ARITHMETIC OF ADJOINT L-VALUES

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Contents

1. Introduction

In this lecture note of the mini-course, we discuss the following five topics:

(1) Introduction to the ordinary (i.e. slope 0) Hecke algebras (the so-called "big Hecke algebra");
(2) Some basic ring theory to deal with Hecke algebras;
(3) "$R = T$" theorem of Wiles–Taylor, and its consequence for the adjoint Selmer groups;
(4) Basics of adjoint L-function (analytic continuation, rationality of the value at $s = 1$);

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(5) Relation of adjoint L-values to congruence of a modular form f and the Selmer group of the adjoint Galois representation $Ad(\rho_f)$ (the adjoint main conjectures).

Let us describe some history of these topics along with the content of the note. Doi and the author started the study of the relation between congruence among cusp forms and L-values in 1976 (the L-value governing the congruence is now called the adjoint L-value $L(1, Ad(f))$). How it was started was described briefly in a later paper [DHI98]. Here is a quote from the introduction of this old article:

"It was in 1976 when Doi found numerically non-trivial congruence among Hecke eigenforms of a given conductor for a fixed weight κ [DO77]. Almost immediately after his discovery, Doi and Hida started, from scratch, numerical and theoretical study of such congruences among elliptic modular forms. We already knew in 1977 that the congruence primes for a fixed primitive cusp form f appear as the denominator of Shimura's critical L-values of the zeta function $D(s, f, g)$ attached to f and another lower weight modular form g. Thus the denominator is basically independent of g."

Though it is not mentioned explicitly in [DHI98] (as it concerns mainly with the adjoint L-value twisted by a non-trivial character), the author realized in 1980 that the rational value $*\frac{D(m,f,g)}{\langle f,f\rangle}$ (with some power $*$ of $2\pi i$) Shimura evaluated in [Sh76] is essentially equal to $*\frac{D(m,f,g)}{L(1,Ad(f))}$ (by Shimura's Theorem 5.1 in the text); so, the author guessed that the adjoint L-value $L(1, Ad(f))$ is the denominator and is the one responsible for the congruence primes of f. This guess is later proven by the author in [H81a], [H81b] and [H88b].

In [H81a, Theorem 6.1], the size of the cohomological congruence module (of the Hecke algebra acting on f) is computed by the square of the L-value (which produces the congruence criterion: "the prime factors of the adjoint L-value give the congruence primes of f"). In [H81b] the converse: "the congruence primes divide the L-value" is proven for ordinary primes, and the work was completed by Ribet in [Ri83] for non-ordinary primes. Most of primes p is expected to be ordinary for f (i.e., $f|T(p) = u \cdot f$ for a p-adic unit u), but it is still an open question if the weight of f is greater than 2 (see [H13a, §7]). The criterion is further made precise in [H88b] as an identity of the order of the ring-theoretic congruence module and the L-value, which implies the if-and-only-if result (as the support of the ring theoretic congruence module of a Hecke algebra is exactly the set of congruence primes). In §2.2 (after some preliminary discussion of ring theory), we give an exposition of an abstract theory of the congruence module and its sibling (the differential module) which has direct relation to the Selmer groups by the Galois deformation theory of Mazur (see Theorem 3.12). These works led the author to propose an analogue of Kummer's criterion [H82] (different from the one by Coates–Wiles [CW77]) for imaginary quadratic fields, which was a precursor of the later proofs of the anti-cyclotomic main conjecture in [T89], [MT90], [HiT94], [H06] and [H09] and is closely related to the proof of the adjoint main conjecture (applied to CM families) stated as Corollary 4.5 in the text.

In the late 1981 (just fresh after being back to Japan from a visit to Princeton for two years, where he had some opportunity to talk to many outstanding senior mathematicians, Coates, Langlands, Mazur, Shimura, Weil,...), the author decided to study, in the current language, p-adic deformations of modular forms and modular Galois representations. Though he finished proving the theorems in [H86a] in the winter of 1981–82, as the results he obtained was a bit unbelievable even to him, he spent another year to get another proof given in [H86b], and on the way, he found p-adic interpolation of classical modular Galois representations. These two articles were published as [H86a] and [H86b] (while the author was in Paris, invited by J. Coates, where he gave many talks, e.g., [H85] and [H86c], on the findings and became confident with his results, besides he had great audiences: Greenberg, Perrin-Riou, Taylor, Tilouine, Wiles, ...). What was given in these two foundational articles is a construction of the big ordinary p-adic Hecke algebra $\mathbf{h}$ along with p-adic analytic families of slope 0 elliptic modular forms (deforming a starting Hecke eigenform) and a p-adic interpolation of modular Galois representations (constructed by Eichler–Shimura and Deligne earlier) in the form of big Galois representations with values in $GL_2(\mathbb{I})$ for a finite flat extension $\mathbb{I}$ of the weight Iwasawa algebra $\Lambda := \mathbb{Z}_p[[T]]$. Here, actually, $\mathrm{Spec}(\mathbb{I})$ is (the normalization of) an irreducible component of $\mathrm{Spec}(\mathbf{h})$ and is the parameter space of a p-adic analytic family of ordinary Hecke eigen cusp forms. See Section 4 for an exposition of the modular deformation theory and p-adic analytic families of modular forms.

Just after the foundation of the modular deformation theory was laid out, Mazur conceived the idea of deforming a given mod p Galois representation without restricting to the modular representations [M89] (which he explained to the author while he was at IHES), constructing the universal Galois deformation rings (in particular the universal ring R whose spectrum parameterizing "p-ordinary" deformations). He then conjectured (under some assumptions) that $\mathrm{Spec}(R)$ should be isomorphic to a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ if the starting mod p representation is modular associated to a mod p elliptic modular form on which $\mathbb{T}$ acts non-trivially; thus, in short, the big ordinary Hecke algebra was expected to be universal. Mazur's Galois deformation theory is described in Section 3 which starts with an interpretation of abelian Iwasawa theory via deformation theory (with a deformation theoretic proof of the classical class number formula of Dirichlet–Kummer–Dedekind in §3.2) reaching to Mazur's theory in two dimensional cases in §3.4. The definition of the adjoint Selmer group and its relation to the congruence modules and the differential modules (of the Galois deformation ring) are also included in this section (§3.5 and §3.6).

We saw a great leap forward in the proof by Wiles and Taylor [W95] of the equality (e.g. Theorem 4.3) of a p-adic Hecke algebra of finite level and an appropriate Galois deformation ring with a determinant condition (that had been conjectured by Mazur). Indeed, follows from their result (combined with an analytic result in [H88b] described in Section 5 in the text), the exact formula connecting the adjoint L-value with the size of the corresponding Selmer group (see §5.4). This identity is called the adjoint non-abelian class number formula. This also implies that the L-value gives the exact size of the congruence module (as the size of the Selmer group is equal to the size of the congruence module by the theorems of Tate and Mazur combined: Theorems 3.5 and 3.12).

By doing all these over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$ of weight variable T, we get the identity "$R = \mathbb{T}$" (see Theorem 4.1), and (the integral part of) the adjoint L-values $L(1, Ad(f))$ are interpolated p-adically by the characteristic element of the congruence module (of $\mathrm{Spec}(\mathbb{I}) \to \mathrm{Spec}(\mathbb{T})$) which forms a one variable adjoint p-adic L-function $L_p \in \mathbb{I}$ of the p-adic analytic family associated to $\mathrm{Spec}(\mathbb{I})$, and the result corresponding to the adjoint non-abelian class number formula in this setting is the proof of the (weight variable) adjoint main conjecture: Corollary 4.5 (see also Corollary 5.8). Urban [U06] proved the two variable adjoint main conjecture, in many cases, adding the cyclotomic variable to the weight variable (by Eisenstein techniques applied to Siegel modular forms on $GSp(4)$).

The last section Section 5 starts with a technical heuristic of why the adjoint L-value is an easiest "integer" to be understood after the residue of the Dedekind zeta function. Guided by this principle, we prove a formula Theorem 5.6 relating the size of the congruence module and the L-value. Out of this theorem and the result exposed in the earlier sections, we obtain the adjoint class number formula Theorem 5.7.

So far, the start step (the congruence criterion by the adjoint L-value) done in [H81a] has been generalized in many different settings (e.g., [U95], [H99], [Gh02], [Di05], [N14] and [BR14]). However, even the converse (i.e., congruence primes give factors of the L-value) proven in [H81b] and [Ri83] has not yet been generalized except for some work of Ghate (e.g., [Gh10]). Even if we now have many general cases of the identification of the Hecke algebra with an appropriate Galois deformation ring (as exposed by some other lecture in this volume, e.g., [Gr15]), the converse might not be an easy consequence of them (as it remains an analytic task to identify the size of the adjoint Selmer group with the integral part of the corresponding adjoint L-value; see Section 5 for the analytic work in the elliptic modular case).

These notes are intended to give not only a systematic exposition (hopefully accessible by graduate students with good knowledge of class field theory and modular forms) of the road to reach the non-abelian class number formula (including a treatment via Galois deformation theory of the classical class number formula in §3.2) but also some results of the author not published earlier, for example, see comments after Corollary 4.4 which is an important step in the proof of the weight variable main conjecture (Corollary 4.5).

Here are some notational conventions in the notes. Fix a prime p. For simplicity, we assume $p \geq 5$ (though $p = 3$ can be included under some modification). Thus $\mathbb{Z}_p^\times = \mu_{p-1} \times \Gamma$ for $\Gamma = 1 + p\mathbb{Z}_p$. Consider the group of p-power roots of unity $\mu_{p^\infty} = \bigcup_n \mu_{p^n} \subset \overline{\mathbb{Q}}^\times$. Then writing $\zeta_n = \exp\left(\frac{2\pi i}{p^n}\right)$, we can identify the group μ_{p^n} with $\mathbb{Z}/p^n\mathbb{Z}$ by $\zeta_n^m \leftrightarrow (m \mod p^n)$. The Galois action of $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ sends ζ_n to $\zeta_n^{\nu_n(\sigma)}$ for $\nu_n(\sigma) \in \mathbb{Z}/p^n\mathbb{Z}$. Then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$ by a character $\nu := \varprojlim_n \nu_n : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$, which is called the p-adic cyclotomic character. Similarly, $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ by ν; so, we get a peculiar identity $\mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \cong \mathbb{Z}_p^\times$. For $x \in \mathbb{Z}_p^\times$, we write $[x, \mathbb{Q}_p] := \nu^{-1}(x) \in \mathrm{Gal}(\mathbb{Q}_p[\mu_{p^\infty}]/\mathbb{Q}_p)$ (the local Artin symbol).

Always W denotes our base ring which is a sufficiently large discrete valuation ring over $\mathbb{Z}_p$ with residue field $\mathbb{F}$ which is an algebraic extension of $\mathbb{F}_p$ (usually we assume that $\mathbb{F}$ is finite for simplicity). Also for the power series ring $W[[T]]$, we write $t = 1 + T$ and for $s \in \mathbb{Z}_p$, $t^s = \sum_{s=0}^\infty \binom{s}{n} T^n$ noting $\binom{s}{n} \in \mathbb{Z}_p \subset W$. Thus for $\gamma = 1 + p \in \mathbb{Z}_p^\times$, $\gamma^s = \sum_{s=0}^\infty \binom{s}{n} p^n \in \Gamma$. For a local W-algebra A sharing same residue field $\mathbb{F}$ with W, we write CL_A the category of complete local A-algebras with sharing residue field with A. Morphisms of CL_A are local A-algebra homomorphisms. For an object $R \in CL_A$, $\mathfrak{m}_R$ denotes its unique maximal ideal.

Here $R \in CL_A$ is complete with respect to the $\mathfrak{m}_R$-adic topology. If A is noetherian, the full subcategory CNL_A of CL_A is made up of noetherian local rings. For an object $R \in CL_A$, $R \otimes_A R$ has maximal ideal $\mathfrak{M} := \mathfrak{m}_R \otimes 1 + 1 \otimes \mathfrak{m}_R$. We define the completed tensor product $R \widehat{\otimes}_A R$ by the $\mathfrak{M}$-adic completion of $R \otimes_A R$. Note that $R \widehat{\otimes}_A R$ is an object in CL_A. The completed tensor product satisfies the usual universality for $\mathfrak{m}_R$-adically continuous A-bilinear maps $B(\cdot,\cdot) : R \times R \to M$. More specifically, for each $\mathfrak{m}_R$-adically continuous A-bilinear map $B(\cdot,\cdot) : R \times R \to M$ for a $\mathfrak{m}_R$-adically complete R-module M, we have a unique morphism $\phi : R \widehat{\otimes}_A R \to M$ such that $B(x,y) = \phi(x \otimes y)$.

2. Some ring theory

We introduce the notion of congruence modules and differential modules for general rings and basic facts about it. We apply the theory to Hecke algebras and deformation rings to express the size of these modules by the associated adjoint L-value.

2.1. Differentials.

We recall here the definition of 1-differentials and some of their properties for our later use. Let R be a A-algebra, and suppose that R and A are objects in CNL_W. The module of 1-differentials $\Omega_{R/A}$ for a A-algebra R ($R, A \in CNL_W$) indicates the module of **continuous** 1-differentials with respect to the profinite topology.

For a module M with continuous R-action (in short, a continuous R-module), let us define the module of A-*derivations* by

$$Der_A(R,M) = \left\{ \delta : R \to M \in \mathrm{Hom}_A(R,M) \middle| \begin{array}{c} \delta\text{: continuous} \\ \delta(ab) = a\delta(b) + b\delta(a) \\ \text{for all } a, b \in R \end{array} \right\}.$$

Here the A-linearity of a derivation δ is equivalent to $\delta(A) = 0$, because

$$\delta(1) = \delta(1 \cdot 1) = 2\delta(1) \Rightarrow \delta(1) = 0.$$

Then $\Omega_{R/A}$ represents the covariant functor $M \mapsto Der_A(R,M)$ from the category of **continuous** R-modules into MOD.

