Adjoint L-value formula and the Tate conjecture for Shimura surfaces over $\mathbb{Q}$

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Abstract: For an elliptic Hecke eigen form f with Galois representation ρ_f, I first present an explicit formula of the adjoint L-value $L(1, Ad(\rho_f)\otimes\chi_E)$ for a real quadratic field E with associated quadratic character χ_E. Then, for an indefinite quaternion algebra D over $\mathbb{Q}$, let $Sh_{E/\mathbb{Q}}$ be the Shimura surface associated to $(D\otimes_{\mathbb{Q}} E)^\times$. Supposing that 2 splits in E, we describe a way to prove that $H^0(K, H^2_{et}(Sh_E, \mathbb{Q}_l(1)))$ is generated by Shimura subcurves of Sh_E coming from all quaternion subalgebras $B \subset D_E$ over $\mathbb{Q}$. Here K is any real abelian extension of $\mathbb{Q}$. If time allows, we sketch how to generalize the argument to all quaternionic Shimura varieties over a totally real field F.

§0. Twisted 4-dimensional quadratic spaces.

Let $D_{/\mathbb{Q}}$ be a quaternion algebra with discriminant ∂. Choose a semi-simple real quadratic extension $E = \mathbb{Q}[\sqrt{\Delta}]_{/\mathbb{Q}}$ $(0 < \Delta \in \mathbb{Z})$ including $E = \mathbb{Q} \times \mathbb{Q}$ with $\sqrt{\Delta} = (1, -1)$, and let $D_E := D \otimes_{\mathbb{Q}} E$. Let $\langle\sigma\rangle = \mathrm{Gal}(E/\mathbb{Q})$ act on D_E through the factor E. Then $D_\sigma := \{v \in D_E | v^\sigma = v^\iota\}$ for $v + v^\iota = \mathrm{Tr}(v)$ is a 4-dimensional quadratic space with a quadratic $\mathbb{Q}$-form induced by the reduced norm $N : D_E \to E$.

We have a decomposition $D_\sigma = Z \oplus D_0$ with $Z = (\mathbb{Q}, z^2)$ and

$$D_0 = Z^\perp := \{v \in D_\sigma | \mathrm{Tr}(v) = 0\} = \{\sqrt{\Delta}w | w \in D, \mathrm{Tr}(w) = 0\}.$$

Write always f for an elliptic cusp form of weight 2 and $\mathcal{F}$ for weight 2 cusp forms on $D_E^\times \backslash D_{E_\mathbb{A}}^\times$.

§**1. Quaternion subalgebras of D_E.** For each $\alpha \in D_\sigma \cap D_E^\times$, define the α-twist σ_α of σ by $v \mapsto \alpha v^\sigma \alpha^{-1} =: v^{\sigma_\alpha}$. Then σ_α is another action of $\mathrm{Gal}(E/\mathbb{Q})$ on D_E, and $D_\alpha = \{\xi^{\sigma_\alpha} = \xi | \xi \in D_E\}$ is a quaternion $\mathbb{Q}$-subalgebra of D_E.

- All quaternion subalgebras $B_{/\mathbb{Q}} \subset D_E$ are realized as D_α for some $\alpha \in D_\sigma$, and $D_z = D \Leftrightarrow z \in Z$;
- $\alpha = \xi^\iota \beta \xi^\sigma$ for $\xi \in D_E^\times \Leftrightarrow D_\alpha \cong D_\beta$ with $\xi D_\alpha \xi^{-1} = D_\beta$;
- $D_\sigma \cong D_{\alpha,\sigma}$ as quadratic spaces (independent of D);
- The even Clifford group G_α of $D_{\alpha,0} = \{v \in D_{\sigma_\alpha} | \mathrm{Tr}(v) = 0\}$ is $D_\alpha^\times$, and $D_E^\times$ is a covering of the similitude group GO_{D_σ} of D_σ.

Let $Sh_E = D_E^\times \backslash D_{E_{\mathbb{A}}}^\times / E_{\mathbb{A}}^\times \mathrm{SO}_2(E_\infty)$ be the Shimura variety for $D_E^\times$ and Sh_α be the image of $D_{\alpha,\mathbb{A}}^\times$ in Sh_E for $\alpha \in D_\sigma$. Let $a = \dim Sh_\alpha \in \{0,1\}$. Regard $Sh_\alpha \in H^0(\mathbb{Q}, H^{2a}(Sh_E, \mathbb{Z}_l(a)))$ and write $(\cdot,\cdot)$ for the Poincaré duality.

§2. **Two formulas.** Write $\theta^*(\phi)(f)$ for the theta lift to Sh_E and $\theta_*(\phi)(\mathcal{F})$ for the theta descent down of a cuspidal harmonic differential form $\mathcal{F}$ on Sh_E of degree $2a$.

