Control of Pro-Limit Mordell–Weil groups.

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Abstract: We give a control theorem of partially ordinary factor of modular jacobians. Then we prove almost constancy of Mordell–Weil rank of Shimura's abelian variety quotients moving along an slope 0 analytic family. We fix a prime p assumed ≥ 5 for simplicity.

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§1. The $U(p)$-operators. We have a well known commutative diagram of $U(p^{s-r})$-operators:

$$\begin{array}{ccc} J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R} \\ \downarrow u & \swarrow u' & \downarrow u'' \\ J_{r,R} & \xrightarrow{\pi^*} & J^r_{s,R}, \end{array} \qquad (1)$$

where the middle u' is given by $U^s_r(p^{s-r})$ and u and u'' are $U(p^{s-r})$. These operators comes from $\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{s-r} \end{smallmatrix}\right) \Gamma'$ for $u : \Gamma = \Gamma_r = \Gamma'$, $u' : \Gamma = \Gamma^r_s, \Gamma' = \Gamma_r$ and $u : \Gamma = \Gamma^r_s = \Gamma'$.

Note that $U(p^n) = U(p)^n$. Define an idempotent $e := \lim_{n\to\infty} U(p)^{n!}$ as an endomorphism of p-torison group $J_r[p^\infty]$ and $J^r_s[p^\infty]$. Then the above diagram implies

$$J_{r/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \cong J^r_{s/\mathbb{Q}}[p^\infty]^{\mathrm{ord}} \quad \text{and} \quad J^{\mathrm{ord}}_{r/\mathbb{Q}}(k) \cong J^{r,\mathrm{ord}}_{s/\mathbb{Q}}(k),$$

where "ord" indicates the image under e.

§2. fppf cohomology.

Since we have $J^{\mathrm{ord}}_{r/\mathbb{Q}}(k) \cong J^{r,\mathrm{ord}}_{s/\mathbb{Q}}$, to get a control of $J^{\mathrm{ord}}_{r/\mathbb{Q}}(k)$, we need to replace the cohomology group in the above commutative diagram by something else.

Suppose that we have morphisms of three varieties schemes $X \xrightarrow{\pi} Y \xrightarrow{g} S = \mathrm{Spec}(k)$. Then we get, for $?_T = ? \times_S T$,

$$\mathrm{Pic}_{X/S}(T) = H^1_{\mathrm{fppf}}(X_T, O^\times_X)$$
$$\mathrm{Pic}_{Y/S}(T) = H^1_{\mathrm{fppf}}(Y_T, O^\times_{Y_T})$$

for any S-scheme T with $H^1_{\mathrm{fppf}}(T, O^\times_T) = 0$ (for example $T = \mathrm{Spec}(K)$ for a field). We suppose that the functors $\mathrm{Pic}_{X/S}$ and $\mathrm{Pic}_{Y/S}$ are representable by smooth group schemes. We then put $J_? = \mathrm{Pic}^0_{?/S}$ ($? = X, Y$). We apply this to $X = X_s$ and $Y = X^r_s$ with cuspidal ∞-sections.

§**3. A spectral sequence under fppf topology.** For an fppf covering $\mathcal{U} \to Y$ and a presheaf $P = P_Y$ on the fppf site over Y, we define via Čech cohomology an fppf presheaf $\mathcal{U} \mapsto \check{H}^q(\mathcal{U}, P)$ denoted by $\underline{\check{H}}^q(P_Y)$. The inclusion functor from the category of fppf sheaves over Y into the category of fppf presheaves over Y is left exact. The derived functor of this inclusion of an fppf sheaf $F = F_Y$ is denoted by $\underline{H}^\bullet(F_Y)$ (see Milne III.1.5 (c)). Thus $\underline{H}^\bullet(\mathbb{G}_{m/Y})(\mathcal{U}) = H^\bullet_{\mathsf{fppf}}(\mathcal{U}, O^\times_{\mathcal{U}})$ for a Y-scheme $\mathcal{U}$ as a presheaf.