The construction of $\Omega_{R/A}$ is easy. The multiplication $a \otimes b \mapsto ab$ induces a A-algebra homomorphism $m : R \widehat{\otimes}_A R \to R$ taking $a \otimes b$ to ab. We put $I = \mathrm{Ker}(m)$, which is an ideal of $R \widehat{\otimes}_A R$. Then we define $\Omega_{R/A} = I/I^2$. It is an easy exercise to check that the map $d : R \to \Omega_{R/A}$ given by $d(a) = a \otimes 1 - 1 \otimes a \mod I^2$ is a continuous A-derivation. Thus we have a morphism of functors: $\mathrm{Hom}_R(\Omega_{R/A}, ?) \to Der_A(R, ?)$ given by $\phi \mapsto \phi \circ d$. Since $\Omega_{R/A}$ is generated by $d(R)$ as R-modules (left to the reader as an exercise), the above map is injective. To show that $\Omega_{R/A}$ represents the functor, we need to show the surjectivity of the above map.

Proposition 2.1. *The above morphism of two functors $M \mapsto \mathrm{Hom}_R(\Omega_{R/A}, M)$ and $M \mapsto Der_A(R,M)$ is an isomorphism, where M runs over the category of continuous R-modules. In other words, for each A-derivation $\delta : R \to M$, there exists a unique R-linear homomorphism $\phi : \Omega_{R/A} \to M$ such that $\delta = \phi \circ d$.*

Proof. Define $\phi : R \times R \to M$ by $(x,y) \mapsto x\delta(y)$ for $\delta \in Der_A(R,M)$. If $a, c \in R$ and $b \in A$, $\phi(ab,c) = ab\delta(c) = b(a\delta(c)) = b\phi(a,c)$ and $\phi(a,bc) = a\delta(bc) = ab\delta(c) = b(a\delta(c)) = b\phi(a,c)$. Thus ϕ gives a continuous A-bilinear map. By the universality of the tensor product, $\phi : R \times R \to M$ extends to a A-linear map $\phi : R\widehat{\otimes}_A R \to M$. Now we see that $\phi(a \otimes 1 - 1 \otimes a) = a\delta(1) - \delta(a) = -\delta(a)$ and

$$\begin{aligned}\phi((a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b)) &= \phi(ab \otimes 1 - a \otimes b - b \otimes a + 1 \otimes ab) \\ &= -a\delta(b) - b\delta(a) + \delta(ab) = 0.\end{aligned}$$

This shows that $\phi|_I$-factors through $I/I^2 = \Omega_{R/A}$ and $\delta = \phi \circ d$, as desired. □

Corollary 2.2. *Let the notation be as in the proposition.*

(i) *Suppose that A is a C-algebra for an object $C \in CL_W$. Then we have the following natural exact sequence:*

$$\Omega_{A/C}\widehat{\otimes}_A R \longrightarrow \Omega_{R/C} \longrightarrow \Omega_{R/A} \to 0.$$

(ii) *Let $\pi : R \twoheadrightarrow C$ be a surjective morphism in CL_W, and write $J = Ker(\pi)$. Then we have the following natural exact sequence:*

$$J/J^2 \xrightarrow{\beta^*} \Omega_{R/A}\widehat{\otimes}_R C \longrightarrow \Omega_{C/A} \to 0.$$

Moreover if $A = C$, then $J/J^2 \cong \Omega_{R/A}\widehat{\otimes}_R C$.

Proof. By assumption, we have algebra morphisms $C \to A \to R$ in Case (i) and $A \to R \twoheadrightarrow C = R/J$ in Case (ii). By the Yoneda's lemma (e.g., [GME, Lemma 1.4.1] or [MFG, Lemma 4.3]), we only need to prove that

$$0 \to Der_A(R, M) \xrightarrow{\alpha} Der_C(R, M) \xrightarrow{\beta} Der_C(A, M)$$

is exact in Case (i) for all continuous R-modules M and that

$$0 \to Der_A(C, M) \xrightarrow{\alpha} Der_A(R, M) \xrightarrow{\beta} \mathrm{Hom}_C(J/J^2, M)$$

is exact in Case (ii) for all continuous C-modules M. The first α is just the inclusion and the second α is the pull back map. Thus the injectivity of α is obvious in two cases. Let us prove the exactness at the mid-term of the first sequence. The map β is the restriction of derivation D on R to A. If $\beta(D) = D|_A = 0$, then D kills A and hence D is actually a A-derivation, i.e. in the image of α. The map β in the second sequence is defined as follows: For a given A-derivation $D : R \to M$, we regard D as a A-linear map of J into M. Since J kills M, $D(jj') = jD(j') + j'D(j) = 0$ for $j, j' \in J$. Thus D induces A-linear map: $J/J^2 \to M$. Then for $b \in A$ and $x \in J$, $D(bx) = bD(x)+xD(b) = bD(x)$. Thus D is C-linear, and $\beta(D) = D|_J$. Now prove the exactness at the mid-term of the second exact sequence. The fact $\beta \circ \alpha = 0$ is obvious. If $\beta(D) = 0$, then D kills J and hence is a derivation well defined on $C = R/J$. This shows that D is in the image of α.

Now suppose that $A = C$ in the assertion (ii). To show the injectivity of β^*, we create a surjective C-linear map: $\gamma : \Omega_{R/A} \otimes C \twoheadrightarrow J/J^2$ such that $\gamma \circ \beta^* = \mathrm{id}$. Let $\pi : R \to C$ be the projection and $\iota : A = C \hookrightarrow R$ be the structure homomorphism giving the A-algebra structure on R. We first look at the map $\delta : R \to J/J^2$ given by $\delta(a) = a - P(a) \mod J^2$ for $P = \iota \circ \pi$. Then

$$\begin{aligned}a\delta(b) + b\delta(a) - \delta(ab) &= a(b - P(b)) + b(a - P(a)) - ab - P(ab) \\ &= (a - P(a))(b - P(b)) \equiv 0 \mod J^2.\end{aligned}$$

Thus δ is a A-derivation. By the universality of $\Omega_{R/A}$, we have an R-linear map $\phi : \Omega_{R/A} \to J/J^2$ such that $\phi \circ d = \delta$. By definition, $\delta(J)$ generates J/J^2 over R, and hence ϕ is surjective. Since J kills J/J^2, the surjection ϕ factors through

$\Omega_{R/A} \otimes_R C$ and induces γ. Note that $\beta(d \otimes 1_C)) = d \otimes 1_C|_J$ for the identity 1_C of C; so, $\gamma \circ \beta^* = \mathrm{id}$ as desired. $\square$

For any continuous R-module M, we write $R[M]$ for the R-algebra with square zero ideal M. Thus $R[M] = R \oplus M$ with the multiplication given by

$$(r \oplus x)(r' \oplus x') = rr' \oplus (rx' + r'x).$$

It is easy to see that $R[M] \in CNL_W$, if M is of finite type, and $R[M] \in CL_W$ if M is a p-profinite R-module. By definition,

$$Der_A(R, M) \cong \left\{\phi \in \mathrm{Hom}_{A-alg}(R, R[M]) \middle| \phi \mod M = \mathrm{id}\right\}, \tag{2.1}$$

where the map is given by $\delta \mapsto (a \mapsto (a \oplus \delta(a))$. Note that $i : R \to R\widehat{\otimes}_A R$ given by $i(a) = a \otimes 1$ is a section of $m : R\widehat{\otimes}_A R \to R$. We see easily that $R\widehat{\otimes}_A R/I^2 \cong R[\Omega_{R/A}]$ by $x \mapsto m(x) \oplus (x - i(m(x)))$. Note that $d(a) = 1 \otimes a - i(a)$ for $a \in R$.

2.2. **Congruence and differential modules.** Let R be an algebra over a normal noetherian domain B. We assume that R is an B-flat module of finite type. Let $\phi : R \to A$ be an B-algebra homomorphism for an integral B-domain A. We define

$$C_1(\phi; A) = \Omega_{R/B} \otimes_{R,\phi} \mathrm{Im}(\phi)$$

which we call the *differential* module of ϕ , and as we will see in Theorem 3.12, if R is a deformation ring, this module is the dual of the associated adjoint Selmer group. If ϕ is surjective, we just have

$$C_1(\phi; A) = \Omega_{R/B} \otimes_{R,\phi} A.$$

We usually suppose ϕ is surjective, but including in the definition for, something like, the normalization of $\mathrm{Im}(\phi)$ as A is useful. We suppose that R is reduced (i.e., having zero nilradical of R). Then the total quotient ring $\mathrm{Frac}(R)$ can be decomposed uniquely into $\mathrm{Frac}(R) = \mathrm{Frac}(\mathrm{Im}(\phi)) \times X$ as an algebra direct product. Write 1_ϕ for the idempotent of $\mathrm{Frac}(\mathrm{Im}(\phi))$ in $\mathrm{Frac}(R)$. Let $\mathfrak{a} = \mathrm{Ker}(R \to X) = (1_\phi R \cap R)$, $S = \mathrm{Im}(R \to X)$ and $\mathfrak{b} = \mathrm{Ker}(\phi)$. Here the intersection $1_\phi R \cap R$ is taken in $\mathrm{Frac}(R) = \mathrm{Frac}(\mathrm{Im}(\phi)) \times X$. Then we put

$$C_0(\phi; A) = (R/\mathfrak{a}) \otimes_{R,\phi} \mathrm{Im}(\phi) \cong \mathrm{Im}(\phi)/(\phi(\mathfrak{a})) \cong 1_\phi R/\mathfrak{a} \cong S/\mathfrak{b},$$

which is called the *congruence* module of ϕ but is actually a ring (cf. [H88a] Section 6). We can split the isomorphism $1_\phi R/\mathfrak{a} \cong S/\mathfrak{b}$ as follows: First note that $\mathfrak{a} = (R \cap (1_\phi R \times 0))$ in $\mathrm{Frac}(\mathrm{Im}(\phi)) \times X$. Then $\mathfrak{b} = (0 \times X) \cap R$, and we have

$$1_\phi R/\mathfrak{a} \cong R/(\mathfrak{a} \oplus \mathfrak{b}) \cong S/\mathfrak{b},$$

where the maps $R/(\mathfrak{a} \oplus \mathfrak{b}) \to 1_\phi R/\mathfrak{a}$ and $R/(\mathfrak{a} \oplus \mathfrak{b}) \to S/\mathfrak{b}$ are induced by two projections from R to $1_\phi R$ and S.

Write $K = \mathrm{Frac}(A)$. Fix an algebraic closure $\overline{K}$ of K. Since the spectrum $\mathrm{Spec}(C_0(\phi; A))$ of the congruence ring $C_0(\phi; A)$ is the scheme theoretic intersection of $\mathrm{Spec}(\mathrm{Im}(\phi))$ and $\mathrm{Spec}(R/\mathfrak{a})$ in $\mathrm{Spec}(R)$:

$$\mathrm{Spec}(C_0(\lambda; A)) = \mathrm{Spec}(\mathrm{Im}(\phi)) \cap \mathrm{Spec}(R/\mathfrak{a}) := \mathrm{Spec}(\mathrm{Im}(\phi)) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/\mathfrak{a}),$$

we conclude that

Proposition 2.3. *Let the notation be as above. Then a prime $\mathfrak{p}$ is in the support of $C_0(\phi; A)$ if and only if there exists an B-algebra homomorphism $\phi' : R \to \overline{K}$ factoring through $R/\mathfrak{a}$ such that $\phi(a) \equiv \phi'(a) \mod \mathfrak{p}$ for all $a \in R$.*

In other words, ϕ mod $\mathfrak{p}$ factors through $R/\mathfrak{a}$ and can be lifted to ϕ'. Therefore, if $B = \mathbb{Z}$ and A is the integer ring of a sufficiently large number field in $\overline{\mathbb{Q}}$, $\bigcup_\phi \mathrm{Supp}(C_0(\phi; A))$ is made of primes dividing the absolute different $\mathfrak{d}(R/\mathbb{Z})$ of R over $\mathbb{Z}$, and each prime appearing in the absolute discriminant of $R/\mathbb{Z}$ divides the order of the congruence module for some ϕ.

By Corollary 2.2 applied to the exact sequence: $0 \to \mathfrak{b} \to R \xrightarrow{\phi} A \to 0$, we know that

$$C_1(\phi; A) \cong \mathfrak{b}/\mathfrak{b}^2. \tag{2.2}$$

Since $C_0(\phi; A) \cong S/\mathfrak{b}$, we may further define $C_n(\phi; A) = \mathfrak{b}^n/\mathfrak{b}^{n+1}$ and call them *higher congruence modules*. The knowledge of all $C_n(\phi; A)$ is almost equivalent to the knowledge of the entire ring R. Therefore the study of $C_n(\phi; A)$ and the graded algebra

$$C(\phi; A) := \bigoplus_n C_n(\phi; A)$$

is important and interesting, when R is a Galois deformation ring. As we will see, even in the most favorable cases, we only know theoretically the cardinality of modules C_0 and C_1 for universal deformation rings R, so far.

3. Deformation rings

We introduce the notion of universal deformation rings of a given Galois representation into $GL_n(\mathbb{F})$ for a finite field.

