm N(v) \mp \frac{[z; v]^2}{2y(z)^2}. \tag{6.20}$$

Thus we choose the canonical Schwartz function of weight $k = n+2$ as follows:

$$\phi_\infty(v) = \Psi_k(v, \mathbf{x}; z, \tau) = \Psi_{k,\sigma_J}(v, \mathbf{x}; z, \tau) = \eta^{1/2}[g_z^{-1} v g_z^{\sigma_1}; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(v)\tau \mp \frac{[z; v]^2\eta\sqrt{-1}}{2y(z)^2}). \tag{6.21}$$

Though the formula is similar in the two cases, the definition of $[z; v]$ is different in Case ID and Case II. In the two cases $\sigma = \sigma_J$ and σ_1, we define a theta series by

$$\theta_{k,\sigma}(\phi) = \theta_k(\phi) = \theta_k(\phi)(\tau; z, \mathbf{x}) = \sum_{v \in D_\sigma} \phi(v) \quad \text{for } \sigma = \sigma_1 \text{ and } \sigma_J. \tag{6.22}$$

6.6. **Invariant pairings and differential operators.** To define the theta descent, for any E-algebra such that $D_{E_A} \cong M_2(A) \times M_2(A)$, we need the $D^\times$-invariant projection $L_E(n; A) \to A$, the D_E^1-invariant self duality of $L_E(n; A)$ and the $\mathrm{SU}_2(\mathbb{R})$-invariant self duality of $L(n^*; \mathbb{C})$ induced by the invariant differential operators. The first projection we describe is the $D^\times$-invariant operator for $L_E(n; A)|_{D^\times}$ which depends on Cases II and ID. Since $[\mathbf{x}]_D \mapsto J[\mathbf{x}]_D = [\mathbf{x}]_I$ gives an isomorphism of quadratic spaces: $D^{\mp}_{\sigma_J,\mathbb{C}} \cong D^{\pm}_{\sigma_1,\mathbb{C}}$, pulling back the invariant differential operator $\triangledown = \frac{\partial^2}{\partial X \partial Y'} - \frac{\partial^2}{\partial Y \partial X'}$ in Case II by this isomorphism, the corresponding invariant differential in Case ID is $\triangledown := J^*(\frac{\partial^2}{\partial X \partial Y'} - \frac{\partial^2}{\partial Y \partial X'}) = \frac{\partial^2}{\partial Y \partial Y'} + \frac{\partial^2}{\partial X \partial X'}$. We record this fact:

$$\triangledown := \begin{cases} \frac{\partial^2}{\partial X \partial Y'} - \frac{\partial^2}{\partial Y \partial X'} & \text{in Case II,} \\ \frac{\partial^2}{\partial X \partial X'} + \frac{\partial^2}{\partial Y \partial Y'} & \text{in Case ID.} \end{cases} \tag{6.23}$$

Thus we have a morphism of sheaves

$$(n!)^{-2}\triangledown^n : L_E^*(n; A)|_{Sh_\delta} \to A|_{Sh_\delta}. \tag{6.24}$$

A canonical $D^1_{E_A}$-invariant pairing is induced by

$$(\cdot,\cdot) = (\cdot,\cdot)_n := (n!)^{-2}\triangledown^n_{\mathrm{id}} \otimes (n!)^{-2}\triangledown^n_\sigma : L_E(n; A) \otimes L_E(n; A) \to A. \tag{6.25}$$

In this case, the left and right factor $\mathrm{GL}_2(A)$ acts by the corresponding embedding (so, no conjugation action of σ involved), and therefore, we do not need to separate two cases II and ID. Writing the variable of the identity (resp. σ) factor of $L_E(n; A)$ as $(X, Y; X', Y')$ (resp. $(S, T; S', T')$), the operators are

$$\begin{aligned} \triangledown_{\mathrm{id}} &:= \frac{\partial^2}{\partial X \partial T} - \frac{\partial^2}{\partial Y \partial S} \\ \triangledown_\sigma &:= \frac{\partial^2}{\partial X' \partial T'} - \frac{\partial^2}{\partial Y' \partial S'}. \end{aligned} \tag{6.26}$$

We later need to have $(\cdot,\cdot)_n$ written in two different ways:

Lemma 6.6. *The following diagram*

$$\begin{array}{ccc} L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(n;\mathbb{C}) & = & L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(n;\mathbb{C}) \\ {\scriptstyle n!^{-2}\triangledown^n \otimes n!^{-2}\triangledown^n}\downarrow & & \downarrow{\scriptstyle n!^{-2}\triangledown^n_{\mathrm{id}} \otimes n!^{-2}\triangledown^n_\sigma} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

is commutative.

Proof. Regarding $L_E(n;\mathbb{C})$ as a $\mathrm{SL}_2(\mathbb{R})$-module, $L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(n;\mathbb{C})$ is a $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$-module by the left and right factor. The left vertical map is a non-zero morphism of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$-modules, and the right hand side is a non-zero morphism of $\mathrm{SL}_2(E_{\mathbb{C}})$-modules, where we identify $\mathrm{SL}_2(E_{\mathbb{C}}) = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. As $\mathrm{SL}_2(\mathbb{R})$-modules, we have $L_E(n;\mathbb{C}) \cong \bigoplus_{k=0}^{2n} L(2k;\mathbb{C})$ and as $\mathrm{SL}_2(E_{\mathbb{C}})$-modules, $L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(n;\mathbb{C}) \cong \bigoplus_{k=0}^{2n}\bigoplus_{l=0}^{2n} L_E(2k\,\mathrm{id} + 2l\sigma;\mathbb{C})$ by Clebsch–Gordan. They have a unique constant quotient $\mathbb{C}$. The canonical projection of $L_E(n;\mathbb{C})|_{\mathrm{SL}_2(\mathbb{R})}$ to $\mathbb{C}$ is given by $n!^{-2}\triangledown^n$. Thus we have $n!^{-2}\triangledown^n \otimes n!^{-2}\triangledown^n = c_n n!^{-2}\triangledown^n_{\mathrm{id}} \otimes n!^{-2}\triangledown^n_\sigma$ for a non-zero constant c_n. Since $n!^{-2}\triangledown^n_{\mathrm{id}} \otimes n!^{-2}\triangledown^n_\sigma$ and $n!^{-2}\triangledown^n \otimes n!^{-2}\triangledown^n$ have equal value 1 at $X^nY^nX'^nY'^nS^nT^nS'^nT'^n$ by the formula (4.9), we find $c_n = 1$. $\square$

We have the second invariant pairing on the $\mathrm{SU}_2(\mathbb{R})$-module $L(n^*;A)$ given by

$$\langle\cdot,\cdot\rangle := (n^*!)^{-2}\nabla'^{n^*} : L(n^*;\mathbb{C}) \otimes L(n^*;\mathbb{C}) \to \mathbb{C}, \tag{6.27}$$

where, writing the variable of the left (resp. right) factor $L(n^*;\mathbb{C})$ as (S,T) (resp. (S',T')),

$$\nabla' := \frac{\partial^2}{\partial S\partial T'} - \frac{\partial^2}{\partial T\partial S'}. \tag{6.28}$$

6.7. **Invariance of** $[\alpha;\mathbf{x}]$**.** For $h \in G^+_{D_\alpha}(\mathbb{R})$, we have $h^{-1}\alpha h^\sigma = \alpha$ and $[\alpha;\mathbf{x}] = [h^{-1}\alpha h^\sigma;\mathbf{x}] = [\alpha;\mathbf{x}h^?]$, where $h^? = h^\iota$ in Case II and $h^? = {}^th$ in Case ID (see (6.2)).

Lemma 6.7. *The polynomial $[\alpha;\mathbf{x}]^{n+1}$ is an element of $H^0(D^1_{\alpha;\mathbb{R}}, L_E(n+1;\mathbb{C}))$, and*

$$[\alpha;\mathbf{x}]^{n+1} \in \begin{cases} \mathbb{C}\left[(X,Y)j^{-1}{}^t\overline{j}^{-1}\begin{pmatrix}X'\\Y'\end{pmatrix}\right]^{n+1} & \text{if } D_{\alpha,\mathbb{R}} = j^{-1}\mathbb{H}j \Leftrightarrow \alpha \doteqdot j^{-1}xj^\sigma,\\ \mathbb{C}\left[(X,Y)j^{-1}J{}^t\overline{j}^{-1}\begin{pmatrix}X'\\Y'\end{pmatrix}\right]^{n+1} & \text{if } D_{\alpha,\mathbb{R}} = jM_2(\mathbb{R})j^{-1} \Leftrightarrow \alpha \doteqdot j^{-1}xj^\sigma\end{cases}$$

for some $j \in \mathrm{SL}_2(\mathbb{C})$, where "$\doteqdot$" means an identity up to a power of $\sqrt{-1}$ and $x = 1$ if $D_\mathbb{R} \cong D_{\alpha,\mathbb{R}}$ and $x = J$ if $D_\mathbb{R} \not\cong D_{\alpha,\mathbb{R}}$. Here $h \in \mathrm{SL}_2(\mathbb{C})$ acts on $L_E(n;\mathbb{C})$ by the pullback action of $(X,Y;X',Y') \mapsto ((X,Y)h;(X',Y')\overline{h})$.

Proof. The identity $H^0(\mathbb{H}^1, L_E(n+1;\mathbb{C})) = H^0(\mathrm{SU}_2(\mathbb{R}), L_E(n+1;\mathbb{C})) = \mathbb{C}(XX'+YY')^{n+1}$ shows the assertion when $D_\mathbb{R} = j^{-1}\mathbb{H}j$. If $D_{\alpha,\mathbb{R}} = jM_2(\mathbb{R})j^{-1}$, this follows again from

$$H^0(\mathrm{SL}_2(\mathbb{R}), L_E(n+1;\mathbb{C})) = \mathbb{C}(XY'-X'Y)^{n+1}.$$

This finishes the proof. □

6.8. **Relation between** $\mathrm{SU}_2(\mathbb{R})$**-polynomial representations.** The automorphic form $f \in M_k(\Gamma)$ gives a global section of $L^*(n^*;\mathbb{C})$. To relate the theta series with values in $L_E(n+1;\mathbb{C})$ and modular forms with values in $L(n^*;\mathbb{C})$, we study the relation of the two modules under the action of $\mathrm{SU}_2(\mathbb{R})$.

Write $\pi : L_E(n+1;\mathbb{C}) \to L(n^*;\mathbb{C})$ for the SU_2-equivariant projection as in [H94, (11.2)].

Lemma 6.8. *If we let $\mathrm{GL}_2(\mathbb{C})$ act on $L_E(n+1;A)$ for a $E_\mathbb{C}$-algebra A by the pull-back action of $(X,Y;X',Y') \mapsto ((X,Y)g;(X',Y')\overline{g})$, the map $\pi : L_E(n+1;A) \to L(n^*;A)$ given by*

$$\pi(\Phi(X,Y;X',Y')) = \begin{cases}\Phi((S,T);(S,T)J) = \Phi(S,T;-T,S) & \textit{in Case II},\\ \Phi(-(S,T)J;(S,T)) = \Phi(T,-S;S,T) & \textit{in Case ID}\end{cases} \tag{6.29}$$

is $\mathrm{SU}_2(\mathbb{R})$-equivariant.

The projection $\pi : L_E(n+1;\mathbb{C}) \twoheadrightarrow L(n^*;\mathbb{C})$ is given by the following variable change:

$$\mathbf{x} = (X,Y;X',Y') \mapsto \widetilde{\mathbf{x}} := \begin{cases}(T,-S;S,T) & \text{in CaseID}\\ (S,T;-T,S) & \text{in CaseII}\end{cases} \tag{6.30}$$

of $\mathrm{SU}_2(\mathbb{R})$-modules unique up to scalar multiplication. Write $\pi(\Phi(\mathbf{x})) = \Phi(\widetilde{\mathbf{x}})$; so,

$$[v;\widetilde{\mathbf{x}}] := \pi([v,\mathbf{x}]). \tag{6.31}$$

Proof. We first deal with Case ID. Since $\overline{u} = J^{-1}uJ$ for $u \in \mathbb{H}^1 = \mathrm{Ker}(N : \mathbb{H}^\times \to \mathbb{R}^\times)$, in Case ID, $[\mathbf{x}]_D = {}^t(X,Y)(X',Y')$ and $[\widetilde{\mathbf{x}}]_D = J^t(S,T)(S,T)$ and

$$\begin{aligned} u^{-1}[\widetilde{\mathbf{x}}]_D u^{\sigma_J} = u^{-1}[\widetilde{\mathbf{x}}]_D u &= u^{-1}J^t(S,T)(S,T)u = JJ^{-1}u^{-1}J^t(S,T)(S,T)u\\ &= J^tu^t(S,T)(S,T)u = J^t((S,T)u)(S,T)u = u\cdot(\pi[\mathbf{x}]_D),\end{aligned}$$

since ${}^tu = Ju^{-1}J^{-1}$. This is compatible as $D^\pm_{\sigma_J} \cong D^\mp_{\sigma_1}$ by $[\mathbf{x}]_D \mapsto J[\mathbf{x}]_D$. Similar to this

$$h^{-1}[\widetilde{\mathbf{x}}]_D h = h\cdot(\pi[\mathbf{x}]_D) \quad \text{for } h \in \mathrm{SL}_2(\mathbb{R}).$$

The case II can be treated similarly. □

We now compute the adjoint $\pi^*: L(n^*;\mathbb{C}) \hookrightarrow L_E(n+1;\mathbb{C})$ of $\pi: L_E(n+1;\mathbb{C}) \twoheadrightarrow L(n^*;\mathbb{C})$ to show integrality of π^* (though we do not use the formula explicitly in this paper). We have a bilinear pairing $(\cdot,\cdot)_{n+1}: L_E(n+1;\mathbb{C})\otimes_{\mathbb{C}} L_E(n+1;\mathbb{C}) \to \mathbb{C}$ given by (6.25) and $\langle\cdot,\cdot\rangle : L(n^*;\mathbb{C})\otimes_{\mathbb{C}} L(n^*;\mathbb{C}) \to \mathbb{C}$ given in (6.27). Then $(P,\pi^*Q) = \langle \pi P, Q\rangle$ for the adjoint π^* of π, and we write

$$\pi^* S^{n+1-i+i'} T^{n+1+i-i'} = \sum_{j,j'} c_{i,i';j,j'} X^j Y^{n+1-j} X'^{j'} Y'^{n+1-j'}.$$

Since highest (resp. lowest) weight of $sym^{\otimes n^*}$ of $\mathrm{SU}_2(\mathbb{R})$ is $a \mapsto a^{n^*}$ (resp. $a \mapsto a^{-n^*}$), we have $\pi^*(S^{n^*}) = cX^{n+1}Y'^{n+1}$ and $\pi^*(T^{n^*}) = c'Y^{n+1}X'^{n+1}$ for non-zero constants c, c'. Thus

$$c_{0,n+1;j,j'} = 0 \ \text{ if } (j,j') \neq (n+1,0), \text{ and similarly, } c_{n+1,0;j,j'} = 0 \ \text{ if } (j,j') \neq (0,n+1). \tag{6.32}$$

Since $\pi\pi^*$ commutes with $\mathrm{SU}_2(\mathbb{R})$-action and $L(n^*;\mathbb{C})$ is irreducible $\mathrm{SU}_2(\mathbb{C})$-module, $\pi\pi^*$ is a scalar multiplication. We have by [H99, page 141]

$$(X^{n+1-i}Y^iX'^{n+1-i'}Y'^{i'}, X^jY^{n+1-j}X'^{j'}Y'^{n+1-j'}) = \delta_{i,j}\delta_{i',j'}(-1)^{i+i'}\left[\binom{n+1}{i}\binom{n+1}{i'}\right]^{-1}.$$

Similar to this computation, $\langle S^{n^*-i}T^i, S^jT^{n^*-j}\rangle = \delta_{i,j}(-1)^i\binom{n^*}{i}^{-1}$. Since

$$\pi(X^{n+1-i}Y^iX'^{n+1-i'}Y'^{i'}) = (-1)^{n+1-i'}S^{n+1-i+i'}T^{n+1+i-i'},$$

by $(\pi^*\pi P, Q) = \langle \pi P, \pi Q\rangle$,

$$\begin{aligned}
&(\pi^*S^{n+1-i+i'}T^{n+1+i-i'}, X^jY^{n+1-j}X'^{j'}Y'^{n+1-j'}) \\
&\qquad = (-1)^{n+1-i'}(\pi^*\pi X^{n+1-i}Y^iX'^{n+1-i'}Y'^{i'}, X^jY^{n+1-j}X'^{j'}Y'^{n+1-j'}) \\
&\quad = (-1)^{j'}\langle S^{n+1-i+i'}T^{n+1+i-i'}, S^{n+1+j-j'}T^{n+1-j+j'}\rangle = (-1)^{j'}\delta_{i-i',j-j'}\binom{n^*}{n+1+i-i'}^{-1}.
\end{aligned}$$