Assuming that f, g and π are all faithfully flat of finite presentation, we use the spectral sequence of Čech cohomology of the flat covering $\pi : X \twoheadrightarrow Y$ in the fppf site over Y (Milne III.2.7):

$$\check{H}^p(X_T/Y_T, \underline{H}^q(\mathbb{G}_{m/Y})) \Rightarrow H^n_{\mathsf{fppf}}(Y_T, O^\times_{Y_T}) \tag{2}$$

for each S-scheme T.

§**4. New fppf commutative diagram.** Suppose that $S = \mathrm{Spec}(k) = T$. We have $H^1(Y, O_Y^\times) \cong \mathrm{Pic}_{Y/S}(T)$. From this spectral sequence, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\check{H}^1(\underline{H}_Y^0) & \hookrightarrow & H^1(Y, O_Y^\times) & \to & \check{H}^0(\frac{X}{Y}, \underline{H}^1(\mathbb{G}_{m,Y})) & \to & \check{H}^2(\underline{H}_Y^0) \\
\uparrow & & \wr\uparrow & & \|\uparrow & & \|\uparrow \\
\check{H}^1(\underline{H}_Y^0) & \to & \mathrm{Pic}_{Y/S}(k) & \to & \check{H}^0(\frac{X}{Y}, \mathrm{Pic}_Y(k)) & \to & \check{H}^2(\underline{H}_Y^0) \\
\uparrow & & \cup\uparrow & & \cup\uparrow & & \uparrow \\
?_1 & \to & J_Y(k) & \to & \check{H}^0(\frac{X}{Y}, J_X(k)) & \to & ?_2,
\end{array}$$

where we have written $J_? = \mathrm{Pic}^0_{?/S}$. Note that

$$\check{H}^0(\frac{X}{Y}, J_X(k)) = J_s[\gamma^{p^{r-1}} - 1](k).$$

§5. Control of ordinary Mordell–Weil groups.

By an explicit computation of Čech cohomology (which we recall later if time allows), from $\deg(U(p)) = p$, we get

Key Lemma. $U(p)^m(\check{H}^q(\underline{H}^0_Y)) = 0$ *for* $m \gg 0$*.*

Thus we get

Theorem 1 (Control)**.** *We have*

$$J_s^{\mathrm{ord}}(k)[\gamma^{p^{r-1}}-1] \cong J_r^{\mathrm{ord}}(k) \text{ and } (J_s/(\gamma^{p^{r-1}}-1)(J_s))^{\text{co-ord}} \cong J_r^{\text{co-ord}}.$$

Here $J_s^{\mathrm{ord}}(k)[\gamma^{p^{r-1}} - 1] := \mathrm{Ker}(\gamma^{p^{r-1}} - 1 : J_s^{\mathrm{ord}}(k) \to J_s^{\mathrm{ord}}(k))$ and the second identity is the **sheaf** identity.

§6. Shimura's abelian variety quotients.

A prime $P \in \mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$ is called *arithmetic* of weight 2 if P factors through $\mathrm{Spec}(\mathbb{Z}_p[\Gamma/\Gamma^{p^r}])$ for some $r \geq 0$. Associated to P is a unique Hecke eigenform of weight 2 on $X_1(Np^r)$ for some minimal $r > 0$. Write B_P (resp. A_P) for the Shimura's abelian quotient (resp. abelian subvariety) associated to f_P of the jacobian J_s (for $s \geq r$). Note that $A_P \subset J_s$ and $J_s \twoheadrightarrow B_P$ are stable under $w_s = \left(\begin{smallmatrix} 0 & -1 \\ Np^s & 0 \end{smallmatrix}\right)$.

If $s > r$, f_P regarded as $S_2(\Gamma_s)$ is still Hecke eigenform; so, $\pi^* : J_r \to J_s$ send A_P of level Np^r isogenous to A_P of level Np^s. The Albanese map $\pi_* : J_s \twoheadrightarrow J_r$ is $T^*(n)$-equivariant; so, B_P of level Np^s does not cover by π_* the B_P of level p^r.