3.1. **One dimensional case.** We can interpret the Iwasawa algebra Λ as a universal Galois deformation ring. Let $F/\mathbb{Q}$ be a number field with integer ring O. We write CL_W for the category of p-profinite local W-algebras A with $A/\mathfrak{m}_A = \mathbb{F}$. We fix a set $\mathcal{P}$ of properties of Galois characters (for example unramifiedness outside a fixed positive integer N). Fix a continuous character $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times$ with the property $\mathcal{P}$. A character $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to A^\times$ for $A \in CL_W$ is called a $\mathcal{P}$-deformation (or just simply a deformation) of $\overline{\rho}$ if ($\rho \mod \mathfrak{m}_A) = \overline{\rho}$ and ρ satisfies $\mathcal{P}$. A couple $(R, \boldsymbol{\rho})$ made of an object R of CL_W and a character $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to R^\times$ satisfying $\mathcal{P}$ is called a *universal couple* for $\overline{\rho}$ if for any $\mathcal{P}$-deformation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to A^\times$ of $\overline{\rho}$, we have a unique morphism $\phi_\rho : R \to A$ in CL_W (so it is a local W-algebra homomorphism) such that $\phi_\rho \circ \boldsymbol{\rho} = \rho$. By the universality, if exists, the couple $(R, \boldsymbol{\rho})$ is determined uniquely up to isomorphisms. The ring R is called the universal deformation ring and $\boldsymbol{\rho}$ is called the universal deformation of $\overline{\rho}$.

For a p-profinite abelian group $\mathcal{G}$ of topologically finite type (so, $\mathcal{G}$ is isomorphic to a product of Γ^m and a finite p-abelian group Δ), consider the group algebra $W[[\mathcal{G}]] = \varprojlim_n W[\mathcal{G}/\mathcal{G}^{p^n}]$. Taking a generator γ_i of the ith factor Γ in Γ^m, we have $W[[\mathcal{G}]] \cong W[\Delta][[T_1, \dots, T_m]]$ by sending $t_i = 1+T_i$ to γ_i. The tautological character $\kappa_\mathcal{G}$ given by $\kappa_\mathcal{G}(g) = g \in W[[\mathcal{G}]]$ is a universal character among all continuous characters $\rho : \mathcal{G} \to A^\times$ ($A \in CL_W$). In other words, for any such ρ, ρ extends to a unique W-algebra homomorphism $\iota_\rho : W[[\mathcal{G}]] \to A$ such that $\iota_\rho \circ \kappa_\mathcal{G} = \rho$. If $\mathcal{G}$ is finite, $\iota_\rho(\sum_g a_g g) = \sum_g a_g \rho(g)$ for $a_g \in W$, and otherwise, ι_ρ is the projective limit of such for finite quotients of $\mathcal{G}$. By the isomorphism $W[[\mathcal{G}]] \cong W[\Delta][[T_1, \dots, T_m]]$, we have $\kappa_\mathcal{G}(\delta \prod_i \gamma_i^{s_i})) = \delta \prod_i t_i^{s_i}$, where $t^s = (1+T)^s = \sum_{n=0}^\infty \binom{s}{n} T^n \in \mathbb{Z}_p$.

Fix an O-ideal $\mathfrak{c}$ prime to p and write $H_{\mathfrak{c}p^n}/F$ for the ray class field modulo $\mathfrak{c}p^n$. Then by Artin symbol, we can identify $\mathrm{Gal}(H_{\mathfrak{c}p^n}/F)$ with the ray class group $Cl_F(\mathfrak{c}p^n)$ (here n can be infinity). Let $C_F(\mathfrak{c}p^\infty)$ for the maximal p-profinite quotient of $Cl_F(\mathfrak{c}p^\infty)$; so, it is the Galois group of the maximal p-abelian extension of F inside $H_{\mathfrak{c}p^\infty}$. If $\overline{\rho}$ has prime-to-p conductor equal to $\mathfrak{c}$, we define a deformation ρ to satisfy $\mathcal{P}$ if ρ is unramified outside $\mathfrak{c}p$ and has prime-to-p conductor a factor of $\mathfrak{c}$. Then we have

Theorem 3.1. *Let $\rho_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to W^\times$ be the Teichmüller lift of $\overline{\rho}$ (by the group embedding $\mathbb{F}^\times \hookrightarrow W^\times$). The couple $(W[[C_F(\mathfrak{c}p^\infty)]], \rho_0\kappa_{\mathcal{G}})$ for $\mathcal{G} = C_F(\mathfrak{c}p^\infty)$ is universal among all $\mathcal{P}$-deformations.*

Proof. By $\mathcal{P}$, $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to A^\times$ factors through $\mathrm{Gal}(H_{\mathfrak{c}p^\infty}/F) = Cl_F(\mathfrak{c}p^\infty)$. Since $Cl_F(\mathfrak{c}p^\infty) = C_F(\mathfrak{c}p^\infty) \times \Delta$ for finite Δ, ρ_0 factors through Δ as $p \nmid |\Delta|$. Then $\rho\rho_0^{-1}$ has image in the p-profinite group $1 + \mathfrak{m}_A$; so, it factors through $C_F(\mathfrak{c}p^\infty)$. Thus the W-algebra homomorphism $\iota : W[[C_F(\mathfrak{c}p^\infty)]]$ extending $\rho\rho_0^{-1}$ is the unique morphism with $\iota \circ \rho_0\kappa_{\mathcal{G}} = \rho$ as ρ_0 has values in the coefficient ring W. □

We make this more explicit assuming $F = \mathbb{Q}$ and $\mathfrak{c} = 1$. Consider the group of p-power roots of unity $\mu_{p^\infty} = \bigcup_n \mu_{p^n} \subset \overline{\mathbb{Q}}^\times$. Note that $\mathbb{Q}[\mu_{p^\infty}] = H_{p^\infty}$. We have $Cl_{\mathbb{Q}}(p^\infty) \cong \mathbb{Z}_p^\times$ and $C_{\mathbb{Q}}(p^\infty) \cong \Gamma$ by the p-adic cyclotomic character ν. The logarithm power series

$$\log(1+x) = \sum_{n=1}^{\infty} -\frac{(-x)^n}{n}$$

converges absolutely p-adically on $p\mathbb{Z}_p$. Note that $\mathbb{Z}_p^\times = \mu_{p-1} \times \Gamma$ for $\Gamma = 1 + p\mathbb{Z}_p$ by $\mathbb{Z}_p^\times \mapsto (\omega(z) = \lim_{n\to\infty} z^{p^n}, \omega(z)^{-1}z) \in \mu_{p-1} \times \Gamma$. We define $\log_p : \mathbb{Z}_p^\times \to \Gamma$ by $\log_p(\zeta s) = \log(s) \in p\mathbb{Z}_p$ for $\zeta\mu_{p-1}$ and $s \in 1+p\mathbb{Z}_p = \Gamma$. Then $\kappa_\Gamma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \Lambda^\times$ is given by $\kappa(\sigma) = t^{\log_p(\nu(\sigma))/\log_p(\gamma)}$ $(\gamma = 1 + p)$. Thus $(W[[T]], \kappa\rho_0)$ is universal among deformations of $\overline{\rho}$ unramified outside p and ∞.

Let $\mathcal{P}$ be a set of properties of n-dimensional representations $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(A)$. For a given n-dimensional representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(\mathbb{F})$ satisfying $\mathcal{P}$, a $\mathcal{P}$-deformation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(A)$ is a continuous representation ssatisfying $\mathcal{P}$ with $\rho \mod \mathfrak{m}_A \cong \overline{\rho}$. Two deformations $\rho, \rho' : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(A)$ for $R \in CL_W$ is equivalent, if there exists an invertible matrix $x \in GL_n(A)$ such that $x\rho(\sigma)x^{-1} = \rho'(\sigma)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. We write $\rho \sim \rho'$ if ρ and ρ' are equivalent. A couple $(R, \boldsymbol{\rho})$ for a $\mathcal{P}$-deformation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(R)$ is called a universal couple over W, if for any given $\mathcal{P}$-deformation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to GL_n(R)$ there exists a unique W-algebra homomorphism $\iota_\rho : R \to A$ such that $\iota_\rho \circ \boldsymbol{\rho} \sim \rho$. There are other variations of the deformation ring depending on $\mathcal{P}$. For example, we can insist a couple $(R, \boldsymbol{\rho})$ is universal either among all everywhere unramified deformations of $\overline{\rho}$ or all deformations unramified outside p and ∞ whose restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is isomorphic to an upper triangular representation whose quotient character is unramified (ordinary deformations).

Let $F_{/\mathbb{Q}}$ be a finite extension inside $\overline{\mathbb{Q}}$ with integer ring O. Let Cl_F be the class group of F in the narrow sense. Put C_F for he p-Sylow subgroup of Cl_F. Identify C_F with the Galois group of p-Hilbert class field H_F over F. Then basically by the same argument proving Theorem 3.1, we get

Theorem 3.2. *Fix a character* $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathbb{F}^\times$ *unramified at every finite place. Then for the Teichmüller lift* $\rho_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to W^\times$ *of* $\overline{\rho}$ *and* $\boldsymbol{\rho}$ *given by* $\boldsymbol{\rho}(\sigma) = \rho_0(\sigma)\sigma|_{H_F} \in W[C_F]$*, the couple* $(W[C_F], \boldsymbol{\rho})$ *is universal among all everywhere unramified deformations of* $\overline{\rho}$*.*

3.2. **Congruence modules for group algebras.** Let H be a finite p-abelian group. We have a canonical algebra homomorphism: $W[H] \to W$ sending $\sigma \in H$ to 1. This homomorphism is called the *augmentation* homomorphism of the group algebra. Write this map $\pi : W[H] \to W$. Then $\mathfrak{b} = \mathrm{Ker}(\pi)$ is generated by $\sigma - 1$ for $\sigma \in H$. Thus

$$\mathfrak{b} = \sum_{\sigma \in H} W[H](\sigma - 1)W[H].$$

We compute the congruence module and the differential module $C_j(\pi, W)$ $(j = 0, 1)$.

Corollary 3.3. *We have* $C_0(\pi; W) \cong W/|H|W$ *and* $C_1(\pi; W) = H \otimes_{\mathbb{Z}} W$*. In particular, if* $\mathbb{Z}[G] \cong \mathbb{Z}[G']$ *for finite abelian groups* G, G'*,* $\mathbb{Z}[G] \cong \mathbb{Z}[G']$ *as algebras implies* $G \cong G'$*.*

Replacing $\mathbb{Z}$ by fields, the last assertion of the corollary is proven by [PW50] and [D56], but the fact is not necessarily true for non abelian groups [C62].

Proof. Let K be the quotient field of W. Then π gives rise to the algebra direct factor $K\varepsilon \subset K[H]$ for the idempotent $\varepsilon = \frac{1}{|H|}\sum_{\sigma \in H} \sigma$. Thus $\mathfrak{a} = K\varepsilon \cap W[H] = (\sum_{\sigma \in H} \sigma)$ and $\pi(W(H))/\mathfrak{a} = (\varepsilon)/\mathfrak{a} \cong W/|H|W$.

Note $C_1(\pi; W) = \mathfrak{b}/\mathfrak{b}^2$ by (2.2). Since $\mathfrak{b}^2$ is generated by $(\sigma - 1)(\tau - 1)$ for $\sigma, \tau \in H$, we find $H \otimes_{\mathbb{Z}} W \cong \mathfrak{b}/\mathfrak{b}^2$ by sending σ to $(\sigma - 1) \in \mathfrak{b}$.

As for last statement, we only need to prove $G_p \cong G'_p$ for the p-Sylow subgroups G_p, G'_p of G and G'. Then $\mathbb{Z}[G] \cong \mathbb{Z}[G']$ implies $\mathbb{Z}_p[G_p] \cong \mathbb{Z}_p[G_p]$ as taking tensor product with $\mathbb{Z}_p$ over $\mathbb{Z}$ and projecting down to $\mathbb{Z}_p[G_p]$ and $\mathbb{Z}_p[G'_p]$ respectively. Then $G_p \cong G_p \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong C_1(\pi; \mathbb{Z}_p[G_p]) \cong C_1(\pi'; \mathbb{Z}_p[G'_p]) \cong G'_p$ for augmentation homomorphisms π, π' with respect to G_p and G'_p, respectively.

There is another deformation theoretic proof of $C_1(\pi; W[H]) = W \otimes_{\mathbb{Z}} H$. Consider the functor $\mathcal{F} : CL_W \to SETS$ given by

$$\mathcal{F}(A) = \mathrm{Hom}_{\mathrm{group}}(H, A^\times) = \mathrm{Hom}_{W\text{-alg}}(W[H], A).$$

Thus $R := W[H]$ and the character $\boldsymbol{\rho} : H \to W[H]$ (the inclusion of H into $W[H]$) are universal among characters of H with values in $A \in CL_W$. Then for any R-module X, consider $R[X] = R \oplus X$ with algebra structure given by $rx = 0$ and $xy = 0$ for all $r \in R$ and $x, y \in X$. Thus X is an ideal of $R[X]$ with $X^2 = 0$. Extend the functor $\mathcal{F}$ to all local W-algebras with residue field $\mathbb{F}$ in an obvious way. Then define $\Phi(X) = \{\rho \in \mathcal{F}(R[X]) | \rho \mod X = \boldsymbol{\rho}\}$. Write $\rho(\sigma) = \boldsymbol{\rho}(\sigma) \oplus u'_\rho(\sigma)$ for $u'_\rho : H \to X$. Since

$$\begin{aligned}\boldsymbol{\rho}(\sigma\tau) \oplus u'_\rho(\sigma\tau) &= \rho(\sigma\tau) \\ &= (\boldsymbol{\rho}(\sigma) \oplus u'_\rho(\sigma))(\boldsymbol{\rho}(\tau) \oplus u'_\rho(\tau)) = \boldsymbol{\rho}(\sigma\tau) \oplus (u'_\rho(\sigma)\boldsymbol{\rho}(\tau) + \boldsymbol{\rho}(\sigma)u'_\rho(\tau)),\end{aligned}$$

we have $u'_\rho(\sigma\tau) = u'_\rho(\sigma)\boldsymbol{\rho}(\tau) + \boldsymbol{\rho}(\sigma)u'_\rho(\tau)$, and thus $u_\rho := \boldsymbol{\rho}^{-1}u'_\rho : H \to X$ is a homomorphism from H into X. This shows $\mathrm{Hom}(H, X) = \Phi(X)$.