This shows

$$\begin{aligned}
(-1)^{k'}\delta_{i-i',k-k'}\binom{n^*}{n+1+i-i'}^{-1} &= (\pi^*S^{n+1-i+i'}T^{n+1+i-i'}, X^kY^{n+1-k}X'^{k'}Y'^{n+1-k'}) \\
&= \sum_{j,j'} c_{i,i';j,j'}(X^jY^{n+1-j}X'^{j'}Y'^{n+1-j'}, X^kY^{n+1-k}X'^{k'}Y'^{n+1-k'}) \\
&= \sum_{j,j'} c_{i,i';j,j'}\delta_{j,k}\delta_{j',k'}(-1)^{j+j'}\left[\binom{n+1}{j}\binom{n+1}{j'}\right]^{-1} \\
&= \sum_{j,j'} c_{i,i';j,j'}(-1)^{j+j'}\left[\binom{n+1}{j}\binom{n+1}{j'}\right]^{-1}.
\end{aligned} \tag{6.33}$$

Take $a \in \mathbb{C}^\times$ with $a\overline{a} = 1$ and put $u_a = \mathrm{diag}[a,\overline{a}] \in \mathrm{SU}_2(\mathbb{R})$. Taking $k = 0$ and $k' = n+1$ in (6.33), we find, if $ii' = 0$

$$(-1)^{n+1} = \sum_{j,j'} c_{i,i';j,j'}(-1)^{j+j'}\left[\binom{n+1}{j}\binom{n+1}{j'}\right]^{-1},$$

and by (6.32), $c_{0,n+1;n+1,0} = 1$ and $c_{n+1,0;0,n+1} = 1$. Therefore

$$\pi^*(S^{n^*}) = X^{n+1}Y'^{n+1} \quad\text{and}\quad \pi^*(T^{n^*}) = Y^{n+1}X'^{n+1}. \tag{6.34}$$

Take $h = \left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbb{R})\mathbb{R}^\times$. Then

$$\pi^*(hP(S,T)) = \pi^*(P((S,T)^th^\iota)) = \pi^*(P)((X,Y)^th^\iota; (X',Y')^t\overline{h}^\iota),$$

and thus

$$\pi^*((aS-bT)^{n^*}) = \pi^*(hS^{n^*}) = hX^{n+1}Y'^{n+1} = (aX-bY)^{n+1}(bX'+aY')^{n+1}.$$

Therefore

$$\sum_{i=0}^{n^*}(-1)^i\binom{n^*}{i}a^{n^*-i}b^i\pi^*(S^{n^*-i}T^i)$$
$$=\sum_{j=0}^{n+1}\sum_{j'=0}^{n+1}(-1)^j\binom{n+1}{j}\binom{n+1}{j'}a^{n^*-j-j'}b^{j+j'}X^{n+1-j}Y^jX'^{j'}Y'^{n+1-j'}.$$

From this we conclude

$$(-1)^i\binom{n^*}{i}\pi^*(S^{n^*-i}T^i)=\sum_{j+j'=i}(-1)^j\binom{n+1}{j}\binom{n+1}{j'}X^{n+1-j}Y^jX'^{j'}Y'^{n+1-j'} \tag{6.35}$$

We record this fact as

Lemma 6.9. *We have the following explicit form of* $\pi^*: L(n^*;\mathbb{C})\hookrightarrow L_E(n+1;\mathbb{C})$*:*

$$\pi^*(S^{n^*-i}T^i)=\sum_{j+j'=i}(-1)^{j'}\binom{i}{j}\binom{n^*-i}{n+1-j}\binom{n^*}{n+1}^{-1}X^{n+1-j}Y^jX'^{j'}Y'^{n+1-j'}. \tag{6.36}$$

Note that $\{\binom{n^*}{i}S^{n^*-i}T^i\}_i$ is an A-basis of the dual lattice $L(n^*;A)^\vee$ of $L(n;A)$ under $\langle\cdot,\cdot\rangle$ for any integral domain A of characteristic 0 and the same for $\{\binom{n+1}{j}\binom{n+1}{j'}X^{n+1-j}Y^jX'^{j'}Y'^{n+1-j'}\}_{j,j'}$ for $L_E(n+1;A)^\vee$ under $(\cdot,\cdot)$. Thus by (6.35), π^* is integral with respect to the dual integral structure of $L_E(n+1;A)$. Write

$$\mathbf{s}^*:=(\pi^*(S^{n^*-i}T^i))_i \tag{6.37}$$

in the sense that $\pi^*P(\mathbf{s})=P(\mathbf{s}^*)\in L_E(n+1;\mathbb{C})$ for $P(\mathbf{s})\in L(n^*;\mathbb{C})$.

6.9. **Vanishing of** $([\alpha;\mathbf{x}]^n,f)$ **if** $D_{\alpha,\mathbb{R}}\cong\mathbb{H}$**.** Write $\mathbf{s}=(S,T)$ for the variables of $L(n^*;\mathbb{C})$. Identifying $D^\times_{E_\mathbb{R}}=\mathrm{GL}_2(\mathbb{C})$, a quaternionic modular form $f(h;\mathbf{s}): D^\times\backslash D^\times_{E_\mathbb{A}}\to L(n^*;\mathbb{C})$ ($k=n+2$ and $n^*=2n+2$) on $\widehat{\Gamma}$ of weight $k\infty+k\infty\sigma$ satisfies

$$f(\gamma zxu;\mathbf{s})=f(x;\mathbf{s}\,{}^tu^\sigma)\quad\text{for}\quad u\in\widehat{\Gamma}\cdot\mathrm{SU}_2(\mathbb{R}), z\in E^\times_\mathbb{A}\text{ and }\gamma\in D^\times_E, \tag{6.38}$$

where ${}^tu^\sigma_\infty=u_\infty^{-1}$ if $\sigma=\sigma_1$ and ${}^tu_\infty$ if $\sigma=\sigma_J$. Here $\mathrm{SU}_2(\mathbb{R})=\{u\in\mathbb{H}^\times|N(u)=1\}$ is the stabilizer in $\mathrm{SL}_2(\mathbb{C})$ of $\varepsilon=-J\in\mathcal{H}$ and $u\in\mathrm{GL}_2(A)$ acts on $P(\mathbf{s})\in L(n^*;A)$ by $P|u(\mathbf{s})=P(\mathbf{s}u^\iota)$. Define for f as above and $j\in\mathrm{SL}_2(\mathbb{C})$,

$$f|_\mathcal{H}j(z;\mathbf{s}^*):=f(j(z);\mathbf{s}^*j^?)\quad\text{and}\quad f|_Gj(h;\mathbf{s}^*):=f(jh;\mathbf{s}^*), \tag{6.39}$$

where $j^?=j^\iota$ in Case II and $j^?={}^tj$ in Case ID. Then writing $jg_z=g_{j(z)}u$ for $u\in\mathrm{SU}_2(\mathbb{R})$

$$f|_Gj(g_z;\mathbf{s}^*g_z^{-?})=f(jg_z;\mathbf{s}^*g_z^{-?})=f(g_{j(z)};\mathbf{s}^*(ug_z^{-1})^?)=f(g_{j(z)};\mathbf{s}^*(g_{j(z)}^{-1}j)^?)=f|_\mathcal{H}j(z;\mathbf{s}^*).$$

Lemma 6.10. *Let* $j\in\mathrm{SL}_2(\mathbb{C})$*. Assume* $f(j^{-1}g_0;\mathbf{s}^*)\neq 0$ *for some* $g_0\in D^1_{E_\mathbb{A}}$*. Then the subspace of* $L_E(n+1;\mathbb{C})$ *spanned by* $\{f|_Gj^{-1}(g_0;\mathbf{s}^*u_\infty^{-1})\}_{u_\infty\in\mathrm{SU}_2(\mathbb{R})}$ *over* $\mathbb{C}$ *in* $L_E(n+1;\mathbb{C})$ *is equal to* $\pi^*(L(n^*;\mathbb{C}))$ *for* $\pi^*:L(n^*;\mathbb{C})\hookrightarrow L_E(n+1;\mathbb{C})$ *as in §6.8. In particular,*

$$([1,\mathbf{x}]^{n+1},f|_Gj^{-1}(g_z;\mathbf{s}^*))=0\quad\textit{if } D_\mathbb{R}\cong\mathbb{H}\textit{ and }\quad([J,\mathbf{x}]^{n+1},f|_Gj^{-1}(g_z;\mathbf{s}^*))=0\quad\textit{if } D_\mathbb{R}\cong M_2(\mathbb{R}).$$

Proof. By a variable change $j^{-1}g\mapsto g$, we may assume that $j=1$. Write V for the subspace of $L_E(n+1;\mathbb{C})$ spanned by $\{f(g_0;\mathbf{s}^*u_\infty^{-1})\}_{u_\infty\in\mathrm{SU}_2(\mathbb{R})}$. As an $\mathrm{SU}_2(\mathbb{R})$-module, $\pi^*(L(n^*;\mathbb{C}))$ is irreducible. Since $\{f(g_0;\mathbf{s}^*u_\infty^{-1})\}_{u_\infty\in\mathrm{SU}_2(\mathbb{R})}\subset\pi^*(L(n^*;\mathbb{C}))$ as $f(g_0;\mathbf{s}^*)=\pi^*(f(g_0;\mathbf{s}))$, by $f(g;\mathbf{s}^*)\neq 0$, $0\neq V\subset\pi^*(L(n^*;\mathbb{C}))$ is a non-trivial subspace stable under the action of $\mathrm{SU}_2(\mathbb{R})$. The irreducibility of $\pi^*(L(n^*;\mathbb{C}))$ as an $\mathrm{SU}_2(\mathbb{R})$-module tells us that $V=\pi^*(L(n^*;\mathbb{C}))$ as desired.

Suppose $D_\mathbb{R}=\mathbb{H}$. Since $[1;\mathbf{x}^th^{\sigma_J}]=[hh^{-\sigma_J};\mathbf{x}]$ for $h\in\mathrm{SU}_2(\mathbb{R})=D^1_\mathbb{R}$ if $D_\mathbb{R}=\mathbb{H}$, $[1,\mathbf{x}]$ is invariant under $\mathrm{SU}_2(\mathbb{R})$ and is orthogonal to $\pi^*(L(n^*;\mathbb{C}))$ under $(\cdot,\cdot)$. This shows the desired vanishing.

Suppose $D_\mathbb{R}=M_2(\mathbb{R})$. Since $[J;\mathbf{x}h^{\sigma_\iota}]=[hJh^{-\sigma_1};\mathbf{x}]$ for $h\in\mathrm{SU}_2(\mathbb{R})$, $[J,\mathbf{x}]$ is invariant under $\mathrm{SU}_2(\mathbb{R})$ and is again orthogonal to $\pi^*(L(n^*;\mathbb{C}))$ under $(\cdot,\cdot)$. Thus the vanishing follows. ☐

We will see later in the proof of Proposition 7.2 that this lemma implies the vanishing of $([\alpha;\mathbf{x}]^{n+1},f)$ if $D_{\alpha,\mathbb{R}}\cong\mathbb{H}$.

6.10. **Theta differential form.** We now interpret $\theta(\phi)$ as a differential form. Let $\Omega^{\nu}_{\mathcal{S}/\mathbb{C}}$ be the sheaf of analytic differential ν-forms on $\mathcal{S} = \Gamma_{\phi}\backslash\mathcal{H}$. Since $\mathcal{S}$ does not have complex structure, we use the symbol $\mathcal{S}$ instead of Sh_E. Note $\Omega^{\nu}_{\mathcal{S}/\mathbb{C}} \cong L^*_E(1;\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{S}}$ (for the sheaf $\mathcal{O}_{\mathcal{S}}$ of analytic functions on $\mathcal{S}$) as we will see. We first prove

Lemma 6.11. *We have a canonical inclusion $\iota : L_E(n+1;\mathbb{C}) \hookrightarrow L_E(n;\mathbb{C}) \otimes L_E(1;\mathbb{C})$ of $\mathrm{GL}_2(E)$-modules given by $\iota([v;\mathbf{x}]^{n+1}) = [v;\mathbf{x}]^n \otimes [v;\mathbf{s}]$ for all $v \in D_\sigma$.*

Proof. We prove that the formula in the lemma gives the linear injection. Write $\mathbf{x} = (X, Y; X', Y')$ (resp. $\mathbf{s} = (S, T; S', T')$) for the variables of $L_E(n;\mathbb{C})$ (resp. $L_E(1;\mathbb{C})$). We write $[v;\mathbf{s}]$ for $[v;\mathbf{x}]|_{\mathbf{x}=\mathbf{s}}$. Then the map $\iota^\circ : [v;\mathbf{x}]^{n+1} \mapsto [v;\mathbf{x}]^n \otimes [v;\mathbf{s}]$ satisfies $\iota^\circ(\mathfrak{z}^{n+1}[v;\mathbf{x}]^{n+1}) = \iota^\circ([\mathfrak{z}v;\mathbf{x}]^{n+1}) = [\mathfrak{z}v;\mathbf{x}]^n \otimes [\mathfrak{z}v;\mathbf{s}] = \mathfrak{z}^{n+1}\iota^\circ([v;\mathbf{x}]^{n+1})$ for scalar $\mathfrak{z}$. Since ι° is a polynomial map in v, by Zariski density of $\{x^{n+1}|x \in \mathbb{G}_m(\mathbb{C})\}$ inside $\mathbb{G}_a(\mathbb{C})$, we find that ι° extends to a linear map $\iota : L_E(n+1;\mathbb{C}) \to L_E(n;\mathbb{C}) \otimes L_E(1;\mathbb{C})$. Since ι plainly sends $[g^{-1}vg^\sigma;\mathbf{x}]$ to $[g^{-1}vg^\sigma;\mathbf{x}]^n \otimes [g^{-1}vg^\sigma;\mathbf{s}]$, by (6.2), ι is $\mathrm{SL}_2(E)$-equivariant. By definition

$$\iota([v;\mathbf{x}]^{n+1}) = [v;\mathbf{x}]^n \otimes [v;\mathbf{s}] \quad \text{for all } v \in D_\sigma.$$

By irreducibility of $L_E(n+1;\mathbb{C})$, ι is an injection. □

This lemma is useful when we compute $n!^{-2}\nabla^n i^*_\nu \iota(\theta(\phi))$. Indeed, we only need to compute $n!^{-2}\nabla^n([v;\mathbf{x}]^n)$ not the derivative of $[v;\mathbf{x}]^{n+1}$ which is a coefficients of $\theta(\phi)$.