Theorem A: $\theta_*(\phi)(\mathcal{F}) = (4\sqrt{-1})^{-1}\sum_{\alpha,N(\alpha)>0}\phi(\alpha)(\mathcal{F},Sh_\alpha)q^{N(\alpha)}$ *for* $q=\exp(2\pi i\tau)$, *if* $D_{\mathbb{R}}\cong M_2(\mathbb{R})$, *where* α *runs over* D_σ *modulo (from the right) the norm* 1 *congruence subgroup* $\Gamma_\phi\subset D_E^\times$ *fixing* ϕ. *Moreover*

$$\{\theta_*(\phi)(\mathcal{F})\}_\phi = 0 \Leftrightarrow \ \mathcal{F} \text{ is non-base-change lift.}$$

Theorem B: $E\frac{L(1,Ad(\rho_f)\otimes\chi_E)}{2^2\pi^3} = (\theta^*(\phi')(f),Sh_1)\neq 0$ *for a special choice* $\phi'=\phi_Z\otimes\phi_0$ *with an explicit constant* $E\neq 0$, *where* ϕ_Z *is a Schwartz-Bruhat function on* $Z_{\mathbb{A}}$ *and* ϕ_0 *on* $D_{0,\mathbb{A}}$.

§3. Sketch of the proof of Theorem B (See-saw).

We have

$$\int_{Sh_1} \theta^*(\phi')(f)(h)dh = \int_{Sh_1}\int_Y f(\tau)\theta(\phi_Z)(\tau)\theta(\phi_0)(\tau,h)d\mu_\tau dh$$
$$= \int_Y f(\tau)\theta(\phi_Z)(\tau)\int_{Sh_1}\theta(\phi_0)(\tau,h)dhd\mu_\tau = \int_Y f(g)\theta(\phi_Z)E(\phi_0)(\tau)d\mu_\tau.$$

Here by Siegel–Weil formula,

$$\int_{Sh_1}\theta(\phi_0)(\tau,h)dh = E(\phi_0)(\tau)$$

for the Siegel–Weil Eisenstein series $E(\phi_0)$.

This type of Rankin convolution integral is the adjoint L-value by Shimura (1975) if we choose ϕ_Z and ϕ_0 well. $\square$

§**4. Theta kernel.** Suppose now that D is indefinite, we take a standard Schwartz function as ϕ_∞ on $D_{\sigma,\mathbb{R}}$. For a Schwartz–Bruhat function $\phi = \phi^{(\infty)}\phi_\infty$ on $D_{\sigma,\mathbb{A}}$, we have the theta series for $g_\tau = \eta^{-1/2}\begin{pmatrix} \eta & \xi \\ 0 & 1 \end{pmatrix}$ $(\tau = \xi + \eta\sqrt{-1} \in \mathfrak{H})$

$$\theta(\phi)(\tau; h) = (\sum_{\alpha \in D} \mathbf{w}(g_\tau)\phi(h^\iota \alpha h^\sigma))d\overline{z} \wedge dw \quad \text{on } (\Gamma_\tau \backslash \mathfrak{H}) \times Sh_E$$

for a congruence subgroup $\Gamma_\tau \subset \mathrm{SL}_2(\mathbb{Z})$. For a cusp form $f \in S_2(\Gamma_\tau)$ and harmonic cuspidal differential form $\mathcal{F}$ of degree 2 in $(z := z_i, w := z_\sigma) \in Sh_E(\mathbb{C}) \doteqdot \varprojlim_\Gamma \Gamma \backslash (\mathfrak{H} \times \mathfrak{H})$ as $E_\mathbb{R} = \mathbb{R} \times \mathbb{R}$, we define for $h_\infty = h_{(z,w)} = (g_z, g_w) \in D^\times_{E_\mathbb{R}}$ and $h = h^{(\infty)}h_\infty \in D^\times_{E_\mathbb{A}}$

$$\theta^*(\phi)(f)(h) = \int_{\Gamma_\tau \backslash \mathfrak{H}} f(\tau)\theta(\phi)(\tau; h)\eta^{-2}d\xi d\eta,$$

$$\theta_*(\phi)(\mathcal{F})(\tau) = \int_{Sh_E} \theta(\phi)(\tau; h) \wedge \mathcal{F}.$$

We call $\theta^*(\phi)(f)$ on Sh_E a theta lift harmonic form and $\theta_*(\phi)(\mathcal{F}) \in M_2(\Gamma_\tau)$ a theta descent.

§5. Rallis inner product formula.