Any W-algebra homomorphism $\xi : R \to R[X]$ with $\xi \mod X = \mathrm{id}_R$ can be written as $\xi = \mathrm{id}_R \oplus d_\xi$ with $d_\xi : R \to X$. Since $(r \oplus x)(r' \oplus x') = rr' \oplus rx' + r'x$ for

$r, r' \in R$ and $x.x' \in X$, we have $d_\xi(rr') = rd_\xi(r') + r'd_\xi(r)$; so, $d_\xi \in Der_W(R, X)$. By universality of $(R, \boldsymbol{\rho})$, we have

$$\Phi(X) \cong \{\xi \in \mathrm{Hom}_{W\text{-alg}}(R, R[X]) | \xi \bmod X = \mathrm{id}\} \\ = Der_W(R, X) = \mathrm{Hom}_R(\Omega_{R/W}, X).$$

Thus taking $X = K/W$, we have

$$\mathrm{Hom}_W(H \otimes_{\mathbb{Z}} W, K/W) = \mathrm{Hom}(H, K/W) = \mathrm{Hom}_R(\Omega_{R/W}, K/W) \\ = \mathrm{Hom}_W(\Omega_{R/W} \otimes_{R,\pi} W, K/W).$$

By taking Pontryagin dual back, we have $H \cong \Omega_{R/W} \otimes_{R,\pi} W = C_1(\pi; W)$. □

We apply the above corollary to $H = C_F$. Suppose that $F/\mathbb{Q}$ is a Galois extension of degree prime to p. Regard F as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, we get an Artin representation $\mathrm{Ind}_F^{\mathbb{Q}} \mathbf{1} = \mathbf{1} \oplus \chi$. Note that as a Galois module with coefficients in $\mathbb{Z}$

$$\chi = \{x \in O | \mathrm{Tr}(x) = 0\}.$$

Define $V(\chi) = \{x \in O | \mathrm{Tr}(x) = 0\} \otimes_{\mathbb{Z}} W$ as Galois module, and put $V(\chi)^* = V(\chi) \otimes_W K/W$. Then for $\Omega_F = \frac{(2\pi)^{r_2} R_F}{w_F \sqrt{D_F}}$ for the regulator R_F of F (up to 2-power), by the class number formula of Dirichlet/Kummer/Dedekind, we have

$$\Omega_F |Cl_F| = \mathrm{Res}_{s=1} \zeta_F(s) = \mathrm{Res}_{s=1} L(s, \mathrm{Ind}_F^{\mathbb{Q}} \mathbf{1}) = L(1, \chi) \mathrm{Res}_{s=1} \zeta(s) = L(1, \chi).$$

Note by Shapiro's lemma, assuming $F/\mathbb{Q}$ is a Galois extension with maximal unramified p-extension F^{ur}/F,

$$\mathrm{Hom}(\mathrm{Gal}(F^{ur}/\mathbb{Q}), K/W) \oplus H^1(\mathrm{Gal}(F^{ur}/\mathbb{Q}), V(\chi)^*) \\ \cong H^1(\mathrm{Gal}(F^{ur}/\mathbb{Q}), \mathrm{Ind}_F^{\mathbb{Q}} K/W) \\ \cong H^1(\mathrm{Gal}(F^{ur}/F), K/W) \cong \mathrm{Hom}(C_F, K/W) \cong C_F \otimes_{\mathbb{Z}} W,$$

where we have written $\mathrm{Ind}_F^{\mathbb{Q}} K/W = (K/W) \oplus V(\chi)^*$ as Galois module. Note that if $p \nmid [F : \mathbb{Q}]$, and $\phi \in \mathrm{Hom}(\mathrm{Gal}(F^{ur}/\mathbb{Q}), K/W)$ factors through $\mathrm{Gal}(H_F/\mathbb{Q}) = \mathrm{Gal}(F/\mathbb{Q}) \ltimes H_F$, and $\mathrm{Gal}(F/\mathbb{Q})$ contains all inertial group, ϕ has to be unramified everywhere; so, $\phi = 0$. Thus we have $\mathrm{Hom}(\mathrm{Gal}(F^{ur}/\mathbb{Q}), K/W) = 0$. We can identify $H^1(\mathrm{Gal}(F^{ur}/F), V(\chi)^*)$ with the Selmer group of χ given by

$$\mathrm{Sel}_{\mathbb{Q}}(\chi) = \mathrm{Ker}(H^1(F, V(\chi)^*) \to \prod_l H^1(I_l, V(\chi)^*)) \tag{3.1}$$

for the inertia group $I_l \subset \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$. We conclude

Theorem 3.4 (Class number formula)**.** *Suppose that $F/\mathbb{Q}$ is Galois and $p \nmid [F : \mathbb{Q}]$. Let $\pi : W[C_F] \to W$ be the augmentation homomorphism. We have, for $r(W) = \mathrm{rank}_{\mathbb{Z}_p} W$,*

$$\left| \frac{L(1, \chi)}{\Omega_F} \right|_p^{r(W)} = |C_1(\pi; W)|^{-1} = |C_0(\pi; W)|^{-1} = \big| |\mathrm{Sel}_{\mathbb{Q}}(\chi)| \big|_p^{r(W)}$$

and $C_1(\pi; W) = C_F \otimes W$ and $C_0(\pi; W) = W/|C_F| W$.

Recall $\Omega_F = \frac{(2\pi)^{r_2} R_F}{w_F\sqrt{D_F}}$, which contains the period $2\pi i = \oint \frac{dt}{t}$ and another transcendental factor R_F. This type of factorization of the transcendental factor also shows up in the $GL(2)$-case (see [H99]) if the adjoint L-value is not critical.

Decompose $C_F = \bigoplus_{i=1}^m C_i$ for cyclic group C_i. Then sending $X_i \mapsto \gamma_i - 1$ for a generators γ_i of C_i, we have $W[C_F] \cong W[[X_1, \dots, X_m]]/(x_1^{|C_1|} - 1, \dots, x_m^{|C_m|} - 1)$ for $x_i = 1 + X_i$. Let A be a complete normal local domain (for example, a complete regular local rings like $A = W$ or $A = W[[T]]$ or $A = W[[T_1, \dots, T_r]]$ (power series ring)). Any local A-algebra R free of finite rank over A has a presentation $R \cong A[[X_1, \dots, X_n]]/(f_1, \dots, f_m)$ for $f_i \in A[[X_1, \dots, X_n]]$ with $m \geq n$. If $m = n$, then R is called a *local complete intersection* over A. There is a theorem of Tate generalizing Corollary 3.3 to local complete inetersection rings. To introduce this, let us explain the notion of pseudo-isomorphisms between torsion A-modules (see [BCM, VII.4.4] for a more detailed treatment). For two A-modules M, N of finite type, a morphism $\phi : M \to N$ is called a pseudo isomorphism if the annihilator of $\mathrm{Ker}(\phi)$ and $\mathrm{Coker}(\phi)$ each has height at least 2 (i.e., the corresponding closed subscheme of $\mathrm{Spec}(A)$ has co-dimension at least 2). If $A = W$, a pseudo-isomorphism is an isomorphism, and if $A = W[[T]]$, it is an isogeny (i.e., having finite kernel and cokernel). The classification theorem of torsion A-modules M of finite type tells us that we have a pseudo isomorphism $M \to \bigoplus_i A/\mathfrak{f}_i$ for finitely many reflexive ideal $0 \neq \mathfrak{f}_i \in A$. An ideal $\mathfrak{f}$ is *reflexive* if $\mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{f}, A), A) \cong \mathfrak{f}$ canonically as A-modules (and equivalently $\mathfrak{f} = \bigcap_{\lambda \in A, (\lambda) \supset \mathfrak{f}} (\lambda)$; i.e., close to be principal). Then the *characteristic ideal* $\mathrm{char}(M)$ of M is defined by $\mathrm{char}(M) := \prod_i \mathfrak{f}_i \subset A$. If A is a unique factorization domain (for example, if A is regular; a theorem of Auslander-Buchsbaum [CRT, Theorem 20.3]), any reflexive ideal is principal. If $A = W$, $|M|$ is a generator of the ideal $\mathrm{char}(M)$ over W.

Theorem 3.5 (J. Tate)**.** *Assume that R is a local complete intersection over a complete normal noetherian local domain A with an algebra homomorphism $\lambda : R \to A$. If after tensoring the quotient field Q of A, $R \otimes_A Q = (\mathrm{Im}(\lambda) \otimes_A Q) \oplus S$ as algebra direct sum for some Q-algebra S, then $C_j(\lambda; A)$ is a torsion A-module of finite type, and we have*

$$\mathrm{Ann}_A(C_0(\lambda; A)) = \mathrm{char}(C_0(\lambda; A)) = \mathrm{char}(C_1(\lambda; A)).$$

In the following section, we prove

$$\mathrm{length}_A(C_0(\lambda; A)) = \mathrm{length}_A(C_1(\lambda; A)), \tag{3.2}$$

assuming that A is a discrete valuation ring (see Proposition 3.10). If A is a normal noetherian domain, $\mathrm{char}_A(M) = \prod_P P^{\mathrm{length}_{A_P} M_P}$ for the localization M_P at height 1-primes P for a given A-torsion module M. Since A_P is a discrete valuation ring if and only if P has height 1, this implies the above theorem. Actually Tate proved a finer result giving the identity of Fitting ideals of $C_0(\lambda; A)$ and $C_1(\lambda; A)$, which is clear from the proof given below (see [MW84, Appendix] for Fitting ideals). This result is a local version of general Grothendieck–Serre duality of proper morphisms studied by Hartshorne (see [ALG, III.7] for more details).

3.3. **Proof of Tate's theorem.** We reproduce the proof from [MR70, Appendix] (which actually determines the Fitting ideal of M more accurate than $\mathrm{char}(M)$; see [MW84, Appendix] for Fitting ideals). We prepare some preliminary results; so, we do not assume yet that R is a local complete inetersection over A. Let A be a normal noetherian integral domain of characteristic 0 and R be a reduced A–algebra free of finite rank r over A. The algebra R is called a *Gorenstein* algebra over A if $\mathrm{Hom}_A(R, A) \cong R$ as R–modules. Since R is free of rank r over A, we choose a base $(x_1, \dots, x_r)$ of R over A. Then for each $y \in R$, we have $r \times r$–matrix $\rho(y)$ with entries in A defined by $(yx_1, \dots, yx_r) = (x_1, \dots, x_r)\rho(y)$. Define $\mathrm{Tr}(y) = \mathrm{Tr}(\rho(y))$. Then $\mathrm{Tr} : R \to A$ is an A–linear map, well defined independently of the choice of the base. Suppose that $\mathrm{Tr}(xR) = 0$. Then in particular, $\mathrm{Tr}(x^n) = 0$ for all n. Therefore all eigenvalues of $\rho(x)$ are 0, and hence $\rho(x)$ and x is nilpotent. By the reducedness of R, $x = 0$ and hence the pairing $(x, y) = \mathrm{Tr}(xy)$ on R is non-degenerate.

Lemma 3.6. *Let A be a normal noetherian integral domain of characteristic 0 and R be an A–algebra. Suppose the following three conditions:*

(1) *R is free of finite rank over A;*
(2) *R is Gorenstein; i.e., we have $i : \mathrm{Hom}_A(R, A) \cong R$ as R–modules;*
(3) *R is reduced.*

Then for an A–algebra homomorphism $\lambda : R \to A$, we have

$$C_0(\lambda; A) \cong A/\lambda(i(\mathrm{Tr}_{R/A}))A.$$

In particular, $\mathrm{length}_A C_0(\lambda; A)$ *is equal to the valuation of $d = \lambda(i(\mathrm{Tr}_{R/A}))$ if A is a discrete valuation ring.*

Proof. Let $\phi = i^{-1}(1)$. Then $\mathrm{Tr}_{R/A} = \delta\phi$. The element $\delta = \delta_{R/A}$ is called the different of R/A. Then the pairing $(x, y) \mapsto \mathrm{Tr}_{R/A}(\delta^{-1}xy) \in A$ is a perfect pairing over A, where $\delta^{-1} \in S = Frac(R)$ and we have extended $\mathrm{Tr}_{R/A}$ to $S \to K = Frac(A)$. Since R is commutative, $(xy, z) = (y, xz)$. Decomposing $S = K \oplus X$, we have

$$C_0(\lambda; A) = \mathrm{Im}(\lambda)/\lambda(\mathfrak{a}) \cong A/R \cap (K \oplus 0).$$

Then it is easy to conclude that the pairing (,) induces a perfect A–duality between $R \cap (K \oplus 0)$ and $A \oplus 0$. Thus $R \cap (K \oplus 0)$ is generated by $\lambda(\delta) = \lambda(i(\mathrm{Tr}_{R/A}))$. □

Next we introduce two A–free resolutions of R, in order to compute $\delta_{R/A}$. We start slightly more generally. Let X be an algebra. A sequence $f = (f_1, \dots, f_n) \in X^n$ is called *regular* if $x \mapsto f_j x$ is injective on $X/(f_1, \dots, f_{j-1})$ for all $j = 1, \dots, n$. We now define a complex $K^\bullet_X(f)$ (called *the Koszul complex*) out of a regular sequence f (see [CRT, Section 16]). Let $V = X^n$ with a standard base $e_1, \dots, e_n$. Then we consider the exterior algebra

$$\bigwedge^\bullet V = \bigoplus_{j=0}^{n} (\wedge^j V).$$

The graded piece $\wedge^j V$ has a base $e_{i_1,\dots,i_j} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}$ indexed by sequences $(i_1, \dots, i_j)$ satisfying $0 < i_1 < i_2 < \cdots < i_j \le n$. We agree to put $\bigwedge^0 V = X$ and $\bigwedge^j V = 0$ if $j > n$. Then we define X–linear differential $d : \bigwedge^j X \to \bigwedge^{j-1} X$ by

$$d(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = \sum_{r=1}^{j} (-1)^{r-1} f_{i_r} e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} \wedge e_{i_{r+1}} \wedge \cdots \wedge e_{i_j}.$$

In particular, $d(e_j) = f_j$ and hence,

$$\bigwedge^0 V/d(\bigwedge^1 V) = X/(f).$$

Thus, $(K^\bullet_X(f), d)$ is a complex and X–free resolusion of $X/(f_1,\dots,f_n)$. We also have

$$d_n(e_1\wedge e_2\wedge\cdots\wedge e_n)=\sum_{j=1}^n(-1)^{j-1}f_je_1\wedge\cdots\wedge e_{j-1}\wedge e_{j+1}\wedge\cdots\wedge e_n.$$

Suppose now that X is a B–algebra. Identifying $\bigwedge^{n-1}V$ with V by

$$e_1\wedge\cdots\wedge e_{j-1}\wedge e_{j+1}\wedge\cdots\wedge e_n\mapsto e_j$$

and $\bigwedge^n V$ with X by $e_1\wedge e_2\wedge\cdots\wedge e_n\mapsto 1$, we have

$$\mathrm{Im}(d_n^*:\mathrm{Hom}_B(\textstyle\bigwedge^{n-1}V,Y)\to\mathrm{Hom}_B(\bigwedge^nV,Y))\cong(f)\mathrm{Hom}_B(X,Y),$$

where $(f)\mathrm{Hom}_B(X,Y)=\sum_j f_j\mathrm{Hom}_B(X,Y)$, regarding $\mathrm{Hom}_B(X,Y)$ as an X–module by $y\phi(x)=\phi(xy)$. This shows that if X is an B–algebra free of finite rank over B, $K^\bullet_X(f)$ is a B–free resolution of $X/(f)$, and

$$\mathrm{Ext}^n_B(X/(f),Y)=H^n(\mathrm{Hom}_B(K^\bullet_X(f),Y))\cong\frac{\mathrm{Hom}_B(X,Y)}{(f)\mathrm{Hom}_B(X,Y)}\tag{3.3}$$

for any B–module Y.