Proposition 6.12. *We have a canonical monomorphism $i_\nu : L^*_E(n+1;\mathbb{C}) \hookrightarrow L^*_E(n;\mathbb{C}) \otimes_{\mathbb{C}} \Omega^{\nu}_{\mathcal{S}/\mathbb{C}}$ for $\nu = 1, 2$ as sheaves over $\mathcal{S}$.*

Proof. By Lemma 6.11, we have an inclusion $\iota : L_E(n+1;\mathbb{C}) \hookrightarrow L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(1;\mathbb{C})$ of $\mathrm{SL}_2(E)$-modules. Thus we need to embed $L^*_E(1;\mathbb{C})$ into $\Omega^{\nu}_{\mathcal{S}/\mathbb{C}}$. Though $L_E(1;\mathbb{C})$ is an irreducible $\mathrm{SL}_2(\mathbb{C})$-module, it is reducible over $\mathrm{SU}_2(\mathbb{R})$. Thus as $\mathrm{SU}_2(\mathbb{R})$-module, we have $L_E(1;\mathbb{C}) \cong \mathbb{C} \oplus L(2;\mathbb{C})$. Since $(X,Y)u^t((X',Y')u^\sigma) = (X,Y)u^t u^{\sigma t}(X',Y')$, on the subspace $\mathbb{C}(XX'+YY') \subset L_E(1,\mathbb{C})$, $\mathrm{SU}_2(\mathbb{R})$ acts trivially. Identifying $L_E(1;\mathbb{C})/(\mathbb{C}(XX'+YY')) \cong \mathbb{C}(-XY') + \mathbb{C}(XX'-YY') + \mathbb{C}X'Y$, since $(X,Y)\left(\begin{smallmatrix} a & b \\ -b^\sigma & a^\sigma \end{smallmatrix}\right) = (aX - b^\sigma Y, bX + a^\sigma Y)$ and $(X',Y')\left(\begin{smallmatrix} a^\sigma & b^\sigma \\ -b & a \end{smallmatrix}\right) = (a^\sigma X' - bY', b^\sigma X' + aY')$, the matrix representation of

$$(-XY', XX'-YY', YX') \mapsto (-XY', XX'-YY', X'Y) \cdot u$$

for $u = \left(\begin{smallmatrix} a & b \\ -b^\sigma & a^\sigma \end{smallmatrix}\right) \in \mathrm{SU}_2(\mathbb{R})$ is given modulo $\mathbb{C}(XX'+YY')$ by

$$\begin{aligned}(-XY', XX'-YY', X'Y) &\cdot \left(\begin{smallmatrix} a & b \\ -b^\sigma & a^\sigma \end{smallmatrix}\right) \\ &= (-XY', XX'-YY', X'Y)\begin{pmatrix} a^2 & 2ab & b^2 \\ -ab^\sigma & aa^\sigma - bb^\sigma & a^\sigma b \\ b^{2\sigma} & -2a^\sigma b^\sigma & a^{2\sigma} \end{pmatrix} = (-XY', XX'-YY', X'Y)\rho_2(u).\end{aligned}$$

On the other hand, Let $\varepsilon = -J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{H}$. Recall the isomorphisms $\mathrm{SO}_{D_\sigma}(\mathbb{R}) = G^+_{D_\sigma}(\mathbb{C})/\mathbb{R}^\times$ and $\mathrm{SO}_{D_\sigma}(\mathbb{R})/\mathrm{SO}_P(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R}) \cong \mathcal{H} = \left\{z = \left(\begin{smallmatrix} x & -y \\ y & \overline{x} \end{smallmatrix}\right) \middle| x \in \mathbb{C}, 0 < y \in \mathbb{R}\right\}$ $(g \mapsto g(\varepsilon))$ and the standard automorphic factor given by $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), z) = \rho(c)z + \rho(d)$. We let $G^+_{D_\sigma}(\mathbb{R}) \subset \mathrm{GL}^+_2(\mathbb{C})$ act on $\mathcal{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = (\rho(a)z + \rho(b))(\rho(c)z + \rho(d))^{-1}$, and identifying $\mathrm{SO}_{D_\sigma}(\mathbb{R})$ with $G^+_{D_\sigma}(\mathbb{R})/\mathbb{R}^\times$, the orthogonal group acts transitively on $\mathcal{H}$. The stabilizer of ε is $\mathrm{SO}_P(\mathbb{R})$ (congruent to $\mathrm{SU}_2(\mathbb{R})$ modulo center). For $u = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R})$,

$${}^t j(u,\varepsilon) = {}^t(\begin{pmatrix} -\overline{b} & 0 \\ 0 & -b \end{pmatrix}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \overline{a} & 0 \\ 0 & a \end{smallmatrix}\right)) = {}^t\begin{pmatrix} \overline{a} & \overline{b} \\ -b & a \end{pmatrix} = u^{-1} \in \mathrm{SU}_2(\mathbb{R}) \Leftrightarrow j(u,\varepsilon) = \overline{u}.$$

As in [H94, (2.4), (2.9b)], writing

$$\omega_1 := (dx, -dy, -d\overline{x}) \quad \text{and} \quad \omega_2 := y^{-1}(dy \wedge dx, -2dx \wedge d\overline{x}, dy \wedge d\overline{x}) \tag{6.40}$$

as the vector valued differential forms, we have ${}^t(u^*\omega_\nu)|_\varepsilon = \rho_2({}^t j(u,\varepsilon))^t\omega_\nu|_\varepsilon$ for $\nu = 1, 2$, and

$$\rho_2\left(\begin{smallmatrix} a & b \\ -b^\sigma & a^\sigma \end{smallmatrix}\right)\begin{bmatrix} S^2 \\ ST \\ T^2 \end{bmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ -ab^\sigma & aa^\sigma - bb^\sigma & a^\sigma b \\ b^{2\sigma} & -2a^\sigma b^\sigma & a^{2\sigma} \end{pmatrix}\begin{bmatrix} S^2 \\ ST \\ T^2 \end{bmatrix}.$$

More generally, as $j(g_z,\varepsilon)=y^{-1/2}$, for $\nu=1,2$,

$$u^*g_z^{*t}\omega_\nu|_\varepsilon=\rho_2({}^tj(g_zu,\varepsilon))^t\omega_\nu|_z=y^{-1t}\omega_\nu|_z\Leftrightarrow u^*g_z^*\overline{\omega}_\nu|_\varepsilon=\overline{\omega}_\nu|_z\rho_2(\overline{{}^tj(g_zu,\varepsilon)})=y^{-1}\overline{\omega}_\nu|_z. \tag{6.41}$$

Thus replacing the basis $(-XY',XX'-YY',X'Y)$ of $L_E(1;\mathbb{C})/\mathbb{C}(XX'+YY')$ by $y^{-1}\overline{\omega}_\nu|_z$, we get a morphism

$$i=i_\nu:L_E^*(n+1;\mathbb{C})\to L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^\nu_{\mathcal{S}/\mathbb{C}}\quad\text{for }\nu=1,2. \tag{6.42}$$

Writing the stalk at ε of $\Omega^\nu_{\mathcal{S}/\mathbb{C}}$ as Ω^ν_ε, consider the morphism $i_\varepsilon:L_E^*(n+1;\mathbb{C})\to L_E^*(n;\mathbb{C})\otimes\Omega^\nu_\varepsilon$ induced by the stalk at ε. By the pull-back action of the stabilizer of ε, $\mathrm{SU}_2(\mathbb{R})$ acts on Ω^ν_ε. We have an isomorphism $\Omega^\nu_\varepsilon\cong L(2;\mathbb{C})\otimes_{\mathbb{C}}\mathcal{O}_\varepsilon$ for the germ $\mathcal{O}_\varepsilon$ of analytic functions around ε of $\mathcal{H}$. By Clebsch-Gordan, we have decomposition into irreducible factors

(1) $L_E(n+1;\mathbb{C})|_{\mathrm{SU}_2(\mathbb{R})}=\bigoplus_{j=0}^{n+1}L(2j;\mathbb{C})$

(2) $L_E(n;\mathbb{C})|_{\mathrm{SU}_2(\mathbb{R})}\otimes_{\mathbb{C}}L(2;\mathbb{C})\cong\bigoplus_{k=0}^{n}L(2k;\mathbb{C})\otimes_{\mathbb{C}}L(2;\mathbb{C})=\bigoplus_{k=0}^{n}\bigoplus_{i=0}^{2k+2}L(2i;\mathbb{C})$.

Since highest weight vectors of $L(2j;\mathbb{C})$ in (1) survive in the product $L_E(n;\mathbb{C})|_{\mathrm{SU}_2(\mathbb{R})}\otimes_{\mathbb{C}}L(2;\mathbb{C})$, i_ε is an injection. By local constancy, the morphism i is a monomorphism of sheaves. □

Corollary 6.13. *We have an analytic global section*

$$\Theta(\phi)(\tau;g;\mathbf{x})=\Theta_\nu(\phi)(\tau;g;\mathbf{x}):=i_{\nu,*}\theta(\phi)(\tau,g;\mathbf{x})$$

of $L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^\nu_{\mathcal{S}/\mathbb{C}}$ by composing i_ν in Proposition 6.12 *with $\theta(\phi)(\tau,g;\mathbf{x})$.*

Since $\pi^*:L(n^*;\mathbb{C})\hookrightarrow L_E(n+1;\mathbb{C})|_{\mathrm{SU}_2(\mathbb{R})}$ is an embedding of $\mathrm{SU}_2(\mathbb{R})$-modules, we have sheaf inclusion $\pi^*:L^*(n^*;\mathbb{C})\hookrightarrow L_E^*(n+1;\mathbb{C})$. In this sense, we consider $\pi^*\circ f$ for $f\in M_k(\Gamma_\phi)$ and regard it as a harmonic global section of $L_E^*(n+1;\mathbb{C})$ over $\mathcal{S}=\Gamma_\phi\backslash\mathcal{H}$. By the isomorphism $L_E^*(n+1;\mathbb{C})\cong\widetilde{L}_E(n+1;\mathbb{C})$, we may also regard f^* as a harmonic global section of $\widetilde{L}_E(n+1;\mathbb{C})$. Further composing $i_{\nu,*}$ for $i_\nu:L_E^*(n+1;\mathbb{C})\hookrightarrow L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^\nu_{\mathcal{S}/\mathbb{C}}$, we may regard

$$\omega_\nu(f):=i_{\nu,*}(\pi^*\circ f) \tag{6.43}$$

as a harmonic closed form with values in $L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^\nu_{\mathcal{S}/\mathbb{C}}$ by (M1–3) [H94, Proposition 2.1] whose cohomology class in de Rham cohomology $H^\nu(\mathcal{S},L_E^*(n;\mathbb{C}))$ is the class of the cohomological form f as in (M1–3) in §6.1. Then we define the 2-dimensional period of f over $Sh_\alpha\cong\Gamma_\alpha\backslash\mathfrak{H}\hookrightarrow\mathcal{S}$ by

$$P_\alpha(f):=\int_{Sh_\alpha}(n!)^{-2}\nabla^n\omega_2(f)=\int_{Sh_\alpha}\omega_2(f)\quad(\nabla=\frac{\partial^2}{\partial X'\partial Y}-\frac{\partial^2}{\partial X\partial Y'}). \tag{6.44}$$

The second identity follows from the fact that the 2-cycle period is nontrivial only for constant sheaves.

6.11. **Compatibility of invariant paring and wedge product.** Consider the pairing $[\cdot,\cdot]_n=(\cdot,\cdot)\otimes\{\cdot,\cdot\}:(L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^2_{\mathcal{S}/\mathbb{C}})\otimes_{\mathbb{C}}(L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^1_{\mathcal{S}/\mathbb{C}})\to\Omega^3_{\mathcal{S}/\mathbb{C}}$ for the invariant pairing $(\cdot,\cdot)_n:L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}L_E^*(n;\mathbb{C})\to\mathbb{C}$ given by $(n!)^{-2}\nabla^n_{\mathrm{id}}\otimes(n!)^{-2}\nabla^n_\sigma$ as in (4.40) and $\{\omega,\omega'\}=\omega\wedge\omega'$. We compare two sheaf pairings $(\cdot,\cdot)_{n+1}$ and $[\cdot,\cdot]_n$.

Lemma 6.14. *The following diagram of sheaf pairing is commutative:*

$$\begin{array}{ccc} L_E^*(n+1;\mathbb{C})\otimes_{\mathbb{C}}L_E^*(n+1;\mathbb{C}) & \xrightarrow{i_2\otimes i_1} & (L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^2_{\mathcal{S}/\mathbb{C}})\otimes_{\mathbb{C}}(L_E^*(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^1_{\mathcal{S}/\mathbb{C}}) \\ {\scriptstyle(\cdot,\cdot)_{n+1}}\downarrow & & \downarrow{\scriptstyle y(z)[\cdot,\cdot]_n} \\ \mathbb{C} & \xrightarrow[i]{} & \mathcal{O}_{\mathcal{S}}d\mu_z, \end{array} \tag{6.45}$$

where i is given by $i(u)=i_0(u)\cdot d\mu_z$ for the inclusion $i_0:\mathbb{C}\hookrightarrow\mathcal{O}_{\mathcal{S}}$.

Proof. The diagram (6.45) induces the diagram of stalks at ε:

$$\begin{array}{ccc} L_E(n+1;\mathbb{C})\otimes_{\mathbb{C}}L_E(n+1;\mathbb{C}) & \xrightarrow{i_2\otimes i_1} & (L_E(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^2_{\mathcal{S},\varepsilon/\mathbb{C}})\otimes_{\mathbb{C}}(L_E(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^1_{\mathcal{S},\varepsilon/\mathbb{C}}) \\ {\scriptstyle(\cdot,\cdot)_{n+1}}\downarrow & & \downarrow{\scriptstyle[\cdot,\cdot]} \\ \mathbb{C} & \xrightarrow[i]{} & \mathcal{O}_{\mathcal{S},\varepsilon}d\mu_z|_\varepsilon. \end{array} \tag{6.46}$$

We prove first commutativity of (6.46): Since $\mathrm{SU}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$ fixes ε, by the construction of i_ν, the left hand side of the top row is a morphism of $\mathrm{SU}_2(\mathbb{R})$-modules via the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}$. Regarding the sheaves in the top line of (6.45) as sheaves over $\mathcal{H}$, it is a right module over $\mathrm{SL}_2(\mathbb{C})$ by the pullback action of the left action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{H}$. Then $i_2 \otimes i_1$ in (6.45) is also $\mathrm{SL}_2(\mathbb{C})$-equivariant. Since the pairings $(\cdot,\cdot)_{n+1}$ and $[\cdot,\cdot]_n$ is $\mathrm{SL}_2(\mathbb{C})$-invariant, we have $[i_2(x), i_1(y)]_n$ is fixed by $\mathrm{SL}_2(\mathbb{C})$. This implies $[i_2(x), i_1(y)]_n = c_n i((x,y)_{n+1})$ with $i((x,y)_{n+1}) = i_0((x,y)_{n+1}) d\mu_z$ for a nonzero constant c_n and $i_0((x,y)_{n+1}) \in \mathbb{C}$.

We need to show that the constant c_n is independent of n and is equal to 1. The construction of i_ν is made in two steps:

$$\text{(6.47)} \qquad \begin{aligned} &(1)\ \iota : L_E(n+1;\mathbb{C}) \hookrightarrow L_E(n;\mathbb{C}) \otimes_{\mathbb{C}} L_E(1;\mathbb{C}), \\ &(2)\ I_\nu : L_E(1;\mathbb{C}) \to \Omega^\nu_{\mathcal{H}/\mathbb{C}}. \end{aligned}$$

The item (1) is the morphism of $\mathrm{SL}_2(\mathbb{C})$-modules. The item (2) at the stalk of ε is given by $(-XY', XX'-YY', YX') \mapsto y^{-1}\overline{\omega}_\nu$ with $(XX'+YY') \mapsto 0$. We have an invariant pairing $(\cdot,\cdot)_1$ on $L_E(1;\mathbb{C}) \otimes_{\mathbb{C}} L_E(1,\mathbb{C})$ given by $\triangledown_{\mathrm{id}} \otimes \triangledown_\sigma$. Writing the left (resp. right) variables as $(X,Y;X',Y')$ (resp. $(S,T;S',T')$). We have

$$(XX', TT') = (YY', SS') = 1 \ \text{ and } \ (XY', TS') = (YX', ST') = -1$$

and all other combinations of monomials vanishes for the pairing. Thus

$$(-XY', TS') = (YX', -ST') = 1 \ \text{ and } \ (XX'-YY', TT'-SS') = 2.$$

Recall $y^{-1}\omega_2 = y^{-2}(dy \wedge dx, -2dx \wedge d\overline{x}, dy \wedge d\overline{x})$ and $y^{-1}\omega_1 = y^{-1}(dx, -dy, -d\overline{x})$. As we have done in the proof of Proposition 6.12, we apply complex conjugation to ω_ν, getting

$$y^{-1}\overline{\omega}_2 = y^{-2}(dy \wedge d\overline{x}, -2d\overline{x} \wedge dx, dy \wedge dx) \text{ and } \ y^{-1}\overline{\omega}_1 = y^{-1}(d\overline{x}, -dy, -dx).$$

Then $i_2(-XY') \wedge i_1(TS') = y^{-3} dy \wedge dx \wedge d\overline{x}$, $i_2(XX'-YY') \wedge i_1(TT'-SS') = 2y^{-3} dy \wedge dx \wedge d\overline{x}$ and $i_2(YX') \wedge i_1(-ST') = y^{-3} dy \wedge dx \wedge d\overline{x}$. Thus $I_2 \otimes I_1$ sends the pairing $(\cdot,\cdot)_1$ to $\{\cdot,\cdot\}$, and $(\cdot,\cdot)_n \otimes (\cdot,\cdot)_1$ is equivalent to $(\cdot,\cdot)_n \otimes \{\cdot,\cdot\}$ under $I_2 \otimes I_1$ at ε.

We now study how ι sends $(\cdot,\cdot)_{n+1}$ to $(\cdot,\cdot)_n \otimes (\cdot,\cdot)_1$. By Clebsch–Gordan [H94, (11.2b)], we have a decomposition of $\mathrm{SL}_2(E)$-modules

$$\begin{aligned} &L_E(n;E) \otimes_{\mathbb{C}} L_E(1;\mathbb{C}) \\ &\quad = L_E(n+1;\mathbb{C}) \oplus L_E((n+1)\,\mathrm{id} + (n-1)\sigma;\mathbb{C}) \oplus L_E((n-1)\,\mathrm{id} + (n+1)\sigma;\mathbb{C}) \oplus L_E(n-1;\mathbb{C}), \end{aligned}$$

where $L_E(2l\,\mathrm{id} + 2k\sigma;\mathbb{C}) = L(2l;\mathbb{C}) \otimes_{\mathbb{C}} L(2k;\mathbb{C})$ with $g \in \mathrm{GL}_2(E)$ acting by $Sym^{\otimes 2i} g$ on the left factor and $Sym^{\otimes 2j} g^\sigma$ on the right factor. The canonical projection of $L_E(n;E) \otimes_{\mathbb{C}} L_E(1;\mathbb{C})$ to the complement of $L_E(n+1;\mathbb{C})$ is given by $\triangledown_{\mathrm{id}} \otimes \triangledown_\sigma$ and the projection to $L_E(n+1;\mathbb{C})$ is given by $\pi : P(X,Y;X',Y';S,T;S',T') \mapsto P(X,Y;X',Y';X,Y;X',Y')$ writing the left variable as $(X,Y;X',Y')$ and the right variable as $(S,T;S',T')$. Thus the inclusion $L_E(n+1;\mathbb{C}) \hookrightarrow L_E(n;E) \otimes_{\mathbb{C}} L_E(1;\mathbb{C})$ is given by the adjoint π^* of π with respect to the invariant perfect pairings $(\cdot,\cdot) := (\cdot,\cdot)_{n+1}$ and $\langle\cdot,\cdot\rangle := (\cdot,\cdot)_n \otimes (\cdot,\cdot)_1$. Basically by construction, the two paring match on the irreducible factors naturally isomorphic to $Sym^{\otimes n+1}$, and hence we conclude $c_n = 1$. Of course we can compute c_n directly by evaluating the two pairings on the standard basis.