This causes some problems.

§7. Albanese functriality.

Let $\iota_s : C_{s/\mathbb{Q}} \subset J_{s/\mathbb{Q}}$ be an abelian subvariety stable under $T(n)$, $U(l)$ and w_s and ${}^t\iota : J_{s/\mathbb{Q}} \twoheadrightarrow {}^tC_{s/\mathbb{Q}}$ be the dual abelian quotient. We then define $\pi : J_s \twoheadrightarrow D_s$ by $D_s := {}^tC_s$ and $\pi = {}^tw_s \circ {}^t\iota_s \circ w_s$ for the map ${}^tw_s \in \mathrm{Aut}({}^tC_{s/\mathbb{Q}[\mu_{p^s}]})$ dual to $w_s \in \mathrm{Aut}(C_{r/\mathbb{Q}[\mu_{p^s}]})$. Then ι and π are defined over $\mathbb{Q}$ and Hecke equivariant (i.e., $T(n)$-equivariant).

Taking C_s to be J_r $r \leq s$, we write π^r_s for $\pi : J_s \to J_r$ and put

$$\widehat{J}^{\mathrm{ord}}_{\infty}(k) = \varprojlim_r J^{\mathrm{ord}}_r(k) \quad \text{with respect to } \pi^r_s.$$

Now let $\iota_s : A_{P,s} := \pi^*(A_P) \subset J_s$ for $s > r$ and $B_{P,s}$ be the quotient abelian variety of J_s.

Then $\pi^* : A^{\mathrm{ord}}_P \cong A^{\mathrm{ord}}_{P,s}$ and $\pi^r_s : B^{\mathrm{ord}}_{P,s} \cong B^{\mathrm{ord}}_P$.

§8. An identity.

By computation, $\pi_s^r \circ \pi_{r,s}^* = p^{s-r}U(p^{s-r})$. To see this, as Hecke operators coming from Γ_s-coset, $\pi_{r,s}^* = [\Gamma_s 1 \Gamma_r]$ (restriction) and $\pi_{r,s,*} = [\Gamma_r]$ (trace). Thus we have

$$\begin{aligned}\pi_s^r \circ \pi_{r,s}^*(x) = x|[\Gamma_s] \cdot w_s \cdot [\Gamma_r] \cdot w_r = x|[\Gamma_s 1 \Gamma_s^r][\Gamma_s^r \cdot [w_s w_r] \cdot [\Gamma_r] \\ = x|[\Gamma_s^r : \Gamma_s][\Gamma_s^r \begin{pmatrix} 1 & 0 \\ 0 & p^{s-r} \end{pmatrix} \Gamma_r] = p^{s-r}(x|U(p^{s-r})).\end{aligned}$$

Corollary 1. *We have the following two commutative diagram for $s' > s$*

$$\begin{array}{ccc} A_{P,s'}^{\mathrm{ord}} & \xleftarrow[\pi_{s,s'}^*]{\sim} & A_{P,s}^{\mathrm{ord}} \\ {\scriptstyle \pi_{s'}^{s}}\downarrow & & \downarrow{\scriptstyle p^{s'-s}U(p)^{s'-s}} \\ A_{P,s}^{\mathrm{ord}} & = & A_{P,s}^{\mathrm{ord}}. \end{array}$$

§9. **Naive control.** Pick a height 1 principal prime $P = (\varpi) \in \mathrm{Spec}(\mathbb{T})$. Suppose $\varpi | (\gamma^{p^r} - 1)$ (prime P with this property is called an *arithmetic* prime). The control result and the corollary before tells us the exactness of

$$0 = \varprojlim_s A^{\mathrm{ord}}_{P,s}(k) \to \widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}} \xrightarrow{\varpi} \widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}} \to B^{\mathrm{ord}}_P(k)$$

is exact. Suppose that $\mathbb{T}$ is an integral domain. We expect that

$$\mathrm{rank}\, B_P(k) = R \cdot [\mathbb{Q}(f_P) : \mathbb{Q}]$$

for almost all P. Here $\mathbb{Q}(f_P)$ is the field generated by the Hecke eigenvalues of f_P (the Hecke field of f_P).