We now suppose that R is a local complete intersection over A. Thus R is free of finite rank over A and $R\cong B/(f_1,\dots,f_n)$ for $B=A[[T_1,\dots,T_n]]$. Write t_j for $T_j \mod (f_1,\dots,f_n)$ in R. Since R is local, t_j are contained in the maximal ideal $\mathfrak{m}_R$ of R. We consider $C=B\otimes_A R\cong R[[T_1,\dots,T_n]]$. Then

$$R=R[[T_1,\dots,T_n]]/(T_1-t_1,\dots,T_n-t_n),$$

and $g=(T_1-t_1,\dots,T_n-t_n)$ is a regular sequence in $C=R[[T_1\dots,T_n]]$. Since C is B–free of finite rank, the two complexes $K^\bullet_B(f)\twoheadrightarrow R$ and $K^\bullet_C(g)\twoheadrightarrow R$ are B–free resolutions of R.

We have an Λ–algebra homomorphism $\Phi:B\hookrightarrow C$ given by $\Phi(x)=x\otimes 1$. We extend Φ to $\Phi^\bullet:K^\bullet_B(f)\to K^\bullet_C(g)$ in the following way. Write $f_i=\sum_{j=1}^n b_{ij}g_j$. Then we define $\Phi^1:K^1_B(f)\to K^1_C(g)$ by $\Phi^1(e_i)=\sum_{j=1}^n b_{ij}e_j$. Then $\Phi^j=\bigwedge^j\Phi^1$. One can check that this map $\Phi^\bullet$ is a morphism of complexes. In particular,

$$\Phi_n(e_1\wedge\cdots\wedge e_n)=\det(b_{ij})e_1\wedge\cdots\wedge e_n.\tag{3.4}$$

Since $\Phi^\bullet$ is the lift of the identity map of R to the B–projective resolutions $K^\bullet_B(f)$ and $K^\bullet_C(g)$, it induces an isomorphism of extension groups computed by $K^\bullet_C(g)$ and $K^\bullet_B(f)$:

$$\Phi^*:H^\bullet(\mathrm{Hom}_B(K^\bullet_C(g),B))\cong\mathrm{Ext}^j_B(R,B)\cong H^\bullet(\mathrm{Hom}_B(K^\bullet_B(f),B)).$$

In particular, identifying $\bigwedge^n B^n=B$, we have from (3.3) that

$$H^n(\mathrm{Hom}_B(K^\bullet_B(f),B))=\mathrm{Hom}_B(B,B)/(f)\mathrm{Hom}_B(B,B)=B/(f)=R$$

and similarly

$$H^n(\mathrm{Hom}_B(K^\bullet_C(g),B))=\frac{\mathrm{Hom}_B(C,B)}{(g)\mathrm{Hom}_B(C,B)}.$$

The isomorphism between R and $\frac{\mathrm{Hom}_B(C,B)}{(g)\mathrm{Hom}_B(C,B)}$ is induced by Φ_n which is a multiplication by $d=\det(b_{ij})$ (see (3.4)). Thus we have

Lemma 3.7. *Suppose that R is a local complete intersection over A. Write $\pi : B = A[[T_1, \ldots, T_n]] \twoheadrightarrow R$ be the projection as above. We have an isomorphism:*

$$h : \frac{\mathrm{Hom}_B(C, B)}{(T_1 - t_1, \ldots, T_n - t_n)\mathrm{Hom}_B(C, B)} \cong R$$

given by $h(\phi) = \pi(\phi(d))$ for $d = \det(b_{ij}) \in C$.

We have a base change map:

$$\iota : \mathrm{Hom}_A(R, A) \longrightarrow \mathrm{Hom}_B(C, B) = \mathrm{Hom}_B(B \otimes_A R, B \otimes_A A),$$

taking ϕ to $\mathrm{id} \otimes \phi$. Identifying C and B with power series rings, $\iota(\phi)$ is just applying the original ϕ to coefficients of power series in $R[[T_1, \ldots, T_n]]$. We define $I = h \circ \iota : \mathrm{Hom}_A(R, A) \to R$.

Lemma 3.8. *Suppose that R is a local complete intersection over A. Then the above map I is an R–linear isomorphism, satisfying $I(\phi) = \pi(\iota(\phi(d))$. Thus the ring R is Gorenstein.*

Proof. We first check that I is an R–linear map. Since $I(\phi) = \pi(\iota(\phi(d))$, we compute $I(\phi \circ b))$ and $rI(\phi)$ for $b \in B$ and $r = \pi(b)$. By definition, we see

$$I(\pi(bx)) = \pi(\iota(\phi(r \otimes 1)d)) \quad \text{and} \quad rI(\phi) = \pi(b\iota(\phi(d)).$$

Thus we need to check $\pi(\iota(\phi)((r \otimes 1 - 1 \otimes b)d)) = 0$. This follows from:

$$r \otimes 1 - 1 \otimes b \in (g) \quad \text{and} \quad \det(b_{ij})g_i = \sum_i b'_{ij} f_i,$$

where b'_{ij} are the (i, j)–cofactors of the matrix (b_{ij}). Thus I is R–linear. Since ι mod $\mathfrak{m}_B$ for the maximal ideal $\mathfrak{m}_B$ of B is a surjective isomorphism from

$$\mathrm{Hom}_A((A/\mathfrak{m}_A)^r, A/\mathfrak{m}_A) = \mathrm{Hom}_A(R, A) \otimes_A A/\mathfrak{m}_A$$

onto

$$\mathrm{Hom}_B((B/\mathfrak{m}_B)^r, B/\mathfrak{m}_B) = \mathrm{Hom}_B(C, B) \otimes_B B/\mathfrak{m}_B,$$

the map ι is non-trivial modulo $\mathfrak{m}_C$. Thus $I \mod \mathfrak{m}_R$ is non-trivial. Since h is an isomorphism, $\mathrm{Hom}_B(C, B) \otimes_C C/\mathfrak{m}_C$ is 1–dimensional, and hence $I \mod \mathfrak{m}_R$ is surjective. By Nakayama's lemma, I itself is surjective. Since the target and the source of I are A–free of equal rank, the surjectivity of I tells us its injectivity. This finishes the proof. □

Corollary 3.9. *Suppose that R is a local complete intersection over A. We have $I(\mathrm{Tr}_{R/A}) = \pi(d)$ for $d = \det(b_{ij})$, and hence the different $\delta_{R/A}$ is equal to $\pi(d)$.*

Proof. The last assertion follows from the first by $I(\phi) = \pi(\iota(\phi(d))$. To show the first, we choose dual basis $x_1, \ldots, x_r$ of R/A and $\phi_1, \ldots, \phi_r$ of $\mathrm{Hom}_A(R, A)$. Thus for $x \in R$, writing $xx_i = \sum_i a_{ij}x_j$, we have $\mathrm{Tr}(x) = \sum_i a_{ii} = \sum_i \phi_i(xx_i) = \sum_i x_i\phi_i(x)$. Thus $\mathrm{Tr} = \sum_i x_i\phi_i$.

Since x_i is also a base of C over B, we can write $d = \sum_j b_j x_i$ with $\iota(\phi_i)(d) = b_i$. Then we have

$$I(\mathrm{Tr}_{R/A}) = \sum_i x_i I(\phi_i) = \sum_i x_i \pi(\iota(\phi_i)(d)) = \sum_i x_i \pi(b_i) = \pi(\sum_i b_i x_i) = \pi(d).$$

This shows the desired assertion. □

We now finish the proof of (3.2):

Proposition 3.10. *Let A be a discrete valuation ring, and let R be a reduced complete intersection over A. Then for an A–algebra homomorphism $R \to A$, we have*

$$\mathrm{length}_A\, C_0(\lambda, A) = \mathrm{length}_A\, C_1(\lambda, A).$$

Actually the assertion of the proposition is equivalent to R being a local complete inetersection over A (see [L95] for a proof).

Proof. Let X be a torsion A–module, and suppose that we have an exact sequence:

$$A^r \xrightarrow{L} A^r \to X \to 0$$

of A–modules. Then we claim $\mathrm{length}_A X = \mathrm{length}_A A/\det(L)A$. By elementary divisor theory applied to L, we may assume that L is a diagonal matrix with diagonal entry $d_1, \dots, d_r$. Then the assertion is clear, because $X = \bigoplus_j A/d_jA$ and length A/dA is equal to the valuation of d.

Since R is reduced, $\Omega_{R/A}$ is a torsion R–module, and hence $\Omega_{R/A} \otimes_R A = C_1(\lambda; A)$ is a torsion A–module. Since R is a local complete inetersection over A, we can write

$$R \cong A[[T_1, \dots, T_r]]/(f_1, \dots, f_r).$$

Then by Corollary 2.2 (ii), we have the following exact sequence for $J = (f_1, \dots, f_r)$:

$$J/J^2 \otimes_{A[[T_1,\dots,T_r]]} A \longrightarrow \Omega_{A[[T_1,\dots,T_r]]/A} \otimes_{A[[T_1,\dots,T_r]]} A \longrightarrow \Omega_{R/A} \otimes_R A \to 0.$$

This gives rise to the following exact sequence:

$$\bigoplus_j A df_j \xrightarrow{L} \bigoplus_j A dT_j \longrightarrow C_1(\lambda; A) \to 0,$$

where $df_j = f_j \mod J^2$. Since $C_1(\lambda; A)$ is a torsion A–module, we see that $\mathrm{length}_A(A/\det(L)A) = \mathrm{length}_A C_1(\lambda; A)$. Since $g = (T_1 - t_1, \dots, T_n - t_n)$, we see easily that $\det(L) = \pi(\lambda(d))$. This combined with Corollary 3.9 and Lemma 3.6 shows the desired assertion. □

3.4. **Two dimensional cases.** Fix a positive integer N prime to p. Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F})$ be the Galois representation unramified outside Np. Let ψ be the Teichmüller lift of $\det(\overline{\rho})$. For simplicity, assume that ψ and $\overline{\psi} := \det(\overline{\rho})$ has conductor divisible by N and $p \nmid \varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$. We consider deformations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(A)$ (over W) satisfying the following three properties:

- (D1) ρ is unramified outside Np,
- (D2) $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \epsilon_\rho & * \\ 0 & \delta_\rho \end{pmatrix}$ with δ_ρ unramified while ϵ_ρ ramified,
- (D3) For each prime $l|N$, writing I_l for the inertia group $\rho|_{I_l} \cong \begin{pmatrix} \psi_l & 0 \\ 0 & 1 \end{pmatrix}$ regarding $\psi_l = \psi|_{\mathbb{Z}_l^\times}$ as the character of I_l by local class field theory.

In particular, we assume (D1–3) for $\overline{\rho}$. Writing $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta} \end{pmatrix}$, we always assume that $\overline{\epsilon}$ is ramified while $\overline{\delta}$ is unramified. We admit the following fact and study its consequences:

Theorem 3.11 (B. Mazur)**.** *We have an universal couple $(R, \boldsymbol{\rho})$ of a W-algebra R and a Galois representation $\boldsymbol{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(R)$ such that $\boldsymbol{\rho}$ is universal among deformations satisfying (D1–3). The algebra R is a $W[[C_{\mathbb{Q}}(Np^\infty)]]$-algebra canonically by the universality of $W[[C_{\mathbb{Q}}(Np^\infty)]]$ applied to $\det(\boldsymbol{\rho})$. In particular,*

R is an algebra over $\Lambda = W[[T]] \subset W[[C_{\mathbb{Q}}(Np^\infty)]]$. If we add one more property to (D1–3)

(det) $\det(\rho) = \psi\epsilon\omega^{-k}\nu^k$ *for the p-adic cyclotomic character ν and $k = k(P)$*

and a character $\epsilon : \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \to \mu_{p^\infty}(W)$, then the residual couple $(R/PR, \boldsymbol{\rho}_P)$ with $\boldsymbol{\rho}_P = \boldsymbol{\rho} \mod P$ for $P = (t - \epsilon([\gamma, \mathbb{Q}_p])\gamma^k) \subset \Lambda$ is universal among deformations satisfying (D1–3) *and* (det).