We finish the proof of Lemma 6.14. The product $\{\cdot,\cdot\}$ is invariant under the center action while $(\cdot,\cdot)_1$ is not (i.e., $(y^{-1/2}\cdot, y^{-1/2}\cdot)_1 = y^{-1}(\cdot,\cdot)_1$). To adjust this, we need to multiply by $y(z)$ to assure the commutativity of the sheaf pairing in (6.45) from the commutativity at ε. □

7. Definite D with imaginary E

In this section, we study the case where D is definite and E is imaginary.

7.1. **L-value formula for definite D and imaginary E.** Recall $\delta_+ = 1$ and that $\Delta_- < 0$ is the square-free part of Δ with $\delta_- = \sqrt{\Delta_-}$. We have the decomposition $D^\pm_\sigma = Z^\pm \oplus D^\pm_0$ so that $Z^\pm = \delta_\pm \mathbb{Q} \subset D^\pm_\sigma$ with $L_Z = N\delta_\pm\mathbb{Z}$. We take $\phi^{(\infty)}_Z := \psi$ on $L^*_Z/L_Z = N^{-1}\mathbb{Z}/N\mathbb{Z}$ for a Dirichlet character $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. We take an Eichler order $R(N_0)$ in D_E for N_0 prime to ∂. Then let

$\phi_0^{(\infty)} \in \mathcal{S}(D_{0,\mathbb{A}^{(\infty)}})$ be as in (4.25) for the characteristic function $\phi_{\widehat{L}}$ of $\delta_\mp \widehat{R}(N_0) \cap D_{0,\mathbb{A}^{(\infty)}}$. Again Remark 4.5 applies; so, we pretend $\phi_0^{(\infty)} = \phi_{\widehat{L}}$. Put

$$\phi^{(\infty)} = \phi_Z^{(\infty)} \otimes \phi_0^{(\infty)} \ \text{ and } \ \phi_0(v) := \phi_0^{(\infty)}(v^{(\infty)})[v_\infty; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(v_\infty)\xi + \frac{P_D[v_\infty]}{2}\eta\sqrt{-1}) \tag{7.1}$$

for $P_D(x, y)$ as in (6.3). Note that Z is positive and D_0 is negative definite and

$$\begin{aligned} &\mathbf{r}_Z(g_\tau)L_Z(g)(\mathfrak{z}_\infty^j\mathbf{e}(|N(\mathfrak{z}_\infty)|\sqrt{-1})) = \eta^{(1+2j)/4}\mathfrak{z}_\infty^j\mathbf{e}(|N(\mathfrak{z}_\infty)|\tau) \quad (\mathfrak{z}_\infty \in Z_{\mathbb{R}}^{\pm}) \\ &\mathbf{r}_{D_0}(g_\tau)L_{D_0}(g)([\mathfrak{x}_\infty; \mathbf{x}]^j\mathbf{e}(-|N(\mathfrak{x}_\infty)|\sqrt{-1})) = \eta^{(3+2j)/4}[\mathfrak{x}_\infty; \mathbf{x}]^j\mathbf{e}(-|N(\mathfrak{x}_\infty)|\overline{\tau}) \quad (\mathfrak{x}_\infty \in D_{0,\mathbb{R}}^{\pm}). \end{aligned} \tag{7.2}$$

Recall $\theta_k(\phi)(\tau; z, \mathbf{x})$ for $\sigma = \sigma_J$ as in (6.22) and $\boldsymbol{\theta}(\phi_j^Z)(\tau) = \sum_{\alpha \in Z}(\mathbf{w}(g_\tau)\phi_j^Z)(\alpha)$. The theta kernel $\theta_k(\phi)(\tau; z, \mathbf{x})$ has values in $L_E^*(n+1; \mathbb{C})$ and is different from the harmonic theta differential form in Corollary 6.13.

By (6.1), we have $[\mathfrak{z} + \mathfrak{x}; \mathbf{x}] = \mathfrak{z}(XX' + YY') + [\mathfrak{x}; \mathbf{x}]$ for $\mathfrak{z} \in Z^{\pm}$ and $\mathfrak{x} \in D_0^{\pm}$. Define

$$\begin{aligned} \phi_j^Z(\tau; \mathfrak{z}) &= \phi_Z^{(\infty)}(\mathfrak{z})\mathfrak{z}_\infty^j\mathbf{e}(|N(\mathfrak{z}_\infty)|\tau), \\ \phi_j^{D_0}(\tau; \mathfrak{x}) &= \phi_{D_0}^{(\infty)}(\mathfrak{x})[\mathfrak{x}_\infty; \mathbf{x}]^j\mathbf{e}(-|N(\mathfrak{x}_\infty)|\overline{\tau}). \end{aligned}$$

We remark $\pi(XX' + YY') = 0$ for $\pi : L_E(n+1; \mathbb{C}) \to L(n^*; \mathbb{C})$ in (6.29). Then we have

$$\eta^{k/2}\theta(\phi)|_{\mathrm{O}_\delta(\mathbb{A})} = \eta^{1+(n/2)}\theta(\phi) = \sum_{j=0}^{n+1}\binom{n+1}{j}(XX' + YY')^j\boldsymbol{\theta}(\phi_j^Z)\boldsymbol{\theta}(\phi_{n+1-j}^{D_0}). \tag{7.3}$$

Recalling we have a twist by $\eta^{1/2}$ in front of (6.6), we now study

$$\eta^{(k+1)/2}\int_{\mathrm{O}_\delta(\mathbb{Q})\backslash\mathrm{O}_\delta(\mathbb{A})/\widehat{\Gamma}_\delta}\theta(\phi)(g)d\mu_g = \mathfrak{m}\sum_{j=0}^{n+1}(XX' + YY')^j\binom{n+1}{j}\eta^{1/2}\boldsymbol{\theta}(\phi_j^Z)E(\phi_{n+1-j}^{D_0}),$$

where $\mathfrak{m}$ is as in (4.23) and for $\Phi \in \mathcal{S}(D_{0,\mathbb{A}})$ and $g \in \mathrm{Mp}(\mathbb{A})$,

$$E(\Phi) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})}|a(g)|^{s-(1/2)}(\mathbf{w}(\gamma g)\Phi)(0)|_{s=\frac{1}{2}}.$$

Since $J^*(YX' - XY') = XX' + YY'$, for $\triangledown$ in Case ID, by (5.7) and (5.8)

$$((n+1)!)^{-2}\triangledown^{n+1}[\mathfrak{z}; \mathbf{x}]^{n+1-j}[\mathfrak{x}; \mathbf{x}]^j = \begin{cases} n+2 = k & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases} \tag{7.4}$$

Take $F = \sum_{m=1}^\infty a_m\mathbf{e}(-m\overline{\tau}) \in S_k^-(M, \psi^{-1}\chi_{D_\sigma})$ for M as in (4.26). Since $(Z^{\pm}, Q^{\pm})$ is positive definite producing holomorphic $\theta(\phi_k^Z)$, F has to be anti-holomorphic. Since $[\mathfrak{x}; \mathbf{x}]|_{\mathfrak{x}=0} = 0$, in the same manner as getting (5.15) from (4.29) and (4.31), we obtain, for $\overline{B} := B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty/B(\widehat{\mathbb{Z}})C_\infty$,

$$\begin{aligned} &\sum_{j=0}^{n+1}\binom{n+1}{j}\int_{\overline{B}}\mathbf{F}(g_\tau)\eta^{k+1/2}\boldsymbol{\theta}(((n+1)!)^{-2}\triangledown^{n+1}\phi_j^Z)(g_\tau)\mathbf{r}(g_\tau)(\phi_{n+1-j}^{D_0})(0)d\mu_\tau \\ &\overset{(7.4)}{=} \int_{\overline{B}}\mathbf{F}(g_\tau)\eta^{k+1/2}\boldsymbol{\theta}(((n+1)!)^{-2}\triangledown^{n+1}\phi_{n+1}^Z)(g_\tau)\mathbf{r}(g_\tau)(\phi_0^{D_0})(0)d\mu_\tau \\ &= \int_0^\infty\int_0^1 F(\tau)\theta((n+1)!)^{-2}\triangledown^{n+1}\phi_{n+1}^Z)(g_\tau)d\xi\eta^{k-(3/2)}d\eta \\ &= 2(4\pi)^{-k+(1/2)}\delta_\pm^{k-1}|\Delta_\pm|^{-k+(1/2)}k\Gamma(k - \frac{1}{2})\sum_{0<m\in\mathbb{Z}}\psi(n)a_{m^2}m^{-k}. \end{aligned} \tag{7.5}$$

Here $\delta_-^{k-1}|\Delta_-|^{-k+(1/2)} = -\sqrt{-1}|\delta_-|^{-k}$. Thus we get in the same manner as in Theorem 4.7

Theorem 7.1. *Suppose that E is imaginary and that D is definite. Let F be a primitive Hecke eigenform in $S_k^-(C, \psi^{-1}\chi_{D_\sigma})$ for $C|M$ with M as in (4.26) and $f = \theta^*(F)$ be the theta lift:*

$$f(g; \mathbf{x}) = \int_{\Gamma_\tau\backslash\mathfrak{H}}\theta_k(\phi)(\tau; g, \mathbf{x})F(\tau)\eta^{k-2}d\xi d\eta.$$

Choose $\phi_Z^{(\infty)}$ *associated to Dirichlet character* ψ *of conductor* $C(\psi)$ *as specified above and* $\phi_0^{(\infty)}$ *as in* (4.25). *Let* ϕ *be a Schwartz-Bruhat function of* $D^{\pm}_{\sigma,\mathbb{A}}$ *as in* (7.1), *and choose the measure* $d\mu_h$ *on* $\mathrm{O}_\delta(\mathbb{A})$ *as in* Theorem 4.2. *Then if* $f(g;\mathbf{x})\neq 0$, *for the mass factor* $\mathfrak{m}_1$ *as in* (4.23) *and* $E^{\pm}(1)$ *as in* Theorem 4.7*,*

$$\pi^2\int_{Sh_\delta}((n+1)!)^{-2}\nabla^{n+1}f(h;\mathbf{x})d\mu_h$$
$$=\mathfrak{m}_1E^{\pm}(1)\delta_{\pm}^{k-1}|\Delta_{\pm}|^{-k+(1/2)}2(4\pi)^{(1/2)-k}k\Gamma(k-\frac{1}{2})L^{(C_s(\psi))}(1,Ad(F)\otimes\chi_E)$$

for the compatible system ρ_F *attached to* F *and* $\widehat{\Gamma}_\delta:=\{u\in\mathrm{SO}_\delta(\mathbb{A})|\phi^{(\infty)}\circ u=\phi^{(\infty)}\}\cong\widehat{R}(N_0)^\times/\widehat{\mathbb{Z}}^\times$. *The Shimura subvariety* Sh_δ *is as in* (4.22).

7.2. **Theta descent.** In this subsection, the choice of the Bruhat function $\phi^{(\infty)}$ is arbitrary. Recall

$$\Gamma=\Gamma_\phi:=\{\gamma\in\mathrm{SO}_{D_\sigma}(\mathbb{Q})=G^+_{D_\sigma}/Z_{G^+_{D_\sigma}}(\mathbb{Q})|\phi^{(\infty)}(\gamma^{-1}x\gamma^\sigma)=\phi^{(\infty)}(x)\text{ for all }x\in D_{\sigma,\mathbb{A}^{(\infty)}}\}.$$

Identifying $D^\times_{E_\mathbb{R}}=\mathrm{GL}_2(\mathbb{C})$, from (6.38), a quaternionic modular form $f(h;\mathbf{s}):D^\times\backslash D^\times_{E_\mathbb{A}}\to L(n^*;\mathbb{C})$ ($k=n+2$ and $n^*=2n+2$) on $\widehat{\Gamma}$ of weight $k\infty+k\infty\sigma$ satisfies

$$f(\gamma zxu;\mathbf{s})=f(x;\mathbf{s}^tu^\sigma)\ \text{ for }\ u\in\widehat{\Gamma}\cdot\mathrm{SU}_2(\mathbb{R}),z\in E^\times_\mathbb{A}\text{ and }\gamma\in D^\times_E. \tag{7.6}$$

where ${}^tu^\sigma_\infty=u_\infty^{-1}$ if $\sigma=\sigma_1$ and ${}^tu_\infty$ if $\sigma=\sigma_J$. Here $\mathrm{SU}_2(\mathbb{R})=\{u\in\mathbb{H}^\times|N(u)=1\}$ is the stabilizer in $\mathrm{SL}_2(\mathbb{C})$ of $\varepsilon=-J\in\mathcal{H}$, and $u\in\mathrm{GL}_2(A)$ acts on $P(\mathbf{s})\in L(n^*;A)$ by $P|u(\mathbf{s})=P(\mathbf{s}u^\iota)$.

Recall the projection $\pi:L_E(n+1;\mathbb{C})\twoheadrightarrow L(n^*;\mathbb{C})$ of $\mathrm{SU}_2(\mathbb{R})$-modules in (6.30) given by

$$\mathbf{x}=(X,Y;X',Y')\mapsto\widetilde{\mathbf{x}}:=(T,-S;S,T).$$

By convention, we write $\pi(\Phi(\mathbf{x}))=\Phi(\widetilde{\mathbf{x}})$. Recall also, for $u\in\mathbb{H}^1=\mathrm{Ker}(N:\mathbb{H}^\times\to\mathbb{R}^\times)$,

$$u^{-1}[\widetilde{\mathbf{x}}]_Du^{\sigma_J}=u\cdot(\pi[\mathbf{x}]_D)\ \text{ and }\ h^{-1}[\widetilde{\mathbf{x}}]_Dh=h\cdot(\pi[\mathbf{x}]_D)\ \text{ for }h\in\mathrm{SL}_2(\mathbb{R}).$$

Recall the adjoint $\pi^*:L(n^*;\mathbb{C})\hookrightarrow L_E(n+1;\mathbb{C})$ of π which is an embedding of $\mathrm{SU}_2(\mathbb{R})$-modules. Write $\pi^*(f(g;\mathbf{s}))$ as $f(g;\mathbf{s}^*)$ (for $\mathbf{s}^*$ as in (6.37)). Thus $h\cdot\pi(f(g;\mathbf{s}^*))=f(g;\mathbf{s}^*h)$ for $h\in\mathrm{GL}_2(E)$.

Recall the pairings $(\cdot,\cdot)=(\cdot,\cdot)_{n+1}$ and $\langle\cdot,\cdot\rangle$ defined in §6.6. Restricting f in (7.6) to $G^+_{D_\sigma}(\mathbb{A})\subset D^\times_{E_\mathbb{A}}$ and taking the measure $d\mu_h$ with $\int_{\widehat{\Gamma}_\phi\mathrm{SU}_2(\mathbb{R})}d\mu_h=1$ and $d\mu_h|_\mathcal{H}=y^{-3}|dy\wedge dx\wedge d\overline{x}|$ on $\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})$, we define the theta descent $\theta_*(f)(\tau)$ by

$$\theta_*(f)(\tau):=\int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})/\mathrm{SU}_2(\mathbb{R})}(\theta(\phi)(\tau;h;\mathbf{x}),f(h;\mathbf{s}^*))d\mu_h \tag{7.7}$$
$$=\int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})/\mathrm{SU}_2(\mathbb{R})}\langle\theta(\phi)(\tau;h;\widetilde{\mathbf{x}}),f(h;\mathbf{s})\rangle d\mu_h$$

for $\widetilde{\mathbf{x}}$ as in (6.30) and $\theta(\phi)(\tau;h;\mathbf{x})$ as in §6.4.