There is a Rallis inner product formula (1987):

$$\langle \theta^*(\phi)(f), \theta^*(\phi)(f)\rangle_{Sh_E} = *L(1, Ad(\rho_f)\otimes\chi_E) \neq 0$$

for most ϕ. Rallis proved this for quadratic space of dimension 4 of Witt index 1 and is generalized by Gan, Qiu and Takeda in 2014 in an Inventiones paper to all reductive pairs. By Rallis, for its automorphic representation $\pi_{\theta^*(f)}$ generated by $\{\theta^*(\phi)(f)\}_\phi$,

$$\{\theta^*(\phi)(f)\}_\phi \neq 0 \text{ if and only if } \pi_{\theta^*(f)} \text{ is locally a theta lift.}$$

Assuming that 2 splits in E, square-integrability of the Jacques-Langlands correspondant $JL(\pi_{\theta^*(f)})$ on $\mathrm{GL}_2(E_{\mathbb{A}})$ at split prime factors of ∂ is the local condition; so, $JL(\pi_{\theta^*(f)})$ is the GL(2)-base-change lift of π_f for the GL(2)-automorphic representation π_f generated by f. In other words, if $\mathcal{F}$ is a base-change lift residing on $D^\times_{E_{\mathbb{A}}}$, we can find f and φ such that $\mathcal{F} = \theta^*(\varphi)(f)$. This combined with the computation of the q-expansion of $\{\theta_*(\phi)(\mathcal{F})\}_\phi$ gives Theorem A.

§**6. Tate conjecture.** If $D_{\mathbb{R}} \cong M_2(\mathbb{R})$ and E is real, $H^0(K, H^2_{et}(Sh_E, \mathbb{Q}_l(1)))$ is conjectured to be generated by algebraic cycles of codimension 1 defined over a number field K.

This is known by Langlands-Harder-Rapoport [LHR] (1986) if $D = M_2(\mathbb{Q})$ (i.e., for Hilbert modular surfaces) if K is real abelian over $\mathbb{Q}$, and Murty–Ramakurishnan extended to imaginary K.

Note that the reflex field of Sh_E is $\mathbb{Q}$; so, the Galois action on $H^2_{et}(Sh_E, \mathbb{Q}_l(1))$ extends to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Theorem C: *Suppose that 2 splits in $E_{/\mathbb{Q}}$. Then for all real abelian extensions $K_{/\mathbb{Q}}$, $H^0(K, H^2_{cusp}(Sh_E(N), \mathbb{Q}_l(1)))$ is spanned by*

$$\{Sh_\alpha \in H^0(K, H^2_{et}(Sh_E(N), \mathbb{Q}_l(1)))|\alpha \in D_\sigma \quad \textit{with} \quad N(\alpha) > 0\}.$$

We can also cover the invariant forms $y_?^{-1} dz_? \wedge d\overline{z}_?$ $(? = i, \sigma)$ by Chern classes of automorphic line bundles as explained in §4.6 of [HLR]; so, we concentrate on the cuspidal part $H^2_{cusp} \subset H^2_{et}$.

§**7. Galois action on automorphic cohomology.** For a quaternion algebra B over a totally real field, let $\mathcal{A}_B$ be the set of all cuspidal automorphic representation holomorphic of weight 2 on $B^\times\backslash B^\times_{\mathbb{A}}$. Since $\langle\sigma\rangle = \mathrm{Gal}(E/\mathbb{Q})$ acts on $D_E^\times$, σ acts on $\mathcal{A}_{D_E}$. Let $\mathcal{A}^\sigma_{D_E} := \{\pi \in \mathcal{A}_{D_E} | \pi^\sigma \cong \pi\}$. Write ρ_π be the two dimensional Galois representation attached to π. Then by Brylinsky-Labesse and Reimann, we have

$$H^2_{cusp}(Sh_E, \overline{\mathbb{Q}}_l(1)) \cong \bigoplus_{\pi\in\mathcal{A}_{D_E}} \pi^{(\infty)} \otimes (\rho_\pi \boxtimes \rho_{\pi,\sigma})(1)$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-modules, where $\rho_{\pi,\sigma}(\tau) = \rho_\pi(\sigma\tau\sigma^{-1})$ for an extension of σ to $\overline{\mathbb{Q}}$. Since $\dim H^0(K, (\rho_\pi \boxtimes \rho_{\pi,\sigma})(1)) = \begin{cases} 1 & \text{if } \rho_{\pi,\sigma} \cong \rho_\pi, \\ 0 & \text{otherwise,} \end{cases}$

for all real abelian extension $K_{/\mathbb{Q}}$, we have

$$H^0(K, H^2_{cusp}(Sh_E, \overline{\mathbb{Q}}_l(1))) \cong \bigoplus_{\pi\in\mathcal{A}^\sigma_{D_E}} \pi^{(\infty)} \otimes (\rho_\pi \boxtimes \rho_{\pi,\sigma})(1).$$

If $K_{/\mathbb{Q}}$ is imaginary abelian, $\dim H^0(K, (\rho_\pi \boxtimes \rho_{\pi,\sigma})(1)) = 2$ can happen, and $\{Sh_\alpha\}_\alpha$ is not enough even for Hilbert modular cases.