See [MFG, §3.2.4] and [HMI, §3.2] for a proof. We write $\boldsymbol{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \left(\begin{smallmatrix} \boldsymbol{\epsilon} & * \\ 0 & \boldsymbol{\delta} \end{smallmatrix}\right)$ with $\boldsymbol{\delta}$ unramified.

3.5. **Adjoint Selmer groups.** We define the adjoint Selmer group over a number field F. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(A)$ be a deformation satisfying (D1–3). Write $V(\rho) = A^2$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via ρ. Since $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ for an unramified δ, we have a filtration $V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}) \hookrightarrow V(\rho) \twoheadrightarrow V_{\mathfrak{p}}(\delta_{\mathfrak{p}})$ stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ for a prime $\mathfrak{p}|p$ of F, where on $V_{\mathfrak{p}}(?)$, $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})$ acts via the character ? and $?_{\mathfrak{p}} = ?|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_{\mathfrak{p}})}$.

We now let $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ act on $M_2(A) = \mathrm{End}_A(V(\rho))$ by conjugation: $x \mapsto \rho(\sigma)x\rho(\sigma)^{-1}$. The trace zero subspace $\mathfrak{sl}(A)$ is stable under this action. This new Galois module of dimension 3 is called the adjoint representation of ρ and written as $Ad(\rho)$. Thus

$$V = V(Ad(\rho)) = \left\{T \in \mathrm{End}_A(V(\rho)) \middle| \mathrm{Tr}(T) = 0\right\}.$$

This space has a three step filtration: $0 \subset V_{\mathfrak{p}}^+ \subset V_{\mathfrak{p}}^- \subset V$ given by

$$(+) \qquad V_{\mathfrak{p}}^+(Ad(\rho)) = \left\{T \in V_{\mathfrak{p}}^-(Ad(\rho)) \middle| T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}}))) = 0\right\},$$

$$(-) \qquad V_{\mathfrak{p}}^-(Ad(\rho)) = \left\{T \in V(Ad(\rho)) \middle| T(V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})) \subset V_{\mathfrak{p}}(\epsilon_{\mathfrak{p}})\right\}.$$

We take a base of $V(\rho)$ so that $\rho|_{D_{\mathfrak{p}}} = \begin{pmatrix} \epsilon_{\mathfrak{p}} & * \\ 0 & \delta_{\mathfrak{p}} \end{pmatrix}$, then we have

$$V_{\mathfrak{p}}^+(Ad(\rho)) = \{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\} \subset V_{\mathfrak{p}}^-(Ad(\rho)) = \left\{\left(\begin{smallmatrix} a & * \\ 0 & b \end{smallmatrix}\right) \middle| a + b = 0\right\} \subset M_2(A).$$

Writing $A^\vee$ for the Pontryagin dual module $\mathrm{Hom}_W(A, K/W) \cong \mathrm{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ for the quotient field K of W. Then for any A-modules M, we put $M^* = M \otimes_A A^\vee$. In particular, $V(Ad(\rho))^*$ and $V_{\mathfrak{p}}(?)^*$ are divisible Galois modules. We define

$$(3.5) \quad \mathrm{Sel}_F(Ad(\rho))$$
$$= \mathrm{Ker}(H^1(F, V(Ad(\rho))^*) \to \prod_{\mathfrak{p}|p} H^1(I_{\mathfrak{p}}, \frac{V(Ad(\rho))^*}{V_{\mathfrak{p}}^+(Ad(\rho))^*}) \times \prod_{\mathfrak{l}\nmid p} H^1(I_{\mathfrak{l}}, V(Ad(\rho)^*)),$$

where $\mathfrak{p}|p$ and $\mathfrak{l} \nmid p$ are primes of F and $I_{\mathfrak{q}}$ is the inertia subgroup at a prime $\mathfrak{q}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

3.6. **Selmer groups and differentials.** Let $(R_{/W}, \boldsymbol{\rho})$ be the universal couple for the deformation satisfying (D1–3) and (det) for $\overline{\rho}$ in Theorem 4.3 (thus $R = \mathbb{T}/P\mathbb{T}$ if $\overline{\rho}$ satisfies the assumption of Theorem 4.3). We recall the argument of Mazur (cf. [MT90]) to relate 1-differentials on $\mathrm{Spec}(R)$ with the dual Selmer group $\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))^\vee$ for a Galois deformation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(W)$ satisfying (D1–3) and (det). Let $\lambda : R \to W$ be the algebra homomorphism inducing ρ (i.e., $\lambda \circ \boldsymbol{\rho} \cong \rho$). Let $\Phi(A)$ be the set of deformations of $\overline{\rho}$ satisfying (D1–3) and (det) with values in $GL_2(A)$.

Let X be a profinite R-module. Then $R[X]$ is an object in CL_W. We consider the W-algebra homomorphism $\xi : R \to R[X]$ with $\xi \mod X = \mathrm{id}$. Then we can write $\xi(r) = r \oplus d_\xi(r)$ with $d_\xi(r) \in X$. By the definition of the product, we get $d_\xi(rr') = rd_\xi(r') + r'd_\xi(r)$ and $d_\xi(W) = 0$. Thus d_ξ is an W-derivation, i.e., $d_\xi \in Der_W(R, X)$. For any derivation $d : R \to X$ over W, $r \mapsto r \oplus d(r)$ is obviously an W-algebra homomorphism, and we get

$$\begin{aligned}(3.6)\quad &\left\{\pi \in \Phi(R[X])\middle|\pi \mod X = \boldsymbol{\rho}\right\} / \approx_X \\ &\cong \left\{\pi \in \Phi(R[X])\middle|\pi \mod X \cong \boldsymbol{\rho}\right\} / \cong \\ &\cong \left\{\xi \in \mathrm{Hom}_{W\text{-alg}}(R, R[X])\middle|\xi \mod X = \mathrm{id}\right\} \\ &\cong Der_W(R, X) \cong \mathrm{Hom}_R(\Omega_{R/W}, X),\end{aligned}$$

where "$\approx_X$" is conjugation under $(1 \oplus M_n(X)) \cap GL_2(R[X])$.

Let π be the deformation in the left-hand-side of (3.6). Then we may write $\pi(\sigma) = \boldsymbol{\rho}(\sigma) \oplus u'_\pi(\sigma)$. We see

$$\boldsymbol{\rho}(\sigma\tau) \oplus u'_\pi(\sigma\tau) = (\boldsymbol{\rho}(\sigma) \oplus u'_\pi(\sigma))(\boldsymbol{\rho}(\tau) \oplus u'_\pi(\tau)) = \boldsymbol{\rho}(\sigma\tau) \oplus (\boldsymbol{\rho}(\sigma)u'_\pi(\tau) + u'_\pi(\sigma)\boldsymbol{\rho}(\tau)),$$

and we have

$$u'_\pi(\sigma\tau) = \boldsymbol{\rho}(\sigma)u'_\pi(\tau) + u'_\pi(\sigma)\boldsymbol{\rho}(\tau).$$

Define $u_\pi(\sigma) = u'_\pi(\sigma)\boldsymbol{\rho}(\sigma)^{-1}$. Then, $x(\sigma) = \pi(\sigma)\boldsymbol{\rho}(\sigma)^{-1}$ has values in $SL_2(R[X])$, and $x = 1 \oplus u \mapsto u = x - 1$ is an isomorphism from the multiplicative group of the kernel of the reduction map $SL_2(R[X]) \twoheadrightarrow SL_2(R)$ given by

$$\{x \in SL_2(R[X]) | x \equiv 1 \mod X\}$$

onto the additive group

$$Ad(X) = \{x \in M_2(X) | \mathrm{Tr}(x) = 0\} = V(Ad(\boldsymbol{\rho})) \otimes_R X.$$

Thus we may regard that u has values in $Ad(X) = V(Ad(\boldsymbol{\rho})) \otimes_R X$.

We also have

$$\begin{aligned}(3.7)\quad u_\pi(\sigma\tau) &= u'_\pi(\sigma\tau)\boldsymbol{\rho}(\sigma\tau)^{-1} \\ &= \boldsymbol{\rho}(\sigma)u'_\pi(\tau)\boldsymbol{\rho}(\sigma\tau)^{-1} + u'_\pi(\sigma)\boldsymbol{\rho}(\tau)\boldsymbol{\rho}(\sigma\tau)^{-1} = Ad(\boldsymbol{\rho})(\sigma)u_\pi(\tau) + u_\pi(\sigma).\end{aligned}$$

Hence u_π is a 1-cocycle unramified outside Np. It is a straightforward computation to see the injectivity of the map:

$$\left\{\pi \in \Phi(R[X])\middle|\pi \mod X \approx \boldsymbol{\rho}\right\} / \approx_X \hookrightarrow H^1_{ct}(F, Ad(X))$$

given by $\pi \mapsto [u_\pi]$ (an exercise). We put $V_p^\pm(Ad(X)) = V_p^\pm(Ad(\boldsymbol{\rho})) \otimes_R X$. Then we see from the fact that $\mathrm{Tr}(u_\pi) = 0$ that

$$(3.8)\qquad u_\pi(I_p) \subset V_p^+(Ad(X)) \Leftrightarrow u'_\pi(I_p) \subset V_p^+(Ad(X)) \Leftrightarrow \delta_\pi(I_p) = 1.$$

For primes $l \nmid Np$, π is unramified at l; so, u_π is trivial on I_l. If $l|N$, we have $\boldsymbol{\rho}|_{I_l} = \epsilon_l \oplus 1$ and $\pi|_{I_l} = \epsilon_l \oplus 1$. Thus $\pi|_{I_l}$ factors through the image of I_l in the maximal abelian quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$ which is isomorphic to $\mathbb{Z}_l^\times$. Thus $u_\pi|_{I_l}$ factors through $\mathbb{Z}_l^\times$. Since $p \nmid \varphi(N)$, $p \nmid l - 1$, which implies $u_\pi|_{I_l}$ is trivial; thus u_π unramified everywhere outside p.

Since $W = \varprojlim_n W/\mathfrak{m}_W^n$ for the finite rings $W/\mathfrak{m}_W^n$, we get $W^\vee = \varinjlim_n (W/\mathfrak{m}_W^n)^\vee$, which is a discrete R-modules, which shows $R[W^\vee] = \bigcup_n R[(W/\mathfrak{m}_W^n)^\vee]$. We put the profinite topology on the individual $R[(W/\mathfrak{m}_W^n)^\vee]$. On $R[W^\vee]$, we give a injective-limit topology. Thus, for a topological space X, a map $\phi : X \to R[W^\vee]$ is continuous if $\phi^{-1}(R[(W/\mathfrak{m}_W^n)^\vee]) \to R[(W/\mathfrak{m}_W^n)^\vee]$ is continuous for all n with respect to the topology on $\phi^{-1}(R[(W/\mathfrak{m}_W^n)^\vee])$ induced from X on the source and the profinite topology on the target (see [HiT94] Chapter 2 for details about continuity). From this, any deformation having values in $GL_2(R[W^\vee])$ gives rise to a continuous 1-cocycle with values in the discrete G-module $V(Ad(\pi))^\vee$. In this way, we get

$$(\Omega_{R/W} \otimes_R W)^\vee \cong \mathrm{Hom}_R(\Omega_{R/W}, W^\vee) \hookrightarrow H^1(\mathbb{Q}, V(Ad(\rho))^*). \tag{3.9}$$

By definition, π is ordinary if and only if u_π restricted to I_p has values in $V_p^+(Ad(\pi))^*$.

From a Selmer cocycle, we recover π by reversing the above argument; so, the image of $(\Omega_{R/A} \otimes_R W)^\vee$ in the Galois cohomology group is equal to $\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))$.

Thus we get from this and (3.8) the following fact:

Theorem 3.12 (B. Mazur)**.** *Suppose $p \nmid \varphi(N)$. Let $(R, \boldsymbol{\rho})$ be the universal couple among deformations satisfying* (D1–3) *and* (det)*. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(W)$ is a deformation, then we have a canonical isomorphism*

$$\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))^\vee \cong \Omega_{R/W} \otimes_{R,\lambda} W \cong C_1(\lambda; W)$$

as W-modules for $\lambda : R \to W$ with $\rho \cong \lambda \circ \boldsymbol{\rho}$.

By a similar argument, considering the universal ring for deformations satisfying only (D1–3), we get

Theorem 3.13 (B. Mazur)**.** *Suppose $p \nmid \varphi(N)$. Let $(R, \boldsymbol{\rho})$ be the universal couple among deformations satisfying* (D1–3)*. Let* $\mathrm{Spec}(\mathbb{I})$ *be an irreducible component of* $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(\widetilde{\mathbb{I}})$ *be the normalization of* $\mathrm{Spec}(\mathbb{I})$*. Writing $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\widetilde{\mathbb{I}})$ for the deformation corresponding to the projection $R \to \widetilde{\mathbb{I}}$, we have a canonical isomorphism*

$$\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho))^\vee \cong \Omega_{R/\Lambda} \otimes_R \widetilde{\mathbb{I}} \cong C_1(\lambda; \widetilde{\mathbb{I}})$$

as $\mathbb{I}$–modules for $\lambda : R \to \widetilde{\mathbb{I}}$ with $\rho \cong \lambda \circ \boldsymbol{\rho}$.

We do not need to assume $p \nmid \varphi(N)$ in the above two theorems, but otherwise, we need some extra care in the proof.

4. Hecke algebras

We recall briefly p-adic Hecke algebras defined over a discrete valuation ring W. We assume that the base valuation ring W flat over $\mathbb{Z}_p$ is sufficiently large so that its residu field $\mathbb{F}$ is equal to $\mathbb{T}/\mathfrak{m}_{\mathbb{T}}$ for the maximal ideal of the connected component $\mathrm{Spec}(\mathbb{T})$ (of our interest) in $\mathrm{Spec}(\mathbf{h})$.