We now show that its Fourier coefficient for $\mathbf{e}(N(\alpha)\tau)$ is given by the period over Sh_α:

$$\int_{Sh_\alpha}(\theta(\phi)(\tau;h;\widetilde{\mathbf{x}}),f(h;\mathbf{s}^*))d\mu_h.$$

For $h\in G^+_{D_\sigma}(\mathbb{R})$ and $u\in\widehat{\Gamma}_\phi\cdot\mathrm{SU}_2(\mathbb{R})$,

$$(\theta(\phi)(\tau;hu;\mathbf{x}),f(hu;\mathbf{s}^*))=\langle\theta(\phi)(\tau;h;\widetilde{\mathbf{x}}^tu^\sigma_\infty),f(h;\mathbf{s}^tu^\sigma_\infty)\rangle=(\theta(\phi)(\tau;h;\mathbf{x}),f(h;\mathbf{s}^*))$$

by (6.9) and (7.6). Thus $(\theta(\phi)(\tau;h;\mathbf{x}),f(h;\mathbf{s}^*))=(\theta(\phi)(\tau;g_z;\mathbf{x}),f(g_z;\mathbf{s}^*))$ for $z=h(\varepsilon)\in\mathcal{H}$, and $h\mapsto(\theta(\phi)(\tau;h;\mathbf{x}),f(h;\mathbf{s}^*))$ factors through $\mathcal{H}=\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{R})$. We know $(P(\mathbf{x}h),Q(\mathbf{s}^*h))=\det(h\overline{h})^{n+1}(P(\mathbf{x}),Q(\mathbf{s}^*))$.

Recall $h^?=h^\iota$ in Case II and $h^?={}^th$ in Case ID for $h\in G^+_{D_{E_\sigma}}(\mathbb{A})$. Even if we are working in Case ID in this section, many formulas here are valid in Case II under the action $h\mapsto h^?$; so, we use this notation $h^?$ to indicate the formula valid in the two cases. Then

$$(\theta(\phi)(\tau;h;\mathbf{x}h^{-?}),f(h;\mathbf{s}^*h^{-?}))=\det(h\overline{h})^{n+1}(\theta(\phi)(\tau;h;\mathbf{x}),f(h;\mathbf{s}^*)).$$

Since $\det(g_z) = 1$, this shows $(\theta(\phi)(\tau; g_z; \mathbf{x}), f(g_z; \mathbf{s}^*)) = (\theta(\phi)(\tau; g_z; \mathbf{x}g_z^{-?}), f(g_z; \mathbf{s}^* g_z^{-?}))$. In sum

$$\theta_*(f)(\tau) := \int_{\mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash \mathrm{SO}_{D_\sigma}(\mathbb{A})/\mathrm{SU}_2(\mathbb{R})} (\theta(\phi)(\tau; g_z; \mathbf{x}g_z^{-?}), f(g_z; \mathbf{s}^* g_z^{-?}))d\mu_z, \tag{7.8}$$

where $d\mu_z = y^{-3}|dy \wedge dx \wedge d\overline{x}|$.

Note ${}^t u_\infty^\sigma = u_\infty^?$ for $u_\infty \in \mathrm{SU}_2(\mathbb{R}) \subset G^+_{D_{E_\sigma}}(\mathbb{R})$. By $f(hu_\infty; \mathbf{s}^* h^{-?}\, {}^t u_\infty^{-\sigma}) = f(h; \mathbf{s}^* h^{-?} u_\infty^{-?}\, {}^t u_\infty^\sigma) = f(h; \mathbf{s}^* h^{-?})$ for $u_\infty \in \mathrm{SU}_2(\mathbb{R})$, we find that $h \mapsto f(h; \mathbf{s}^* h^{-?})$ factors through $D^\times \backslash D^\times_{E_{\mathbb{A}}}/\mathrm{SU}_2(\mathbb{R})$. Thus $f(z; \mathbf{s}^*) := f(g_z; \mathbf{s}^* g_z^{-?})$ is a well defined function on $\mathcal{H}$. We have for $\gamma \in \Gamma_\phi$. writing $\gamma g_z = g_{\gamma(z)} u$ with $u \in \mathrm{SU}_2(\mathbb{R})$,

$$\begin{aligned}(7.9)\quad f(z; \mathbf{s}^*\gamma^{-?}) &= f(g_z; \mathbf{s}^*(\gamma g_z)^{-?}) = f(\gamma g_z; \mathbf{s}^*(\gamma g_z)^{-?}) \\ &= f(g_{\gamma(z)}u; \mathbf{s}^*(g_{\gamma(z)}u)^{-?}) = f(g_{\gamma(z)}; \mathbf{s}^* g_{\gamma(z)}^{-?}) = f(\gamma(z), \mathbf{s}^*).\end{aligned}$$

This formula also applies to $\theta(\tau; z; \mathbf{x}) := \theta(\tau; g_z; \mathbf{x}g_z^{-?})$ in place of $f(g; \mathbf{s}^*)$:

$$\theta(\tau; z; \mathbf{x}\gamma^{-?}) = \theta(\tau; \gamma(z), \mathbf{x}). \tag{7.10}$$

By (6.2): $[g_z^{-1}\alpha g_z^\sigma; \mathbf{x}g_z^{-?}] = [\alpha; \mathbf{x}]$, and by definition

$$\theta(\tau; z; \mathbf{x}) = \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm} \phi^{(\infty)}(\alpha)[\alpha; \mathbf{x}]^{n+1} \mathbf{e}(\pm N(\alpha)\xi + \frac{\eta P[g_z^{-1}\alpha g_z^\sigma]}{2}\sqrt{-1}). \tag{7.11}$$

7.3. **Vanishing of Fourier coefficients of $\mathbf{e}(N(\alpha)\tau)$ when $D_{\alpha,\mathbb{R}} \cong \mathbb{H}$.** Since $G^+_{D_\sigma} \subset D_E^\times$ and $D_E^\times$ satisfies the strong approximation theorem, we have $G^+_{D_\sigma}(\mathbb{A}) = \bigsqcup_{a \in \mathcal{A}_G} G^+_{D_\sigma}(\mathbb{Q}) a \widehat{\Gamma} G^+_{D_\sigma}(\mathbb{R})$ for a finite set $\mathcal{A}_G = \mathcal{A}_{G,\phi}$ on which the reduced norm map induces a bijection $N : \mathcal{A}_G \cong \widehat{\mathbb{Z}}^\times/(N(\widehat{\Gamma}) \cap \mathbb{Q}^\times)$. Since $\mathrm{SO}_{D_\sigma} = G^+_{D_\sigma}/Z_{G^+_{D_\sigma}}$, we can choose a finite set $\mathcal{A} = \mathcal{A}_\phi \subset \mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)})$ such that $N : \mathcal{A}_\phi \cong \widehat{\mathbb{Z}}^\times/((\widehat{\mathbb{Z}}^\times)^2 N(\widehat{\Gamma}) \cap \mathbb{Q}^\times)$ and

$$\mathrm{SO}_{D_\sigma}(\mathbb{A}^{(\infty)}) = \bigsqcup_{a \in \mathcal{A}} \mathrm{SO}_{D_\sigma}(\mathbb{Q}) a \widehat{\Gamma}_\phi.$$

If ϕ is as in Theorem 7.1, $\widehat{\Gamma}_\phi \supset \widehat{R}(N_0)^\times$ and hence $\mathcal{A}_\phi = \{1\}$. We thus assume $\mathcal{A} = \{1\}$. In the following computation, to treat Case ID and Case II uniformly, we put $\tau_D = \overline{\tau}$ in Case ID, $\tau_I = \tau$ in Case II, $\eta_I = \eta$ in Case ID and $\eta_I = -\eta$. Thus $\tau_* = \tau_I$ or τ_D and $\eta_* = \eta_I$ or η_D depending on cases. The "$-$" sign of $\eta_I = -\eta$ is to present the following computation uniformly in the two cases II and ID as (6.18) and (6.21) has an opposite sign in front of $[z; v]^2/2y(z)^2$. Using (6.18) in Case ID and (6.21) in Case II, we compute (recalling $h^? = h^\iota$ in Case II and $h^? = {}^t h$ in Case ID), for a fundamental domain $\mathcal{F}$ of Γ_ϕ in $\mathcal{H}$,

$$\begin{aligned}(7.12)\quad \theta_*(f)(\tau) &\overset{(7.11),(6.18),(6.21)}{=} \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm/\Gamma_\phi} \phi^{(\infty)}(\alpha) \\ &\quad \times \sum_{\gamma \in \Gamma_\phi} \int_{\mathcal{F}} ([\gamma^{-1}\alpha\gamma^\sigma; \mathbf{x}]^{n+1}, f(z; \mathbf{s}^*)) \mathbf{e}(\pm N(\alpha)\tau_* \pm \frac{[z; \gamma^{-1}\alpha\gamma^\sigma]^2 \eta_* \sqrt{-1}}{2y(z)^2}) d\mu_z \\ &\overset{(6.17)}{=} \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm/\Gamma_\phi} \phi^{(\infty)}(\alpha) \sum_{\gamma \in \Gamma_\phi} \int_{\mathcal{F}} ([\alpha; \mathbf{x}\gamma^?]^{n+1}, f(z; \mathbf{s}^*)) \mathbf{e}(\pm N(\alpha)\tau_* \pm \frac{[\gamma(z); \alpha]^2 \eta_* \sqrt{-1}}{2y(\gamma(z))^2}) d\mu_z \\ &= \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm/\Gamma_\phi} \phi^{(\infty)}(\alpha) \sum_{\gamma \in \Gamma_\phi} \int_{\mathcal{F}} ([\alpha; \mathbf{x}]^{n+1}, f(z; \mathbf{s}^*\gamma^{-?})) \mathbf{e}(\pm N(\alpha)\tau_* \pm \frac{[\gamma(z); \alpha]^2 \eta_* \sqrt{-1}}{2y(\gamma(z))^2}) d\mu_z \\ &\overset{(7.9)}{=} \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm/\Gamma_\phi} \phi^{(\infty)}(\alpha) \sum_{\gamma \in \Gamma_\phi} \int_{\mathcal{F}} ([\alpha; \mathbf{x}]^{n+1}, f(\gamma(z); \mathbf{s}^*)) \mathbf{e}(\pm N(\alpha)\tau_* \pm \frac{[\gamma(z); \alpha]^2 \eta_* \sqrt{-1}}{2y(\gamma(z))^2}) d\mu_z \\ &= \eta^{1/2} \sum_{\alpha \in D_\sigma^\pm/\Gamma_\phi} \phi^{(\infty)}(\alpha) \int_{\Gamma_\alpha\backslash\mathcal{H}} ([\alpha; \mathbf{x}]^{n+1}, f(z; \mathbf{s}^*)) \mathbf{e}(\pm N(\alpha)\tau_* \pm \frac{[z; \alpha]^2 \eta_* \sqrt{-1}}{2y(z)^2}) d\mu_z.\end{aligned}$$

If $D_{\alpha,\mathbb{R}} = j^{-1}\mathbb{H}j$ ($\Leftrightarrow \alpha = j^{-1}j^{\sigma_J}$ up to non-zero scalars) for $j \in \mathrm{SL}_2(\mathbb{C})$, writing $\doteqdot$ for the equality up to non-zero scalars,

$$([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*)) \doteqdot ([j^{-1}j^{\sigma_J};\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*)) = ([1;\mathbf{x}j^{?}]^{n+1}, f(z;\mathbf{s}^*)) = ([1;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*j^{-?})).$$

The constant is given by $\sqrt{-1}^{\epsilon}$ for $\epsilon = (1\mp 1)/2$ for the parity of $D^{\pm}_{\sigma}$. Then noting $|\Gamma_\alpha| < \infty$,

$$\begin{aligned}
(7.13)\quad &\int_{\Gamma_\alpha\backslash\mathcal{H}} ([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(N(\alpha)\tau_* \pm \frac{[z;\alpha]^2\eta_*\sqrt{-1}}{2y(z)^2})d\mu_z\\
&= |\Gamma_\alpha|^{-1}\int_{\mathcal{H}} ([\sqrt{-1}^{\epsilon};\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*j^{-?}))\mathbf{e}(N(\alpha)\tau_* \pm \frac{[z;j^{-1}j^{\sigma_J}]^2\eta_*\sqrt{-1}}{2y(z)^2})d\mu_z\\
&\overset{z\mapsto j^{-1}(z)}{=} |\Gamma_\alpha|^{-1}\int_{\mathcal{H}} ([\sqrt{-1}^{\epsilon};\mathbf{x}]^{n+1}, f(j^{-1}(z);\mathbf{s}^*j^{-?}))\mathbf{e}(N(\alpha)\tau_* \pm \frac{[j^{-1}(z);j^{-1}j^{\sigma_J}]^2\eta_*\sqrt{-1}}{2y(j^{-1}(z))^2})d\mu_z\\
&\overset{(6.17)}{=} |\Gamma_\alpha|^{-1}\int_{\mathcal{H}} ([\sqrt{-1}^{\epsilon};\mathbf{x}]^{n+1}, f|_{\mathcal{H}}j^{-1}(z;\mathbf{s}^*))\mathbf{e}(N(\alpha)\tau_* \pm \frac{[z;1]^2\eta_*\sqrt{-1}}{2y(z)^2})d\mu_z\\
&= |\Gamma_\alpha|^{-1}\int_{\mathcal{H}} ([\sqrt{-1}^{\epsilon};\mathbf{x}]^{n+1}, f|_{G}j^{-1}(g_z;\mathbf{s}^*))\mathbf{e}(N(\alpha)\tau_* \pm \frac{[z;1]^2\eta_*\sqrt{-1}}{2y(z)^2})d\mu_z.
\end{aligned}$$

Proposition 7.2. *Suppose $D_{\alpha,\mathbb{R}} \cong \mathbb{H}$. Then in Cases ID and II, we have*

$$\int_{\Gamma_\alpha\backslash\mathcal{H}} ([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(\pm\frac{[z;\alpha]^2\eta_*\sqrt{-1}}{2y(z)^2})d\mu_z = 0.$$

Proof. Under $D_{\alpha,\mathbb{R}} \cong \mathbb{H}$, $\alpha \doteqdot jxj^{-\sigma}$ for $j \in \mathrm{SL}_2(\mathbb{C})$ and $x = 1$ if $D_{\mathbb{R}} \cong \mathbb{H}$ and $x = J$ if $D_{\mathbb{R}} \cong M_2(\mathbb{R})$. Here "$\doteqdot$" means up to a power of $\sqrt{-1}$. Then

$$[\alpha;\mathbf{x}] \doteqdot [jxj^{-\sigma};\mathbf{x}] = [x;\mathbf{x}j^{-?}],$$

and $([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*)) \doteqdot ([x;\mathbf{x}]^{n+1}, f|_{\mathcal{H}}j^{-1}(z;\mathbf{s}^*)) = 0$ by Lemma 6.10. □

7.4. **Fourier coefficients of theta descent.** We now assume that D is definite and $D_{\alpha,\mathbb{R}} \cong M_2(\mathbb{R})$.

We first interpret

$$\int_{\Gamma_\alpha\backslash\mathcal{H}} ([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(\pm N(\alpha)\overline{\tau} \pm \frac{[z;\alpha]^2\eta\sqrt{-1}}{2y(z)^2})d\mu_z$$

as an integral of differential form. In this case $|\Gamma_\alpha| = \infty$ with finite volume $\Gamma_\alpha\backslash\mathrm{SL}_2(\mathbb{R})$. By (7.12)

$$(7.14)\quad \theta_*(f)(\tau) = \eta^{1/2}\sum_{\alpha\in D^{\pm}_{\sigma}/\Gamma_\phi}\phi^{(\infty)}(\alpha)\int_{\Gamma_\alpha\backslash\mathcal{H}} ([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(\pm N(\alpha)\overline{\tau} \pm \frac{[z;\alpha]^2\eta\sqrt{-1}}{2y(z)^2})d\mu_z.$$

Let $S_\phi := \Gamma_\phi\backslash\mathcal{H}$. We consider $\Theta_2(\phi;\tau;z;\mathbf{x}) \in H^0(S_\phi, L^*_E(n;\mathbb{C})\otimes\Omega^2_{S_\phi/\mathbb{C}})$ as in Corollary 6.13 and $\omega_1(f)(z;\mathbf{s}^*) := i_1(f(g_z;\mathbf{s})) \in H^0(S_\phi, L^*_E(n;\mathbb{C})\otimes\Omega^1_{S_\phi/\mathbb{C}})$ for a cusp form f on $\Gamma = \Gamma_\phi$ as in (M1–3) in §6.1. Here $i_\nu : L^*_E(n+1;\mathbb{C}) \to L^*_E(n;\mathbb{C})\otimes_{\mathbb{C}}\Omega^\nu_{S/\mathbb{C}}$ is as in Proposition 6.12, and we write $i_0 : \mathbb{C} \hookrightarrow \mathcal{O}_{\mathcal{S}}$ for the constant inclusion for the structure sheaf $\mathcal{O}_{\mathcal{S}}$ of the real 3-dimensional analytic manifold $\mathcal{S}$.