§**8. Proof of Theorem C.** Let

$\mathcal{A}^{sq}_{M_2(\mathbb{Q})} = \{\pi \in \mathcal{A}_{M_2(\mathbb{Q})} | \pi$ is square integrable at $v|\partial$ splitting in $E\}$.

By the proof of the Rallis inner product formula, the project started by Rallis to know the injectivity of the theta correspondence on automorphic representations is finished. This shows, under the splitting condition of 2,

the base-change map $\theta^* : \mathcal{A}^{sq}_{M_2(\mathbb{Q})} \to \mathcal{A}^{\sigma}_{D_E}$ is onto and 2 to 1.

Suppose $(\mathcal{F}, Sh_\alpha) = 0$ for all $\alpha \in D_\sigma$ with $N(\alpha) > 0$. Then $\theta_*(\varphi)(\mathcal{F}) = 0$ for all φ. Since $\mathcal{F} = \theta^*(\phi)(f)$ for some ϕ (i.e., $\mathcal{F}$ generates an element in $\mathcal{A}^{\sigma}_{D_E}$), we conclude $\mathcal{F} = 0$. Thus the space generated by Sh_α's contains $H^0(K, H^2_{cusp}(Sh_E, \overline{\mathbb{Q}}_l(1)))$. $\square$

§**9. Quaternionic Shimura varieties.** Let E be a general totally real field. By a result of Getz and Hahn for $M_2(E)$ (or more precisely, by a generalization of their computation of $\mathbb{T} := H^0(K, H^{2a}(Sh_E, \mathbb{Q}_l(1)))$ for quaternionic Shimura varieties), this space $\mathbb{T}$ is non-trivial only when E has a subfield F with $[E : F] = 2$, and all quaternion subalgebras over F in D_E for all such F contributes. Therefore the argument is combinatrially demanding, and I have not finished the details.

Even if $E_{/\mathbb{Q}}$ is real quadratic, for any quaternion algebra $B_{/E}$, we have $\mathcal{A}_B^\sigma := \{\pi \in \mathcal{A}_B | L(s, \rho_\pi) = L(s, \rho_{\pi,\sigma})\}$. However, if B cannot descends to $\mathbb{Q}$, Shimura subcurves do not exist; so, the Tate classes in such cases are mysterious.

§**10. Hecke equivariance.** For a split prime $(p) = \mathfrak{p}\mathfrak{p}^\sigma$,

$$D_{\sigma,\mathbb{Q}_p} = \{(v, v^\iota) \in D_{\mathfrak{p}} \times D_{\mathfrak{p}^\sigma} | v \in D_{\mathfrak{p}}\} \cong D_{\mathbb{Q}_p} \quad ((v, v^\iota) \leftrightarrow v).$$

If $D_{\mathbb{Q}_p} = M_2(\mathbb{Q}_p)$, the Hecke operator given by the action of $h \in D^\times_{E_{\mathfrak{p}}}$ on D_{σ,E_p} and another given by the Weil representation, if $\phi = \phi_p\phi^{(p)}$ with ϕ_p given by the characteristic function of $M_2(\mathbb{Z}_p) = M_2(O_{E_{\mathfrak{p}}})$, we find $\phi|T(p) = \phi|T(\mathfrak{p})$. This shows

Theorem D. *For almost all primes $p = \mathfrak{p}\mathfrak{p}^\sigma$ split in E with $D_{\mathbb{Q}_p} \cong M_2(\mathbb{Q}_p)$, $\theta_*(\phi)(\mathcal{F})|T(p) = \theta_*(\mathcal{F}|T(\mathfrak{p}))$ and $\theta_*(\phi)(\mathcal{F})|\langle N_{E/\mathbb{Q}}(z)\rangle = \theta_*(\mathcal{F}|\langle z\rangle)$ for the central element $z \in E^\times_{\mathbb{A}}$. In particular, if $\mathcal{F}$ is a Hecke eigenform, $\theta_*(\phi)(\mathcal{F}) \neq 0$ implies $\mathcal{F}$ belongs to the base-change lift of π_f to $D^\times_E$ for a Hecke eigen form f on $D^\times_{\mathbb{A}}$.*

In other word, $\theta_*(\phi)(\mathcal{F}) = 0$ if $\mathcal{F}$ is not a base-change lift. For SL(2), Hecke equivariance essentially follows from the solution of the Howe conjecture by Gan–Takeda in 2015, which is not sufficient to prove $(\mathcal{F}, Sh_\alpha) = 0$ for non-base-change $\mathcal{F}$.