The base ring W may not be finite over $\mathbb{Z}_p$. For example, if we want to study Katz p-adic L-functions, the natural ring of definition is the Witt vector ring $W(\overline{\mathbb{F}}_p)$ of over an algebraic closure $\overline{\mathbb{F}}_p$ (realized in $\mathbb{C}_p$), though the principal ideal generated by a blanch of the Katz p-adic L-function descends to an Iwasawa algebra over a finite extension W of $\mathbb{Z}_p$ (and in this sense, we may assume finiteness over $\mathbb{Z}_p$ of W just to understand our statement as it only essentially depends on the ideal in the Iwasawa algebra over W).

4.1. **Finite level.** Fix a field embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ and a positive integer N prime to p. Here $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ is an algebraic closure of $\mathbb{Q}_p$. Though a p-adic analytic family $\mathcal{F}$ of modular forms is intrinsic, to identify members of the family with classical modular forms with coefficients in $\mathbb{C}$, we need to have the fixed embeddings $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We write $|\alpha|_p$ for the p-adic absolute value (with $|p|_p = 1/p$) induced by i_p. Take a Dirichlet character $\psi : (\mathbb{Z}/Np^{r+1}\mathbb{Z})^\times \to W^\times$ with $(p \nmid N, r \geq 0)$, and consider the space of elliptic cusp forms $S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$ of weight $k+1$ with character ψ as defined in [IAT, (3.5.4)]. Let the ring $\mathbb{Z}[\psi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$ be generated by the values ψ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively.

We assume that the $\psi_p = \psi|_{(\mathbb{Z}/p^{r+1}\mathbb{Z})^\times}$ has conductor p^{r+1} if non-trivial and $r = 0$ if trivial. Since we will consider only $U(p)$-eigenforms with p-adic unit eigenvalues (under $|\cdot|_p$), this does not pose any restriction. For simplicity we assume that N is cube-free. We often write N_ψ for Np^{r+1} if confusion is unlikely.

The Hecke algebra over $\mathbb{Z}[\psi]$ is the subalgebra of the linear endomorphism algebra of $S_{k+1}(\Gamma_0(N_\psi), \psi)$ generated by Hecke operators $T(n)$:

$$(4.1)\quad \begin{aligned} h = h_{k,\psi} &= h_k(\Gamma_0(N_\psi), \psi; \mathbb{Z}[\psi]) \\ &= \mathbb{Z}[\psi][T(n)|n = 1, 2, \cdots] \subset \mathrm{End}(S_{k+1}(\Gamma_0(N_\psi), \psi)), \end{aligned}$$

where $T(n)$ is the Hecke operator as in [IAT, §3.5] and p^{r+1} is the conductor of the restriction ψ_p of ψ to $\mathbb{Z}_p^\times$ unless ψ_p is trivial in which case $r = 0$. We put $h_{k,\psi/A} = h \otimes_{\mathbb{Z}[\psi]} A$ for any $\mathbb{Z}[\psi]$-algebra A. When we need to indicate that our $T(l)$ is the Hecke operator of a prime factor l of N_ψ, we write it as $U(l)$, since $T(l)$ acting on a subspace $S_{k+1}(\Gamma_0(N'), \psi) \subset S_{k+1}(\Gamma_0(N_\psi), \psi)$ of level N' prime to l does not coincide with $U(l)$ on $S_{k+1}(\Gamma_0(N_\psi), \psi)$.

For any ring $A \subset \mathbb{C}$, put

$$(4.2)\quad \begin{aligned} S_{k+1,\psi/A} &= S_{k+1}(\Gamma_0(N_\psi), \psi; A) \\ &:= \{f = \sum_{n=1}^{\infty} a(n,f)q^n \in S_{k+1}(\Gamma_0(N_\psi), \psi) | a(n,f) \in A\} \end{aligned}$$

for $q = \exp(2\pi i z)$ with $z \in \mathfrak{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$. As we have a good $\mathbb{Z}$-integral structure on the modular curves $X_0(N_\psi)$, we have

$$S_{k+1,\psi/A} = S_{k+1,\psi/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} A \text{ and } S_{k+1}(\Gamma_0(N_\psi), \psi) \cong S_{k+1,\psi/\mathbb{Z}[\psi]} \otimes_{\mathbb{Z}[\psi]} \mathbb{C}$$

as long as $k \geq 1$ (this is because the corresponding line bundle is not very unple if $k = 1$). Thus hereafter for any ring A (not necessarily in $\mathbb{C}$), we define $S_{k+1,\psi/A}$ by the above formula. Then we have

$$h_{k,\psi/A} \cong A[\psi][T(n)|n = 1, 2, \cdots] \subset \mathrm{End}_A(S_{k+1,\psi/A})$$

as long as A is a $\mathbb{Z}[\psi]$-algebra and $k \geq 1$.

For p-profinite ring A, the ordinary part $\mathbf{h}_{k,\psi/A} \subset h_{k,\psi/A}$ is the maximal ring direct summand on which $U(p)$ is invertible. Writing e for the idempotent of $\mathbf{h}_{k,\psi/A}$, we have $e = \lim_{n\to\infty} U(p)^{n!}$ under the p-profinite topology of $h_{k,\psi/A}$. By the fixed embedding $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, the idempotent e not only acts on the space of modular forms with coefficients in W but also on the classical space $S_{k+1}(\Gamma_0(N_\psi), \psi)$. We write the image of the idempotent as $S^{ord}_{k+1,\psi/A}$ (as long as e is defined over A). Note $U(p)$ is a

$\mathbb{Z}[\psi]$-integral operator; so, if either A is p-adically complete or contains all eigenvalues of $U(p)$, e is defined over A. Note here if $r = 0$ (i.e., $\psi_p = 1$), the projector e (actually defined over $\overline{\mathbb{Q}}$) induces a surjection $e : S_{k+1}(\Gamma_0(N), \psi) \to S^{ord}_{k+1}(\Gamma_0(Np), \psi)$ if $k > 1$. Define a pairing $(\cdot, \cdot) : h_k(\Gamma_0(N_\psi), \psi; A) \times S_{k+1}(\Gamma_0(N_\psi), \psi; A) \to A$ by $(h, f) = a(1, f|h)$. By the celebrated formula of Hecke:

$$a(m, f|T(n)) = \sum_{0<d|(m,n),(d,N_\psi)=1} \psi(d)d^k a(\frac{mn}{d^2}, f),$$

it is an easy exercise to show that, as long as A is a $\mathbb{Z}[\psi]$-algebra,

$$\begin{aligned}
&\mathrm{Hom}_A(S_{k+1,\psi/A}, A) \cong h_{k,\psi/A},\ \ \mathrm{Hom}_A(S^{ord}_{k+1,\psi/A}, A) \cong \mathbf{h}_{k,\psi/A}\\
&\mathrm{Hom}_A(h_{k,\psi/A}, A) \cong S_{k+1,\psi/A},\ \ \mathrm{Hom}_A(\mathbf{h}_{k,\psi/A}, A) \cong S^{ord}_{k+1,\psi/A}\\
&\mathrm{Hom}_{A\text{-alg}}(h_{k,\psi/A}, A)\\
&\cong \{f \in S_{k+1,\psi/A} : f|T(n) = \lambda(n)f \ \text{ with } \ \lambda(n) \in A \text{ and } \ a(1, f) = 1\}.
\end{aligned} \tag{4.3}$$

The second isomorphism can be written as $\phi \mapsto \sum_{n=1}^{\infty} \phi(T(n))q^n$ for $\phi : h_{k,\psi/A} \to A$ and the last is just this one restricted to A-algebra homomorphisms.

4.2. **Ordinary of level Np^∞.** Fix ψ, and assume now that $\psi_p = \psi|_{\mathbb{Z}_p^\times}$ has conductor at most p and $\psi(-1) = 1$. Let ω be the modulo p Teichmüller character. Recall the multiplicative group $\Gamma := 1+p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ and its topological generator $\gamma = 1+p$. Then the Iwasawa algebra $\Lambda = W[[\Gamma]] = \varprojlim_n W[\Gamma/\Gamma^{p^n}]$ is identified with the power series ring $W[[T]]$ by a W-algebra isomorphism sending $\gamma \in \Gamma$ to $t = 1 + T$. As constructed in [H86a], [H86b] and [GME], we have a unique 'big' ordinary Hecke algebra $\mathbf{h}$. The algebra $\mathbf{h}$ is characterized by the following two properties (called Control theorems; see [H86a] Theorem 3.1, Corollary 3.2 and [H86b] Theorem 1.2 for $p \geq 5$ and [GME] Theorem 3.2.15 and Corollary 3.2.18 for general p):

(C1) $\mathbf{h}$ is free of finite rank over Λ equipped with $T(n) \in \mathbf{h}$ for all $1 \leq n \in \mathbb{Z}$ (so $U(l)$ for $l|Np$),

(C2) if $k \geq 1$ and $\epsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}$ is a character,
$$\mathbf{h} \otimes_{\Lambda,t\mapsto\epsilon(\gamma)\gamma^k} W[\epsilon] \cong \mathbf{h}_{k,\epsilon\psi_k}\ (\gamma = 1 + p) \text{ for } \psi_k := \psi\omega^{-k},$$
sending $T(n)$ to $T(n)$ (and $U(l)$ to $U(l)$ for $l|Np$). Here $W[\epsilon]$ is the W-subalgebra in $\mathbb{C}_p$ generated by the values of ϵ. If $W[\epsilon] = W$, the above identity becomes
$$\mathbf{h}/(t - \epsilon(\gamma)\gamma^k)\mathbf{h} \cong \mathbf{h}_{k,\epsilon\psi_k}.$$

Let $\mathrm{Spec}(\mathbb{I})$ be a reduced irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. Write $a(n)$ for the image of $T(n)$ in $\mathbb{I}$ (so, $a(p)$ is the image of $U(p)$). If a point P of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ kills $(t - \epsilon(\gamma)\gamma^k)$ with $1 \leq k \in \mathbb{Z}$ (i.e., $P((t - \epsilon(\gamma)\gamma^k)) = 0$), we call it an *arithmetic* point and we write $\epsilon_P = \epsilon$, $\psi_P = \epsilon_P\psi\omega^{-k}$, $k(P) = k \geq 1$ and $p^{r(P)}$ for the order of ϵ_P. If P is arithmetic, by (C2), we have a Hecke eigenform $f_P \in S_{k+1}(\Gamma_0(Np^{r(P)+1}), \epsilon\psi_k)$ such that its eigenvalue for $T(n)$ is given by $a_P(n) := P(a(n)) \in \overline{\mathbb{Q}}_p$ for all n. Thus $\mathbb{I}$ gives rise to a family $\mathcal{F} = \{f_P|\text{arithemtic } P \in \mathrm{Spec}(\mathbb{I})\}$ of Hecke eigenforms. We define a *p-adic analytic family of slope* 0 (with coefficients in $\mathbb{I}$) to be the family as above of Hecke eigenforms associated to an irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. We call this family slope 0 because $|a_P(p)|_p = 1$ for the p-adic absolute value $|\cdot|_p$ of $\overline{\mathbb{Q}}_p$ (it is also often called an ordinary family). Such a family is called analytic because the Hecke eigenvalue $a_P(n)$ for $T(n)$ is given by an analytic function $a(n)$ on (the rigid analytic space associated to) the p-profinite formal spectrum $\mathrm{Spf}(\mathbb{I})$. Identify $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{W\text{-alg}}(\mathbb{I},\overline{\mathbb{Q}}_p)$ so that each element $a \in \mathbb{I}$ gives rise to a "function" $a : \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ whose value at $(P : \mathbb{I} \to \overline{\mathbb{Q}}_p) \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ is $a_P := P(a) \in \overline{\mathbb{Q}}_p$. Then a is an analytic function of the rigid analytic space associated to $\mathrm{Spf}(\mathbb{I})$. Taking a finite covering $\mathrm{Spec}(\widetilde{\mathbb{I}})$ of $\mathrm{Spec}(\mathbb{I})$ with surjection $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p) \twoheadrightarrow \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, abusing slightly the definition, we may regard the family $\mathcal{F}$ as being indexed by arithmetic points of $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$, where arithmetic points of $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ are made up of the points above arithmetic points of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$. The choice of $\widetilde{\mathbb{I}}$ is often the normalization of $\mathbb{I}$ or the integral closure of $\mathbb{I}$ in a finite extension of the quotient field of $\mathbb{I}$.

4.3. **Modular Galois representation.** Associated to each connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ is a 2-dimensional continuous representation $\rho_{\mathbb{T}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in the total quotient ring of $\mathbb{T}$ (the representation was first constructed in [H86b]). The representation $\rho_{\mathbb{T}}$ restricted to the p-decomposition group D_p is reducible with unramified *quotient* character (e.g., [GME, §4.2]). We write $\rho_{\mathbb{T}}^{ss}$ for its semi-simplification over D_p. As is well known now (e.g., [GME, §4.2]), $\rho_{\mathbb{I}}$ is unramified outside Np and satisfies
(Gal)
$$\mathrm{Tr}(\rho_{\mathbb{T}}(Frob_l)) = a(l) \ \ (l \nmid Np),\ \ \rho_{\mathbb{T}}^{ss}([\gamma^s,\mathbb{Q}_p]) \sim \left(\begin{smallmatrix} t^s & 0 \\ 0 & 1\end{smallmatrix}\right) \text{ and } \rho_{\mathbb{I}}^{ss}([p,\mathbb{Q}_p]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & a(p)\end{smallmatrix}\right),$$
where $\gamma^s = (1+p)^s = \sum_{n=0}^{\infty}\binom{s}{n}p^n \in \mathbb{Z}_p^\times$ and $t^s = (1+p)^s = \sum_{n=0}^{\infty}\binom{s}{n}T^n \in \mathbb{Z}_p[[T]]^\times$ for $s \in \mathbb{Z}_p$ and $[x,\mathbb{Q}_p]$ is the local Artin symbol. For each prime ideal P of $\mathrm{Spec}(\mathbb{T})$, writing $\kappa(P)$ for the residue field of P, we also have a semi-simple Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$ unramified outside Np such that $\mathrm{Tr}(\rho_P(Frob_l)) = a(l) \mod P$ for all primes $l \nmid Np$. If P is the maximal ideal $\mathfrak{m}_{\mathbb{T}}$, we write $\overline{\rho}$ for ρ_P which is called the residual representation of $\rho_{\mathbb{T}}$. We sometimes write $\overline{\rho}$ as $\overline{\rho}_{\mathbb{T}}$ to indicate the corresponding connected component. If $\overline{\rho}$ is absolutely irreducible, ρ_P has values in $GL_2(\mathbb{T}/P)$; in particulr, $\rho_{\mathbb{T}}$ has values in $GL_2(\mathbb{T})$. We assume irreducibility of $\overline{\rho}$.