From (7.14) and Lemma 6.14 combined with Proposition 7.2, $\eta^{-1/2}\theta_*(f)(\tau)$ is equal to

$$(7.15)\quad \sum_{\alpha\in D^{\pm}_{\sigma}/\Gamma_\phi, D_{\alpha,\mathbb{R}}\cong M_2(\mathbb{R})}\phi^{(\infty)}(\alpha)\int_{\Gamma_\alpha\backslash\mathcal{H}} y[i_{\nu,*}([\alpha;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f(z;\mathbf{s}^*))]\mathbf{e}(\pm N(\alpha)\overline{\tau} \pm \frac{[z;\alpha]^2\eta\sqrt{-1}}{2y^2})d\mu_z.$$

Choose $j \in \mathrm{SL}_2(\mathbb{C})$ as in Lemma 6.7 so that $D_{\alpha,\mathbb{R}} = j^{-1}M_2(\mathbb{R})j$, where $\alpha = j^{-1}J^{\pm}j^{\sigma_J}$ with $J^+ = \sqrt{-1}J$ if $\alpha \in D^+_\sigma$ and $J^- = J$ if $\alpha \in D^-_\sigma$.

Assuming $D_{\alpha,\mathbb{R}} \cong M_2(\mathbb{R})$, we need to compute

$$(7.16)\quad I^{\pm} := \int_{\Gamma_\alpha\backslash\mathcal{H}} y(z)[i_{\nu,*}([\alpha;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f(z;\mathbf{s}^*))]\mathbf{e}(\pm\frac{[z;\alpha]^2\eta\sqrt{-1}}{2y(z)^2})d\mu_z,$$

as $I^{\pm}$ is the coefficient of $\mathbf{e}(\pm N(\alpha)\overline{\tau})$. Here $f|_G j^{-1}$ is as in (6.39). We also note

$$(7.17)\quad \pm N(\alpha) = \pm N(j^{-1}J^{\pm}j^{\sigma_J}) < 0$$

which is consistent with $\mp \mathrm{Im}(\pm\overline{\tau}) > 0$. By (6.1) in Case ID, we have $[J;\mathbf{x}] = YX' - XY'$; so,

$$y(z)[i_{\nu,*}([J;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f|_G j^{-1}(g_z;\mathbf{s}^*))] = y(z)[i_{\nu,*}((YX'-XY')^{n+1}), i_{3-\nu,*}(f|_G j^{-1}(g_z;\mathbf{s}^*))].$$

First suppose $n = 0$. Then writing $x = u + v\sqrt{-1}$, by Proposition 6.12

$$i_{\nu,*}((YX'-XY')) = \begin{cases} y^{-1}(d\overline{x} - dx) = -2\sqrt{-1}y^{-1}dv & \text{if } \nu = 1, \\ y^{-2}(dy \wedge dx + dy \wedge d\overline{x}) = 2y^{-2}(dy \wedge du) & \text{if } \nu = 2. \end{cases} \tag{7.18}$$

Since $L_E(0;\mathbb{C}) = \mathbb{C}$ and $L_E(1;\mathbb{C}) = \mathbb{C}SS' + \mathbb{C}ST' + \mathbb{C}TS' + \mathbb{C}TT'$, the function

$$f^j(g;\mathbf{s}) := f|_G j^{-1}(g_z;\mathbf{s}) = f(j^{-1}g_z;\mathbf{s}) : j\Gamma j^{-1}\backslash \mathrm{SL}_2(\mathbb{C}) \to L(2;\mathbb{C}) = \mathbb{C}S^2 + \mathbb{C}ST + \mathbb{C}T^2 \tag{7.19}$$

has the form $f^j(g;\mathbf{s}) = f_0^j S^2 + 2f_1^j ST + f_2^j T^2$. By (6.36),

$$(-1)^i \binom{n^*}{i} \pi^*(S^{n^*-i}T^i) = \sum_{j+j'=i} (-1)^j \binom{n+1}{j}\binom{n+1}{j'} X^{n+1-j}Y^j X'^{j'} Y'^{n+1-j'},$$

and applying this to $n = 0$, $\mathbf{s}^* = (XY', (YY'-XX')/2, -YX')$. Thus we have $f^j(g;\mathbf{s}^*) = f_0^j XY' + f_1^j(YY'-XX') - f_2^j YX'$. Therefore

$$i_{\nu,*}(f^j(g;\mathbf{s}^*)) = \begin{cases} -f_0^j y^{-1}d\overline{x} + f_1^j y^{-1}dy + f_2^j y^{-1}dx & \text{if } \nu = 1, \\ -f_0^j y^{-2}(dy \wedge d\overline{x}) + 2f_1^j y^{-2}(d\overline{x} \wedge dx) - f_2^j y^{-2}(dy \wedge dx) & \text{if } \nu = 2. \end{cases} \tag{7.20}$$

Since $d\overline{x} \wedge dx = 0$ on $\mathfrak{H}$, if $\nu = 2$,

$$i_{2,*}(f^j(g;\mathbf{s}^*))|_{\mathfrak{H}} = -f_0^j y^{-2}(dy \wedge d\overline{x}) + 2f_1^j y^{-2}(d\overline{x} \wedge dx) - f_2^j y^{-2}(dy \wedge dx) = j^{-*}\omega_2(f). \tag{7.21}$$

Identifying $\mathcal{H} = \mathfrak{H} \times \mathbb{R}$ by $z \mapsto (u + y\sqrt{-1}, v)$, we get from (7.18),

$$\begin{aligned} y(z)[i_{\nu,*}((YX'-XY')), i_{3-\nu,*}(f^j)] &= -(f_0^j + f_2^j)y^{-2}dy \wedge dx \wedge d\overline{x} \\ &= 2\sqrt{-1}(f_0^j + f_2^j)y^{-2}dy \wedge du \wedge dv = -2\sqrt{-1}j^{-*}\omega_2(f)|_{\mathfrak{H}} \wedge dv \end{aligned} \tag{7.22}$$

which is independent of $\nu = 1, 2$.

Write $\Gamma_\alpha^j := j\Gamma_\alpha j^{-1} \subset \mathrm{SL}_2(\mathbb{R})$. Then $\Gamma_\alpha^j\backslash\mathcal{H} = \Gamma_\alpha^j\backslash\mathfrak{H} \times \mathbb{R}$ by $z \mapsto (u + y\sqrt{-1}, v)$. This shows $\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H} \cong \mathbb{R}$. By (6.17), we have

$$\frac{[z;h^{-1}vh^{\sigma_J}]^2}{2y(z)^2} = \frac{[h(z);v]^2}{2y(h(z))^2} \quad \text{and} \quad \frac{[z;h^{-1}Jh^{\sigma_J}]^2}{2y(z)^2} = \frac{[h(z);J]^2}{2y(h(z))^2}$$

as $h^{-1}Jh^{\sigma_J} = h^{-1}JJ^{-1}hJ = J$ for $h \in \mathrm{SL}_2(\mathbb{R})$. This shows that the function $z \mapsto \mathbf{e}(\frac{[z;\sqrt{-1}J]^2\eta\sqrt{-1}}{2y(z)^2})$ factors through $\mathrm{SL}_2(\mathbb{R})\backslash\mathfrak{H}$.

Recall $p_+(z) = \left(\begin{smallmatrix} 1+x\overline{x} & -xy \\ -y\overline{x} & y^2 \end{smallmatrix}\right)$ and $[z;v] = \mathrm{Tr}_{D_\mathbb{C}/\mathbb{C}}(p_+(z)v^\iota)$ as in (6.16) for $\partial = -1$. Thus $[z;J] = y(\overline{x} - x) = -2\sqrt{-1}yv$, and $\mathbf{e}(\pm\frac{[z;J^\pm]^2\eta\sqrt{-1}}{2y(z)^2}) = \mathbf{e}(2v^2\eta\sqrt{-1})$. Thus we get

$$I^\pm = c_\pm \int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}} \mathbf{e}(2v^2\eta\sqrt{-1})dv \int_{\Gamma_\alpha^j\backslash\mathrm{SL}_2(\mathbb{R})(z)} (f_0^j + f_2^j)y^{-2}dy \wedge du,$$

where $c_+ = 2$ and $c_- = -2\sqrt{-1}$. Since $z \mapsto \int_{\Gamma_\alpha^j\backslash\mathrm{SL}_2(\mathbb{R})(z)} (f_0^j + f_2^j)y^{-2}dy \wedge du$ is independent of z (as the integrand is a closed 2-form [H99, §3]), we have

$$I^\pm = c_\pm \int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}} \mathbf{e}(2v^2\eta\sqrt{-1})dv \cdot \int_{\Gamma_\alpha^j\backslash\mathrm{SL}_2(\mathbb{R})(\varepsilon)} (f_0^j + f_2^j)y^{-2}dy \wedge du. \tag{7.23}$$

We now deal with the general case of $n > 0$. Similarly to (7.20), we write

$$I_{2,*}(n!^{-2}\nabla^n\iota(f^j(g;\mathbf{s}^*))) = -f_0^j y^{-2}(dy \wedge d\overline{x}) + 2f_1^j y^{-2}(d\overline{x} \wedge dx) - f_2^j y^{-2}(dy \wedge dx) = (n!)^{-2}\nabla^n\omega_2(f^j), \tag{7.24}$$

where ι is as in Lemma 6.11 and ∇ is with respect to the factor $L_E(n;\mathbb{C})$ (acting trivially the factor $L_E(1;\mathbb{C})$). The down-to-earth explicit form of $n!^{-2}\nabla^n\iota(f^j(g;\mathbf{s}^*))$ can be found in [H99, page 141].

We decompose $\mathcal{H} = \mathfrak{H} \times \mathbb{R}$ by $z \mapsto (u + y\sqrt{-1}, v)$. By (5.7), $n!^{-2}\nabla^n[J;\mathbf{x}]^n = (n+1)$. Then we get, for $P_\alpha(f)$ in (6.44),

$$I^{\pm} = c_{\pm,n}\int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}} \mathbf{e}(2v^2\eta\sqrt{-1})dv \cdot \int_{\Gamma^j_\alpha\backslash \mathrm{SL}_2(\mathbb{R})(\varepsilon)} (f_0^j + f_2^j)y^{-2}dy\wedge du$$
$$= c_{\pm,n}\int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}} \mathbf{e}(2v^2\eta\sqrt{-1})dv \cdot P_\alpha(f).$$

where $c_{+,n} = 2\sqrt{-1}^n(n+1)$ and $c_{-,n} = -2\sqrt{-1}(n+1)$. The power of $\sqrt{-1}$ comes from

$$[\sqrt{-1}J;\mathbf{x}]^{n+1} = \sqrt{-1}^{n+1}[J;\mathbf{x}]$$

as $\sqrt{-1}J \in D^+_{\sigma_J,\mathbb{R}}$, while $J \in D^-_{\sigma_J,\mathbb{R}}$. As is well known [HMI, (2.5.5)], we have

$$\int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}} \mathbf{e}(2v^2\eta\sqrt{-1})dv = \int_{-\infty}^{\infty}\exp(-4\pi\eta v^2)dv = 2\eta^{-1/2}.$$

Thus

$$I^{\pm} = 2c_{\pm,n}\eta^{-1/2}\cdot P_\alpha(f). \tag{7.25}$$

Thus we conclude

Theorem 7.3. *Suppose that $f : \mathcal{H} \to L(n^*;\mathbb{C})$ is a cusp form on $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ of weight $k = n+2 \geq 2$ satisfying* (M1–3) *in* §6.1 *for $\Gamma = \Gamma_\phi$ with an arbitrary $\phi^{(\infty)}$. Then we have*

$$\theta_*(f) = 2c_{\pm,n}\sum_{\alpha\in D^{\pm}_\sigma/\Gamma_\phi,\, D_{\alpha,\mathbb{R}}\cong M_2(\mathbb{R})} \phi^{(\infty)}(\alpha)P_\alpha(f)\mathbf{e}(\pm N(\alpha)\overline{\tau}),$$

where $c_{+,n} = 2\sqrt{-1}^n(n+1)$ and $c_{-,n} = -2\sqrt{-1}(n+1)$ and $P_\alpha(f)$ as in (6.44).

8. Indefinite D with E imaginary

In his section, we assume that D is indefinite. We regard $\mathcal{S} = \mathrm{SO}_{D_\sigma}(\mathbb{Q})\backslash\mathrm{SO}_{D_\sigma}(\mathbb{A})/\widehat{\Gamma}_\phi$ as the automorphic manifold of $G^+_{D_\sigma}$ and D^1_E for the derived group $D^1_E = \mathrm{Ker}(N) \subset G^+_{D_\sigma}$.

8.1. **Analytic theta differential form.** For each $f(g;\mathbf{x}) \in M_k(\Gamma)$, we pick $g \in \mathrm{O}^+_{D_\sigma}(\mathbb{R})$ with $z = g(\varepsilon) \in \mathcal{H}$ and define as in [H94, §2.2] $f_\infty(z;\mathbf{x}) := f(g;\mathbf{x}j(g,\varepsilon)^\iota)$. For $u \in \mathrm{SU}_2(\mathbb{R})$,

$$f_\infty(gu(\varepsilon);\mathbf{x}) = f(gu;\mathbf{x}j(gu,\varepsilon)^\iota) = f(g,\mathbf{x}uj(u,\varepsilon)^\iota j(g,\varepsilon)^\iota) = f(g,\mathbf{x}j(g,\varepsilon)^\iota). \tag{8.1}$$

At the end of [H94, §2.2], there is a typographical error, and "$f(z;{}^tj_J(\gamma,z)\mathbf{x})j_J(\gamma,z)^{k_B}$" should be "$f(z;j_J(\gamma,z)\mathbf{x})j_J(\gamma,z)^{k_B}$". Here the left action $\mathbf{x} \mapsto j(\gamma,z)\mathbf{x}$ there is replaced by the left action $\mathbf{x} \mapsto \mathbf{x}j(\gamma,z)^\iota$. So $f_\infty(z;\mathbf{x})$ is well-defined independent of the choice of g with $g(\varepsilon) = z$.

We compute $\nabla^n\Theta(\tau;g;\mathbf{x})$ ($g \in \mathrm{SO}_{D_0}(\mathbb{R})$) for Θ as in Corollary 6.13, where $\nabla = \frac{\partial^2}{\partial X'\partial Y} - \frac{\partial^2}{\partial X\partial Y'}$. Note $[\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right);\mathbf{x}] = dYX' + bXX' - cYY' - aXY'$, and for $v = \mathfrak{z}1_2 + \mathfrak{x}$ with $\mathfrak{z}1_2 = \left(\begin{smallmatrix} \mathfrak{z} & 0\\ 0 & \mathfrak{z}\end{smallmatrix}\right)$ and $\mathfrak{x} = \left(\begin{smallmatrix} a & b\\ c & -a\end{smallmatrix}\right)$,

$$[v;\mathbf{x}] = [\mathfrak{z}1_2;\mathbf{x}] + [\mathfrak{x};\mathbf{x}] = \mathfrak{z}(YX' - XY') - a(YX' + XY') + bXX' - cYY'.$$

Note $[v;\mathbf{x}g^\iota] = [g^\iota vg;\mathbf{x}]$ for $g \in G^+_{D_0}(\mathbb{R}) = D^\times_{\mathbb{R}}$ and $\nabla^n[v;\mathbf{x}g]^n = N(g)^n\nabla^n[v;\mathbf{x}]^n$ [H94, page 498]. If D is a division algebra, we can choose always $(g,g) \in (D\otimes_{\mathbb{Q}}\mathbb{C})^\times = \mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})$ so that $g^{-1}\mathfrak{x}g = \mathrm{diag}[a,-a]$. Since we can find g so that $g^{-1}\mathfrak{x}g = \mathrm{diag}[a,-a]$ for $\mathfrak{x}$ in a Zariski open non-empty subset of D_0 (and the function $\nabla^n[v;\mathbf{x}]^n$ is a polynomial in v), we may assume that $v = \mathfrak{z}1_2 + \mathrm{diag}[a,-a]$ to compute $\nabla^n[v;\mathbf{x}]^n$. Since $[\mathrm{diag}[a,-a];\mathbf{x}] = -a(YX' + XY')$, we have

$$[v;\mathbf{x}]^n = (\mathfrak{z}(YX' - XY') - a(YX' + XY'))^n = ((\mathfrak{z} - a)YX' - (\mathfrak{z} + a)XY')^n$$
$$= \sum_{j=0}^n \binom{n}{j}(-1)^j(a+\mathfrak{z})^j(\mathfrak{z} - a)^{n-j}(YX')^{n-j}(XY')^j.$$

By [H99, page 141], we have $(n!)^{-2}\nabla^n[(YX')^{n-j}(XY')^j] = (-1)^j\binom{n}{j}^{-1}$, and hence,

$$(n!)^{-2}\nabla^n[\left(\begin{smallmatrix}\mathfrak{z}+a & 0\\ 0 & \mathfrak{z}-a\end{smallmatrix}\right);\mathbf{x}]^n = \sum_{j=0}^{n}(\mathfrak{z}+a)^{n-j}(\mathfrak{z}-a)^j = (\mathfrak{z}+a)^n\sum_{j=0}^{n}\left(\frac{\mathfrak{z}-a}{\mathfrak{z}+a}\right)^j$$
$$= (2a)^{-1}\left[(\mathfrak{z}+a)^{n+1}-(\mathfrak{z}-a)^{n+1}\right] = \sum_{j=0}^{\lceil (n-1)/2\rceil}\binom{n+1}{2j+1}\mathfrak{z}^{n-2j}a^{2j}.$$