This follows if $E^\infty(k) := \mathrm{Coker}(\widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}} \to B^{\mathrm{ord}}_P(k))$ is finite for $R = \mathrm{rank}_{\mathbb{T}}\, \widehat{J}^{\mathrm{ord}}_\infty(k)_{\mathbb{T}}$.

§10. Control theorem.

Theorem 2. *For almost all principal arithmetic prime* $P = (\varpi) \in \mathrm{Spec}(\mathbb{T})$*, we have the following exact sequence (of* p*-profinite* Λ*-modules):*

$$0 \to \widehat{J}^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \xrightarrow{\varpi} \widehat{J}^{\mathrm{ord}}_{\infty,\mathbb{T}}(k) \xrightarrow{\rho_\infty} B^{\mathrm{ord}}_P(k) \to E^{\infty}_2(k) \to 0.$$

If k *is a number field, the error term* $E^{\infty}_2(k)$ *is finite under* $|\underline{\mathrm{III}}^1(k^S/k, T_pB^{\mathrm{ord}}_P)| < \infty$*. If* $k/\mathbb{Q}_l$ *is a finite extension with* $l \neq p$*, for any principal* (ϖ)*,* $E^{\infty}_2(k) = 0$*, and if* $k/\mathbb{Q}_p$ *is a finite extension,* $E^{\infty}_2(k)$ *is finite if* A_r *hs good reduction over* $\mathbb{Z}_p[\mu_{p^\infty}]$*.*

This tells us that if $|\underline{\mathrm{III}}^1(k^S/k, T_pB^{\mathrm{ord}}_{P_0})| < \infty$ ($\Leftarrow |\underline{\mathrm{III}}(k^S/k, B^{\mathrm{ord}}_{P_0})| < \infty$ and $\dim_{\mathbb{Q}(f_{P_0})} B_{P_0}(k) \otimes_{\mathbb{Z}} \mathbb{Q} \leq 1$) for one point P_0, the generic Λ-rank of $\widehat{J}^{\mathrm{ord}}_{\infty,\mathbb{T}}(k)$ is constant equal to $\mathrm{rank}\, B_{P_0}(k) = \dim B_{P_0}$ or 0; so, if the constancy of the rank for almost all principal P.

§11. Error term and Tate–Shafarevich groups.

Let k be a number field. Put $E^s(k) = \mathrm{Coker}(J_s^{\mathrm{ord}}(k) \to B_P^{\mathrm{ord}}(k))$. From the exact sequence

$$0 \to \varpi(J_s^{\mathrm{ord}})(k^S) \to J_s^{\mathrm{ord}}(k^S) \to B_P^{\mathrm{ord}}(k^S) \to 0,$$

we have $E^s(k) \hookrightarrow H^1(\varpi(J_s^{\mathrm{ord}}))[\varpi]$ for $H^1(?) = H^1(k^S/k, ?)$.

It is easy to show $E^\infty(k_v) = 0$ if $v \nmid p$. By a result of P. Schneider on universal norm from $k_v[\mu_{p^\infty}]/k_v$ for $p|v$, we have $E^\infty(k_v)$ is finite if $v|p$. Since $\prod_{v \in S} E^s(k_v) = \prod_{v|p} E^s(k_v)$ is finite, essentially

$$E^s(k) \hookrightarrow \underline{\text{Ш}}^1(k^S/k, \varpi(J_s^{\mathrm{ord}}))[\varpi].$$

We can also show that

$$\underline{\text{Ш}}^1(k^S/k, T_pA_P^{\mathrm{ord}}) \twoheadrightarrow \underline{\text{Ш}}^1(k^S/k, \varpi(J_s^{\mathrm{ord}}))[\varpi]$$

up to finite error.