Since $(n!)^{-2}\nabla^n[\left(\begin{smallmatrix}\mathfrak{z}+a & 0\\ 0 & \mathfrak{z}-a\end{smallmatrix}\right);\mathbf{x}]^n$ depends only on $\mathfrak{z}$ and $N(\mathfrak{x}) = -a^2$ (as $\mathrm{Tr}(\mathfrak{x}) = 0$), this implies

$$(n!)^{-2}\nabla^n[\mathfrak{z}1_2+\mathfrak{x};\mathbf{x}]^n = \sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\mathfrak{z}^{n-2j}N(\mathfrak{x})^j. \tag{8.2}$$

In view of Corollary 6.13, we need to compute for i_ν in (6.42)

$$i_\nu\circ[\mathfrak{z}1_2+\mathfrak{x};\mathbf{x}] = i_\nu([\mathfrak{z}1_2;\mathbf{x}]+[\mathfrak{x};\mathbf{x}]) = i_\nu(\mathfrak{z}(YX'-XY')-a(YX'+XY')+bXX'-cYY').$$

Since $\mathrm{Ker}(i_\nu) = \mathbb{C}(XX'+YY')$, writing $XX' = \frac{1}{2}((XX'+YY')+(XX'-YY'))$ and $YY' = \frac{1}{2}((XX'+YY')-(XX'-YY'))$, we have from (6.41)

$$i_\nu\circ[\mathfrak{z}1_2+\mathfrak{x};\mathbf{x}] = i_\nu((\mathfrak{z}-a)YX'+(\mathfrak{z}+a)(-XY')+\frac{b}{2}(XX'-YY')+\frac{c}{2}((XX'-YY'))) \tag{8.3}$$
$$= \begin{cases}(\mathfrak{z}+a)y^{-1}dx-(\mathfrak{z}-a)y^{-1}d\overline{x}+\frac{b+c}{2}y^{-1}dy & \text{if } \nu=1,\\ y^{-2}\{(\mathfrak{z}+a)dy\wedge dx+(\mathfrak{z}-a)dy\wedge dx-(b+c)dx\wedge d\overline{x}\} & \text{if } \nu=2.\end{cases}$$

Since $D_{\mathbb{R}}^\times\to D_{E_{\mathbb{R}}}^\times$ induces $\mathfrak{H}\ni x+y\sqrt{-1}\mapsto\left(\begin{smallmatrix}x & -y\\ y & \overline{x}\end{smallmatrix}\right)\in\mathcal{H}$ with $x\in\mathbb{R}$, we find

$$i_\nu\circ[\mathfrak{z}1_2+\mathfrak{x};\mathbf{x}]|_{\mathfrak{H}} = \begin{cases}2ay^{-1}dx+\frac{b+c}{2}y^{-1}dy & \text{if } \nu=1,\\ 2\mathfrak{z}(y^{-2}dy\wedge dx) & \text{if } \nu=2.\end{cases} \tag{8.4}$$

Recall (6.11)

$$\theta(\tau;z;\mathbf{x}) = \eta^{1/2}\sum_{\alpha\in D_\sigma^\pm}\phi^{(\infty)}(\alpha)[\alpha;\mathbf{x}g_z]^{n+1}\mathbf{e}(\pm N(\alpha)\xi+\frac{\eta P[g_z^{-1}\alpha g_z^\sigma]}{2}\sqrt{-1}).$$

As before, we decompose $D_\sigma^\pm = Z^\pm\oplus D_0^\pm$ so that $Z^\pm = \sqrt{\Delta_\pm}\mathbb{Q}\subset D_\sigma^\pm$ with $Q(x) = x^2$ and $L_Z = N\sqrt{\Delta_\pm}\mathbb{Z}$. To compute the factorization of $P[\mathfrak{z}_\infty+\mathfrak{x}_\infty]$, we write $v_\infty = \left(\begin{smallmatrix}a & \sqrt{-1}b\\ \sqrt{-1}c & \pm\overline{a}\end{smallmatrix}\right)$ as in (3.3). Then $\mathfrak{z} = \frac{1}{2}\mathrm{Tr}(v_\infty)$ and $\mathfrak{x} = \left(\begin{smallmatrix}a & \sqrt{-1}b\\ \sqrt{-1}c & -a\end{smallmatrix}\right)$; so, $v_\infty = \left(\begin{smallmatrix}\mathfrak{z}+\sqrt{-1}a & \sqrt{-1}b\\ \sqrt{-1}c & \mathfrak{z}-\sqrt{-1}a\end{smallmatrix}\right) = \mathfrak{z}+\sqrt{-1}\left(\begin{smallmatrix}a & b\\ c & -a\end{smallmatrix}\right)$ where $\mathfrak{z},a,b,c\in\mathbb{R}$. Thus $P[v_\infty]/2 = \mathfrak{z}^2+a^2+b^2+c^2$ and $N(v_\infty) = \mathfrak{z}^2+a^2-bc$. This shows $P_Z = P|_Z = |N|_Z|$ and $P = N\oplus P_0$ for $P_0 = P|_{D_0}$ which is a positive majorant of $s_\pm|_{D_0}$. Thus we have a factorization

$$\mathbf{e}(\pm N(v_\infty)\xi+\frac{\eta P[g^{-1}v_\infty g]}{2}\sqrt{-1}) = \mathbf{e}(\pm N(\mathfrak{z}_\infty)\xi+\eta|N(\mathfrak{z}_\infty)|\sqrt{-1})\mathbf{e}(\pm N(\mathfrak{x}_\infty)+\frac{\eta P_0[g^{-1}\mathfrak{x}_\infty g]}{2}\sqrt{-1}).$$

Suppose that $\phi^{(\infty)} = \phi_Z^{(\infty)}\otimes\phi_0^{(\infty)}$ for $\phi_Z\in\mathcal{S}(Z_{\mathbb{A}^{(\infty)}})$ and $\phi_0\in\mathcal{S}(D_{0,\mathbb{A}^{(\infty)}})$, with $\phi_{Z,\infty}(\mathfrak{z};\tau) = \mathbf{e}(\pm\mathfrak{z}^2\xi+\eta\mathfrak{z}^2\sqrt{-1})$ and $\phi_{0,\infty}(\mathfrak{x};\tau;z) = \mathbf{e}(\pm N(\mathfrak{x})\xi+\frac{\eta P[g_z^{-1}\mathfrak{x}g_z^\sigma]}{2}\sqrt{-1})$, where $g_z\in D_\infty^1$ such that $g_z(\sqrt{-1}) = z\in\mathfrak{H}\subset\mathcal{H}$. Then

$$\begin{aligned}\theta_j(\phi_Z^{(\infty)})(\tau) &= \sum_{\alpha\in Z}\phi_Z^{(\infty)}(\alpha)\alpha^j\mathbf{e}(\pm N(\alpha)\xi+\eta N(\alpha)\sqrt{-1}),\\ \theta_j(\phi_0^{(\infty)})(\tau;z) &= \sum_{\alpha\in D_0}\phi_0^{(\infty)}(\alpha)N(\alpha)^j\mathbf{e}(\pm N(\alpha)\xi+\frac{\eta P_0[g_z^{-1}\alpha g_z^\sigma]}{2}\sqrt{-1}).\end{aligned} \tag{8.5}$$

Since $\Omega^2_{\mathcal{S}/\mathbb{C}}\cong L_E(1;\mathbb{C})\otimes_{\mathbb{C}}\mathcal{O}_{\mathcal{S}}$ and the differential operator $n!^{-2}\nabla^n$ acts on the factor $[v;\mathbf{x}]^n$ of $\iota([v;\mathbf{x}]^{n+1}) = [v;\mathbf{x}]^n\otimes[v;\mathbf{s}]$ under the notation of (6.11), the factor $[v;\mathbf{s}]$ ($v = \mathfrak{z}1_2+\mathfrak{x}$) is sent to $2\mathfrak{z}y^{-2}dy\wedge dx$ over Sh_δ by (8.3).

Proposition 8.1. *Let the assumption and notation be as above. Suppose that $\nu = 2$. Then for the image Sh_δ of $(\Gamma_\phi \cap D^\times)\backslash\mathfrak{H}$ in $\Gamma_\phi\backslash\mathcal{H}$, writing $z = x + y\sqrt{-1}$ and $\overline{z} = x - y\sqrt{-1}$ for the variable on Sh_δ,*

$$(n!)^{-2}\nabla^n\Theta(\phi)(\tau; z)|_{Sh_\delta} = 2\eta^{1/2}\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\theta_{n-2j+1}(\phi_Z^{(\infty)})(\tau)\theta_j(\phi_0^{(\infty)})(\tau; z)y^{-2}dy \wedge dx,$$

which is an analytic 2-form on Sh_δ.

Here we have θ_{n-2j+1} in place of θ_{n-2j} because of (8.4).

To prove Proposition 8.1, we used the realization of the sheaf $L_E^*(n;\mathbb{C})$ (the quotient by the $\mathrm{SU}_2(\mathbb{R})$-action). In [H94], we used $\widetilde{L}_E(n;\mathbb{C})$ (the quotient by the Γ_ϕ-action). Of course the outcome is the same but the computation via $\widetilde{L}_E$ is more complicated. In the following remark, we shall give a brief outline of the use of $\widetilde{L}_E(n;\mathbb{C})$ without the computation of its image under $(n!)^{-2}\nabla^n$. For $\gamma \in \Gamma_\phi$, we have $\theta(\phi)(\tau; \gamma g; \mathbf{x}) = \theta(\tau; g; \mathbf{x})$. Thus

$$\begin{aligned}\theta_\infty(\gamma(z); \mathbf{x}) = \theta(\tau; \gamma g; \mathbf{x}j(\gamma g, \varepsilon)^\iota) &= \theta(\tau; g; \mathbf{x}j(\gamma g, \varepsilon)^\iota)\\ &= \theta_\infty(z; \mathbf{x}j(g, \varepsilon)^\iota j(\gamma, z)^\iota) = \rho_{n^*}(j(\gamma, z)^\iota)\theta_\infty(\tau; z; \mathbf{x}).\end{aligned}$$

Here again the remark after (8.1) applies again and ${}^tj(\gamma, z)$ in [H94] is replaced by $j(\gamma, z)^\iota$.

Remark 8.2. Regard $L_E(n;\mathbb{C}) = L(n^*;\mathbb{C}) = \sum_{j=0}^{n^*}\mathbb{C}S^{n^*-j}T^j$ as a $\mathrm{SU}_2(\mathbb{R})$ module for the coordinate vector $\mathbf{s}^* = {}^t(S^{n^*}, S^{n^*-1}T, \cdots, T^{n^*})$, and define the matrix expression $\rho_{n^*}(g)$ of symmetric n^*-power by

$$\rho_{n^*}\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\mathbf{s}^{n^*} = {}^t(aS + bT, cS + dT)^{n^*}.$$

Now following [H94, (2.8a–e)], we express $\theta(\tau; g; \mathbf{x}) = \theta(\tau; g) : \mathrm{O}^+_{D_\sigma}(\mathbb{R}) \to L(n^*;\mathbb{C})$ as $\{\theta_j : \mathrm{O}^+_{D_\sigma}(\mathbb{R}) \to L_E(n;\mathbb{C})\}_{j=0,\dots,n^*}$ by $\sum_j S^{n^*-j}T^j\theta_j(\tau; g; \mathbf{s}; \mathbf{x}) = \theta'(\tau; g)\cdot\mathbf{s}^{n^*}$ via the coordinate $\mathbf{s}^{n^*}$. Take the polynomial column vector $\Psi(\mathbf{x}; \mathbf{a}) \in \mathbb{Z}[X, Y, X', Y', A, B]^{n^*}$ as in [H94, (2.8a–b)] and put $\theta(\tau, g; \mathbf{x}; \mathbf{a}) = \sum_{i=0}^2\theta^{(i)}(\tau; g; \mathbf{x})A^{2-i}B^i = \theta'(\tau; g)\cdot\Psi(\mathbf{x}; \mathbf{a})$. Then we define a differential form with values in $\widetilde{L}_E(n;\mathbb{C}) \otimes \Omega^2_{\mathcal{S}/\mathbb{C}}$ by

$$\widetilde{\Theta}(\tau, z; \mathbf{x}) = y^{-1}(\theta^{(0)}(\tau; g; \mathbf{x})dy \wedge dx - 2\theta^{(1)}(\tau; g; \mathbf{x})dx \wedge d\overline{x} + \theta^{(2)}(\tau; g; \mathbf{x})dy \wedge d\overline{x})$$

for $g \in \mathrm{O}^+_{D_\sigma}(\mathbb{R})$ with $g(\varepsilon) = z$. Then we have by [H94, Proposition 2.1]

$$\textit{For } \gamma \in \Gamma_\phi,\ \gamma^*\widetilde{\Theta}(\tau; g; \mathbf{x}) := \widetilde{\Theta}(\tau; \gamma(z); \mathbf{x}) = \widetilde{\Theta}(\tau; z; \mathbf{x}\gamma).$$

The direct computation of $\nabla^n\widetilde{\Theta}(\tau; g; \mathbf{x})$ is more involved, and anyway $\nabla^n\widetilde{\Theta} = \nabla^n\Theta$ as they have values in the constant sheaf; so, it follows from Proposition 8.1. We omit the details.

8.2. **L-value formula in the indefinite imaginary case.** In this subsection, we assume that D is division indefinite. We now assume that D is an indefinite quaternion algebra over $\mathbb{Q}$. Recall the decomposition $D^\pm_\sigma = Z^\pm \oplus D^\pm_0$ so that $Z^\pm = \delta_\pm\mathbb{Q} \subset D^\pm_\sigma$ with $Q(x) = x^2$ and $L_Z = N\delta_\pm\mathbb{Z}$. We take $\phi_Z^{(\infty)} := \psi$ on $L_Z^*/L_Z = N^{-1}\mathbb{Z}/N\mathbb{Z}$ for a Dirichlet character $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. We take an Eichler order $R_E(N_0)$ in D_E for N_0 prime to ∂. Then let $\phi_0^{(\infty)} \in \mathcal{S}(D_{0,\mathbb{A}^{(\infty)}})$ be as in (4.25) for the characteristic function $\phi_{\widehat{L}}$ of $\widehat{L} := \widehat{R}_0(N_0) \cap D_{0,\mathbb{A}^{(\infty)}}$. Again Remark 4.5 applies. We put $\phi^{(\infty)} = \psi \otimes \phi_0^{(\infty)}$ and

$$\phi(v) := \phi^{(\infty)}(v^{(\infty)})[v_\infty; \mathbf{x}]^{n+1}\mathbf{e}(\pm N(v_\infty)\xi + \frac{P_I[v_\infty]}{2}\eta\sqrt{-1}) \tag{8.6}$$

for $P_I(x, y)$ as in (6.4). Note

$$\begin{aligned}&\mathbf{r}_Z(g_\tau)L_Z(g)(\mathfrak{z}_\infty^j\mathbf{e}(\mathfrak{z}_\infty^2\sqrt{-1})) = \eta^{(1+2j)/4}\mathfrak{z}_\infty^j\mathbf{e}(\pm N(\mathfrak{z}_\infty)\xi + |N(\mathfrak{z}_\infty)|\eta\sqrt{-1}),\\ &\mathbf{r}_{D_0}(g_\tau)L_{D_0}(g)(N(\mathfrak{x})^j\mathbf{e}(N(\mathfrak{x}_\infty)\sqrt{-1})) = \eta^{j+3/4}N(\mathfrak{x})^j\mathbf{e}(\pm N(\mathfrak{x}_\infty)\xi + \frac{P_0(g^{-1}\mathfrak{x}g)}{2}\eta\sqrt{-1})\end{aligned} \tag{8.7}$$

for $P_0 = P_I|_{D_0}$. Recall again $\boldsymbol{\theta}(\phi_j^Z)(\tau) = \sum_{\alpha\in Z}(\mathbf{w}(g_\tau)\phi_j^Z)(\alpha)$.

By Proposition 8.1, $(n!)^{-2}\nabla^n\Theta(\phi)(\tau;z)|_{Sh_\delta}$ is equal to

$$2\eta^{1/2}\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\theta_{n-2j+1}(\phi_Z^{(\infty)})(\tau)\theta_j(\phi_0^{(\infty)})(\tau;z)y^{-2}dy\wedge dx,$$

and therefore it is equal to

$$2\eta^{1/2}\eta^{-1-(n/2)}\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\boldsymbol{\theta}_{n-2j+1}(\phi_Z^{(\infty)})(\tau)\boldsymbol{\theta}_j(\phi_0^{(\infty)})(\tau;z)y^{-2}dy\wedge dx. \tag{8.8}$$

We now compute $\eta^{k/2}\int_{Sh_\delta}(n!)^{-2}\nabla^n\Theta(\phi)(\tau;z)y^{-2}dy\wedge dx$ which is equal to

$$2\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\int_{Sh_\delta}\eta^{1/2}\boldsymbol{\theta}(\phi^Z_{n-2j+1})\boldsymbol{\theta}(\phi_j^{D_0})d\mu_g$$
$$=2\mathfrak{m}\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\eta^{1/2}\boldsymbol{\theta}(\phi^Z_{n-2j+1})E(\phi_j^{D_0}),$$

where $\mathfrak{m}$ is as in (4.23) and for $\Phi\in\mathcal{S}(D_{0,\mathbb{A}})$ and $g\in\mathrm{Mp}(\mathbb{A})$,

$$E(\Phi)=\sum_{\gamma\in B(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{Q})}|a(g)|^{s-(1/2)}(\mathbf{w}(\gamma g)\Phi)(0)|_{s=\frac12}.$$

Take $F=\sum_{m=1}^\infty a_m\mathbf{e}(-m\tau^{\mp})\in S_k^{\mp}(M,\psi^{-1}\chi_{D_\sigma})$ for M as in (4.26). In the same manner as getting (4.32) from (4.29) and (4.31), for $\overline{B}:=B(\mathbb{Q})\backslash B(\mathbb{A})C_\infty/B(\widehat{\mathbb{Z}})C_\infty$,

$$2\sum_{j=0}^{\lceil (n-1)/2\rceil}(-1)^j\binom{n+1}{2j+1}\int_{\overline{B}}F(\tau)\eta^{k/2}\boldsymbol{\theta}(\phi^Z_{n-2j+1})(g_\tau)\eta^{1/2}\mathbf{r}(g_\tau)(\phi_j^{D_0})(0)d\mu_\tau \tag{8.9}$$
$$=2(n+1)\int_{\overline{B}}F(\tau)\eta^{1/2}\theta(\phi^Z_{n+1})(g_\tau)\eta^{k-2}d\xi d\eta=2(k-1)\int_0^\infty\int_0^1F(\tau)\theta(\phi^Z_{n+1})(g_\tau)d\xi\eta^{k-(3/2)}d\eta$$
$$=4(k-1)\delta_\pm^{k-1}\int_0^\infty\sum_{m>0}\psi(m)m^{k-1}\exp(-4\pi|\Delta_\pm|m^2\eta)\eta^{k-(3/2)}d\eta$$
$$=4(4\pi)^{(1/2)-k}(k-1)\delta_-^{k-1}|\Delta_\pm|^{-k+(1/2)}\Gamma(k-\frac12)\sum_{0<m\in\mathbb{Z}}\psi(n)a_{m^2}m^{-k}.$$

As before $\delta_-^{k-1}|\Delta_-|^{-k+(1/2)}=-|\delta_-|^{-k}\sqrt{-1}$. Thus we get, in the same manner as in Theorem 4.7,

Theorem 8.3. *Suppose that E is imaginary and D is division indefinite. Let ϕ be a Schwartz-Bruhat function of $D^\pm_{\sigma,\mathbb{A}}$ as in (8.6). Choose $\phi_Z^{(\infty)}$ associated to Dirichlet character ψ of conductor $C(\psi)$ and $\phi_0^{(\infty)}$ as in (4.25). Let F be a primitive Hecke eigenform in $S_k^{\mp}(C,\psi^{-1}\chi_{D_\sigma})$ for the conductor C as in (4.26) and define the $L_E(n;\mathbb{C})$-valued harmonic form by*

$$\omega(F)=\int_{\Gamma_\tau\backslash\mathfrak{H}}\Theta(\phi)(\tau;g)F(\tau)\eta^{k-2}d\xi d\eta$$

for the analytic differential form $\Theta(\phi)$ as in Proposition 8.1. *Then if $\omega(F)\neq 0$, for the mass factor $\mathfrak{m}_1$ as in* (4.23) *and $E^\pm(1)$ as in* Theorem 4.7,

$$\pi\int_{Sh_\delta}(n!)^{-2}\nabla^n\omega(F)=\mathfrak{m}_1E^\pm(1)\delta_\pm^{k-1}|\Delta_\pm|^{-k+\frac12}4(4\pi)^{\frac12-k}(k-1)\Gamma(k-\frac12)L^{(C_s(\psi))}(1,Ad(F)\otimes\chi_E)$$

for Sh_δ as in (4.22) *with $\Gamma_\delta:=\{u\in\mathrm{O}_\delta(\mathbb{A}^{(\infty)})|\phi^{(\infty)}\circ u=\phi^{(\infty)}\}$. The measure is induced by $y^{-2}dxdy$ identifying $Sh_\delta=D^\times\backslash D^\times_{\mathbb{A}}/\widehat{\Gamma}_\phi\mathbb{A}^\times C_\infty(D^\times_\infty)$ with $\Gamma_\phi\backslash\mathcal{H}$.*

8.3. **Fourier coefficients of theta descent.** We assume $D_{\alpha,\mathbb{R}} \cong M_2(\mathbb{R})$, and we include the case where $D \cong M_2(\mathbb{Q})$ (since the coefficient of $\mathbf{e}(\pm N(\alpha)\tau)$ vanishes if $D_{\alpha,\mathbb{R}} \cong \mathbb{H}$ as seen in §7.3). Though the line of the argument is almost identical to the one in Case ID, we give some details as we need to make subtle modifications.

We first interpret the integral for the Schwartz function as in (6.21)

$$\int_{\Gamma_\alpha\backslash\mathcal{H}}([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(\pm N(v)\tau \mp \frac{[z;v]^2\eta\sqrt{-1}}{2y(z)^2})d\mu_z$$

as an integral of differential form. In this case $|\Gamma_\alpha| = \infty$ with finite volume $\Gamma_\alpha\mathfrak{H}$. By (7.12) in which we replace the Schwartz function in (6.18) by the one in (6.21) in Case II

$$\theta_*(f)(\tau) = \eta^{1/2}\sum_{\alpha\in D_\sigma^\pm/\Gamma_\phi}\phi^{(\infty)}(\alpha)\int_{\Gamma_\alpha\backslash\mathcal{H}}([\alpha;\mathbf{x}]^{n+1}, f(z;\mathbf{s}^*))\mathbf{e}(\pm N(v)\tau \mp \frac{[z;v]^2\eta\sqrt{-1}}{2y^2})d\mu_z. \tag{8.10}$$

From (8.10) and Lemma 6.14 combined with Proposition 7.2, $\eta^{-1/2}\theta_*(f)(\tau)$ is equal to

$$\sum_{\alpha\in D_\sigma^\pm/\Gamma_\phi, D_{\alpha,\mathbb{R}}\cong M_2(\mathbb{R})}\phi^{(\infty)}(\alpha)\int_{\Gamma_\alpha\backslash\mathcal{H}} y[i_{\nu,*}([\alpha;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f(z;\mathbf{s}^*))]\mathbf{e}(\pm N(v)\tau \mp \frac{[z;v]^2\eta\sqrt{-1}}{2y^2})d\mu_z.$$

Choose $j \in \mathrm{SL}_2(\mathbb{C})$ as in Lemma 6.7 so that $D_{\alpha,\mathbb{R}} = j^{-1}M_2(\mathbb{R})j$, where $\alpha = j^{-1}\beta^\pm j^{\sigma_1}$ with $\beta^+ = 1$ if $\alpha \in D_\sigma^+$ and $\beta^- = \sqrt{-1}$ if $\alpha \in D_\sigma^-$.

Assuming $D_{\alpha,\mathbb{R}} \cong M_2(\mathbb{R})$, we need to compute

$$I^\pm := \int_{\Gamma_\alpha\backslash\mathcal{H}} y(z)[i_{\nu,*}([\alpha;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f(z;\mathbf{s}^*))]\mathbf{e}(\mp\frac{[z;\alpha]^2\eta\sqrt{-1}}{2y(z)^2})d\mu_z. \tag{8.11}$$

Similarly to (7.17), we have

$$\pm N(\alpha) = \pm N(j^{-1}\beta^\pm j^{\sigma_1}) > 0. \tag{8.12}$$

By (6.1) in Case II, we have $[1;\mathbf{x}] = YX' - XY'$; so,

$$y(z)[i_{\nu,*}([1;\mathbf{x}]^{n+1}), i_{3-\nu,*}(f|_G j^{-1}(g_z;\mathbf{s}^*))] = y(z)[i_{\nu,*}((YX'-XY')^{n+1}), i_{3-\nu,*}(f|_G j^{-1}(g_z;\mathbf{s}^*))].$$

Now by the same computation in §7.4 starting from (7.18) ending by (7.24), replacing J by 1, we reach the equation (7.24) which we repeat:

$$I_{2,*}(n!^{-2}\nabla^n\iota(f^j(g;\mathbf{s}^*))) = -f_0^j y^{-2}(dy\wedge d\overline{x}) + 2f_1^j y^{-2}(d\overline{x}\wedge dx) - f_2^j y^{-2}(dy\wedge dx) = (n!)^{-2}\nabla^n\omega_2(f^j).$$

For $P_\alpha(f)$ in (6.44), we get

$$\begin{aligned}I^\pm = c_{\mp,n}\int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}}\mathbf{e}(2v^2\eta\sqrt{-1})dv\cdot\int_{\Gamma_\alpha^j\backslash\mathrm{SL}_2(\mathbb{R})(\varepsilon)}(f_0^j + f_2^j)y^{-2}dy\wedge du\\ = c_{\mp,n}\int_{\mathrm{SL}_2(\mathbb{R})\backslash\mathcal{H}}\mathbf{e}(2v^2\eta\sqrt{-1})dv\cdot P_\alpha(f).\end{aligned}$$

where $c_{+,n} = 2\sqrt{-1}^n(n+1)$ and $c_{-,n} = -2\sqrt{-1}(n+1)$. Thus

$$I^\pm = 2c_{\pm,n}\eta^{-1/2}\cdot P_\alpha(f). \tag{8.13}$$

Thus we conclude

Theorem 8.4. *Suppose that $f : \mathcal{H} \to L(n^*;\mathbb{C})$ is a cusp form on $\mathrm{SO}_{D_\sigma}(\mathbb{A})$ of weight $k = n+2 > 0$ satisfying* (M1–3) *in* §6.1 *for $\Gamma = \Gamma_\phi$ with an arbitrary $\phi^{(\infty)}$. Then we have*

$$\theta_*(f) = 2c_{\mp,n}\sum_{\alpha\in D_\sigma^\pm/\Gamma_\phi, D_{\alpha,\mathbb{R}}\cong M_2(\mathbb{R})}\phi^{(\infty)}(\alpha)P_\alpha(f)\mathbf{e}(\pm N(\alpha)\tau),$$

where $c_{+,n} = 2\sqrt{-1}^n(n+1)$ and $c_{-,n} = -2\sqrt{-1}(n+1)$ and $P_\alpha(f)$ as in (6.44).

References

Books

[AQF] G. Shimura, *Arithmetic of Quadratic Forms*, Springer, 2010.

[BNT] A. Weil, *Basic Number Theory*, Springer, 1974.

[EMI] H. Hida, *Elementary Modular Iwasawa Theory, World Scientific*, World Scientific, 2022.

[HMI] H. Hida, *Hilbert modular forms and Iwasawa theory*, Oxford University Press, 2006.

[IAT] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton University Press and Iwanami Shoten, 1971, Princeton-Tokyo.

[LFE] H. Hida, *Elementary Theory of L–functions and Eisenstein Series*, LMSST **26**, Cambridge University Press, Cambridge, 1993.

[MSS] S. S. Kudla, M. Rapoport and T. Yang, *Modular Forms and Special Cycles on Shimura Curves*, Annals of Math. Studies **161**, 2006.

[PAF] H. Hida, *p-Adic Automorphic Forms on Shimura Varieties*, Springer Monographs in Mathematics, Springer, 2004.

[SL2] S. Lang, $\mathrm{SL}_2(\mathbf{R})$, Graduate Texts in Mathematics, **105**, Springer-Verlag, New York, 1985.

[WRS] S. S. Gelbart, *Weil's Representation and the Spectrum of the Metaplectic Group*, Lecture notes in Math. **530** (1976), Springer.

Articles

[A78] T. Asai, On the Doi-Naganuma lifting associated with imaginary quadratic fields, Nagoya Math. J. **71** (1978), 149–167.

[DHI] K. Doi, H. Hida and H. Ishii, Discriminant of Hecke fields and twisted adjoint L–values for $GL(2)$, Inventiones Math. **134** (1998), 547–577.

[F83] S. Friedberg, On the imaginary quadratic Doi-Naganuma lifting of modular forms of arbitrary level, Nagoya Math. J. **92** (1983), 1–20.

[GT16] W.-T. Gan and S Takeda, A proof of the Howe duality conjecture, Journal of the American Mathematical Society **29** (2016), 473–493.

[Ha05] J. Hanke, An exact mass formula for quadratic forms over number fields, J. Reine Angew. Math. **584** (2005), 1–27.

[H88] H. Hida, On p–adic Hecke algebras for GL_2 over totally real fields, Ann. of Math. **128** (1988), 295–384.

[H94] H. Hida, On the critical values of L–functions of $GL(2)$ and $GL(2) \times GL(2)$, Duke Math. J. **74** (1994), 431–529.

[H99] H. Hida, Non-critical values of adjoint L–functions for $SL(2)$, Proc. Symp. Pure Math. **66** (1999) Part I, 123–175.

[H05] H. Hida, The integral basis problem of Eichler, International Mathematics Research Notices, Vol. **2005**, no.34, 2101–2122.

[H06] H. Hida, Anticyclotomic main conjectures, Documenta Math. Volume Coates (2006), 465–532.

[H10] H. Hida, Central critical values of modular Hecke L-functions, Kyoto Journal of Mathematics, **50** (2010), 777–826.

[H22] H. Hida, The universal ordinary deformation ring associated to a real quadratic field. Proc. Indian Acad. Sci. Math. Sci. **132** (2022), no. 1, Paper No. 17, 51 pp.

[K67] T. Kubota, Topological covering of $SL(2)$ over a local field, J. Math. Soc. Japan **19** (1967), 114–121.

[K78] S. S. Kudla, Theta-functions and Hilbert modular forms, Nagoya Math. J. **69** (1978), 97–106.

[K84] S. S. Kudla, Seesaw dual reductive pairs, Automorphic forms of several variables (Katata, 1983), Progr. Math., **46** (1984) 244-268.

[O77] T. Oda, On modular forms associated with indefinite forms of signature $(2, n-2)$, Math. Ann. **231** (1977), 97—144.

[Sh65] H. Shimizu, On Zeta Functions of Quaternion Algebras, Annals of Math. **81** (1965), 166–193.

[Sh73] G. Shimura, On modular forms of half integral weight, Annals of Math. **97** (1973), 440–481.

[Sh75] G. Shimura, On the holomorphy of certain Dirichlet series. Proc. London Math. Soc. (3) **31** (1975), 79-98.

[Sh81] G. Shimura, On certain zeta functions attached to two Hilbert modular forms: I. The case of Hecke characters, II. The case of automorphic forms on a quaternion algebra, I: Annals of Math. **114** (1981), 127–164; II: ibid. 569–607.

[Sh82] G. Shimura, The period of certain automorphic forms of arithmetic type, J. Fac. Sci. Univ. Tokyo, Sec. IA **28** (9182), 605–632.

[Sh99] G. Shimura, An exact mass formula for orthogonal groups, Duke Math. J. **97** (1999), 1–66.

[S75] T. Shintani, On construction of holomorphic cusp forms of half integral weight. Nagoya Math. J. **58** (1975), 83–126.

[Sw90] W. J. Sweet, The metaplectic case of the Weil–Siegel formula, Thesis (Ph.D.)-University of Maryland, College Park. 1990, 214 pp.

[TU22] J. Tilouine and E. Urban, Integral Period Relations and Congruences, to appear in Algebra & Number Theory.

[W64] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. **111** (1964), 143-211.

[W65] A. Weil, Sur la formule de Siegel dans la théorie des groupes classiques, Acta Math. **113** (1965), 1–87.

[W81] J.-L. Waldspurger, Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. Pures Appl. (9) **60** (1981), 375–484.

[W85] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L-automorphes en leur centre de symétrie. Compositio Mathematica, **54** (1985), 173–242.

[W90] J.-L. Waldspurger, Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. **2**, Weizmann, Jerusalem, 1990, pp. 267–324.

[Y95] H. Yoshida, On a conjecture of Shimura concerning periods of Hilbert modular forms, American J. Math **117** (1995), 1019–1038.

Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, U.S.A.
E-mail address: `hida@math.ucla.edu`