By (Gal) and Chebotarev density, $\mathrm{Tr}(\rho_{\mathbb{I}})$ has values in $\mathbb{I}$; so,
$$P \circ \mathrm{Tr}(\rho_{\mathbb{I}}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \overline{\mathbb{Q}}_p\ (P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p))$$
gives rise to a pseudo-representation of Wiles (e.g., [MFG, §2.2]). Then by a theorem of Wiles, we can make a unique 2-dimensional semi-simple continuous representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$ unramified outside Np with $\mathrm{Tr}(\rho_P(Frob_l)) = a_P(l)$ for all primes l outside Np (though the construction of ρ_P does not require the technique of pseudo representation and was known before the invention of the technique; see [MW86, §9 Proposition 1]). This is the Galois representation associated to the Hecke eigenform f_P (constructed earlier by Eichler–Shimura and Deligne) if P is arithmetic (e.g., [GME, §4.2]). More generally, for any algebra homomorphism $\lambda \in \mathrm{Hom}_{\mathbb{Z}[\psi]\text{-alg}}(h_{k,\psi/\mathbb{Z}[\psi]},\overline{\mathbb{Q}})$, they associated a Galois representation $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Q}_p(\lambda))$ unramified outside Np with $\mathrm{Tr}(\rho_\lambda(Frob_l)) = \lambda(T(l))$ for primes $l \nmid Np$ and $\det(\rho_\lambda) = \psi\nu^k$ for the p-adic cyclotomic character ν. Thus moving around primes, ρ_λ form a compatible system ϱ_λ of Galois representations with coefficients in $\mathbb{Q}(\lambda)$.

4.4. **Hecke algebra is universal.** Start with a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h})$ of level Np^∞ with character ψ. For simplicity, as before, assume that $N = C$ for the prime-to-p conductor C of $\overline{\psi} = \det(\overline{\rho})$. Recall the deformation properties (D1–3):

(D1) ρ is unramified outside Np,

(D2) $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \left(\begin{smallmatrix}\epsilon & * \\ 0 & \delta\end{smallmatrix}\right)$ with δ unramified while ϵ ramified,

(D3) For each prime $l|N$, regarding $\psi_l = \psi|_{\mathbb{Z}_l^\times}$ as the character of I_l by local class field theory, we have $\rho|_{I_l} \cong \left(\begin{smallmatrix}\psi_l & 0 \\ 0 & 1\end{smallmatrix}\right)$,

Theorem 4.1 (Wiles et al)**.** *If $\overline{\rho}$ is absolutely irreducible over* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_p])$*, $(\mathbb{T}, \rho_{\mathbb{T}})$ is a local complete intersection over Λ and is universal among deformations satisfying* (D1–3).

See [W95] (see also [MFG, Section 3.2] and [HMI, Chapter 3]) for a proof. Absolute irreducibility of $\overline{\rho}$ over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_p])$ is equivalent to absolute irreducibility over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{(-1)^{(p-1)/2}p}])$. Also we note that the assumption $p \geq 5$ we made for simplicity in this note can be eased to $p \geq 3$ for the theorem above and below (in Wiles' proof of Fermat's last theorem, the case $p = 3$ has absolute importance).

Let $\pi : \mathbb{T} \to \mathbb{I}$ be the projection map inducing $\mathrm{Spec}(\mathbb{I}) \hookrightarrow \mathrm{Spec}(\mathbb{T})$ and write $\widetilde{\mathbb{I}}$ for the normalization of $\mathbb{I}$ (that is, the integral closure of $\mathbb{I}$ in its quotient field $Q(\mathbb{I})$). Replace $\mathbb{T}$ by $\mathbb{T}_{\mathbb{I}} = \mathbb{T}\otimes_\Lambda \widetilde{\mathbb{I}}$ and π by the composite $\boldsymbol{\lambda} := m\circ(\pi\otimes 1) : \mathbb{T}_{\mathbb{I}} \to \mathbb{I}\otimes_\Lambda\widetilde{\mathbb{I}} \xrightarrow{m} \widetilde{\mathbb{I}}$. We would like to apply Tate's theory (Theorem 3.5) to this setting. Note that $\widetilde{\mathbb{I}}$ is free of finite rank over Λ as Λ is a regular local ring of dimension 2. Therefore the local complete intersection property of $\mathbb{T}$ over Λ implies that $\mathbb{T}_{\mathbb{I}}$ is a local complete intersection over the normal noetherian integral domain $\widetilde{\mathbb{I}}$. Then we get the following facts for the projection $\boldsymbol{\lambda} : \mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}}$:

Corollary 4.2. *We have the following equalities:*

1. $C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) = \widetilde{\mathbb{I}}/(L_p)$ *for some* $L_p \in \widetilde{\mathbb{I}}$ *(i.e.,* $\mathrm{Ann}_{\widetilde{\mathbb{I}}}(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}))$ *is principal).*
2. $\mathrm{char}(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}})) = \mathrm{char}(C_1(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}))$ *as ideals of* $\widetilde{\mathbb{I}}$.

The assertion (2) is the consequence of Tate's theorem (Theorem 3.5).

Proof. The fact (1) can be shown as follows. Write $\mathfrak{b} = \mathrm{Ker}(\boldsymbol{\lambda})$; i.e., we have an exact sequence

$$0 \to \mathfrak{b} \to \mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}} \to 0.$$

Since $\mathfrak{b}$ is the $\widetilde{\mathbb{I}}$-direct summand of $\mathbb{T}_{\mathbb{I}}$, by taking $\widetilde{\mathbb{I}}$-dual (indicated by superscript $*$), we have another exact sequence $0 \to \widetilde{\mathbb{I}}^* \to \mathbb{T}_{\mathbb{I}}^* \to \mathfrak{b}^* \to 0$. Since $\mathbb{T}_{\mathbb{I}}$ is a local complete intersection, by Lemma 3.8, it is Gorenstein: $\mathbb{T}_{\mathbb{I}}^* \cong \mathbb{T}_{\mathbb{I}}$ as $\mathbb{T}_{\mathbb{I}}$-modules. Thus for $Q = \mathrm{Frac}(\Lambda)$, $\mathrm{Frac}(\mathbb{T}_{\mathbb{I}}) = \mathbb{T}_{\mathbb{I}} \otimes_\Lambda Q = Q(\mathbb{I}) \oplus X$ for $X := \mathfrak{b}^* \otimes_\Lambda Q$ which is an algebra direct sum and the projection to $Q(\mathbb{I}) = Q(\widetilde{\mathbb{I}})$ is induced by $\boldsymbol{\lambda}$. Thus $\mathrm{Im}(\widetilde{\mathbb{I}}^* \hookrightarrow \mathbb{T}_{\mathbb{I}}^* \cong \mathbb{T}_{\mathbb{I}}) \subset \mathbb{T}_{\mathbb{I}}$ is the ideal $\mathfrak{a} = (Q(\mathbb{I}) \oplus 0) \cap \mathbb{T}$, and $\mathfrak{a} \cong \widetilde{\mathbb{I}}^* \cong \widetilde{\mathbb{I}}$ is principal. Since $C_0(\boldsymbol{\lambda}; \mathbb{I}) = \widetilde{\mathbb{I}}/\mathfrak{a}$, the result follows. □

For an arithmetic point P of $\mathrm{Spec}(\Lambda)$, recall $\psi_P = \epsilon_P\psi\omega^{-k(P)}$.

Theorem 4.3 (Wiles ét al)**.** *If $\overline{\rho}$ is absolutely irreducible over* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_p])$*, $(\mathbb{T}/P\mathbb{T}, (\rho_{\mathbb{T}} \bmod P))$ is a local complete intersection over W and is universal among deformations ρ satisfying* (D1–3) *and* (det) $\det(\rho) = \psi_P \nu^k$ *for the p-adic cyclotomic character ν and $k = k(P)$.*

Actually Wiles–Taylor proved the first cases of Theorem 4.3 which actually implies Theorem 4.1 (see [MFG, Theorem 5.29] for this implication).

For any arithmetic prime $\mathfrak{P} \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ above $P \in \mathrm{Spec}(\Lambda)$, we see from the definition

$$\mathbb{T}_{\mathbb{I}} \otimes_{\widetilde{\mathbb{I}}} \widetilde{\mathbb{I}}/\mathfrak{P} = \mathbb{T} \otimes_\Lambda \widetilde{\mathbb{I}} \otimes_{\widetilde{\mathbb{I}}} \widetilde{\mathbb{I}}/\mathfrak{P} = \mathbb{T} \otimes_\Lambda \widetilde{\mathbb{I}}/\mathfrak{P} = (\mathbb{T}/P\mathbb{T}) \otimes_{\Lambda/P} \widetilde{\mathbb{I}}/\mathfrak{P}.$$

Thus we have the Hecke eigenform $f_{\mathfrak{P}}$ and its Galois representation and the projection $\lambda : \mathbb{T}/P\mathbb{T} \hookrightarrow ((\mathbb{T}/P\mathbb{T}) \otimes_{\Lambda/P} \widetilde{\mathbb{I}}/\mathfrak{P}) \to \widetilde{\mathbb{I}}/\mathfrak{P}$ given by $\boldsymbol{\lambda} \otimes 1$. Then, in the same way as Corollary 4.2, we can prove

Corollary 4.4. *We have $C_0(\lambda; \widetilde{\mathbb{I}}/\mathfrak{P}) = (\widetilde{\mathbb{I}}/\mathfrak{P})/(L_p(\mathfrak{P}))$ for $L_p \in \widetilde{\mathbb{I}}$ in* Corollary 4.2*, where $L_p(\mathfrak{P}) = \mathfrak{P}(L_p)$.*

This corollary is essentially [H88b, Theorem 0.1]; however, the assumptions (0.8a,b) made in [H88b] is eliminated as these are proven later in [H13b, Lemma 4.2].

Proof. From the $\widetilde{\mathbb{I}}$-split exact sequences $0 \to \mathfrak{b} \to \mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}} \to 0$, its $\widetilde{\mathbb{I}}$-dual $0 \leftarrow \mathfrak{b}^* \leftarrow \mathbb{T}_{\mathbb{I}}^* \leftarrow \widetilde{\mathbb{I}}^* \leftarrow 0$ is isomorphic to $0 \to \mathfrak{a} \to \mathbb{T}_{\mathbb{I}} \to S \to 0$ for the image S in X of $\mathbb{T}_{\mathbb{I}}$. Thus the latter sequence is also $\widetilde{\mathbb{I}}$-split. Tensoring $\widetilde{\mathbb{I}}/\mathfrak{P}$, we get the following two exact sequence:

$$0 \to \mathfrak{b}/\mathfrak{P}\mathfrak{b} \to \mathbb{T}_{\mathbb{I}}/\mathfrak{P}\mathbb{T}_{\mathbb{I}} \to \widetilde{\mathbb{I}}/\mathfrak{P} \to 0 \quad \text{and} \quad 0 \to \mathfrak{a}/\mathfrak{P}\mathfrak{a} \to \mathbb{T}_{\mathbb{I}}/\mathfrak{P}\mathbb{T}_{\mathbb{I}} \to S/\mathfrak{P}S \to 0.$$

Then we get

$$C_0(\lambda; \widetilde{\mathbb{I}}/\mathfrak{P}) = (\widetilde{\mathbb{I}}/\mathfrak{P})/(\mathfrak{a}/\mathfrak{P}\mathfrak{a}) = (\widetilde{\mathbb{I}}/\mathfrak{a}) \otimes_{\mathbb{T}_{\mathbb{I}}} (\widetilde{\mathbb{I}}/\mathfrak{P}) = C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) \otimes_{\widetilde{\mathbb{I}}} \widetilde{\mathbb{I}}/\mathfrak{P} = (\widetilde{\mathbb{I}}/\mathfrak{P})/(L_p(\mathfrak{P}))$$

as desired. $\square$

Assume $p \nmid \varphi(N)$. By Mazur's theorem (Theorem 3.13),

$$C_1(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}) \cong \mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}}))^\vee, \ \mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}})^\vee) = \mathrm{char}(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}})) = (L_p)$$
$$\text{and } \mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\mathfrak{P}})^\vee) = \mathrm{char}(C_0(\boldsymbol{\lambda}; \widetilde{\mathbb{I}}/\mathfrak{P})) = (L_p(\mathfrak{P})).$$

In the following section, we relate $L_p(\mathfrak{P})$ with $L(1, Ad(\rho_{\mathfrak{P}}))$; so, we get the one variable adjoint main conjecture:

Corollary 4.5. *Then there exists a p-adic L-function $L_p \in \widetilde{\mathbb{I}}$ such that we have*

$$\mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\widetilde{\mathbb{I}}})^\vee) = (L_p) \quad \textit{and} \quad \mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\mathfrak{P}})^\vee) = (L_p(\mathfrak{P}))$$

for all arithmetic points $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$.

Though we assumed $p \nmid \varphi(N)$ in our proof, this condition is not necessary for the validity of the above corollary.

In [H90], the author constructed a two variable p-adic L-function $L \in \widetilde{\mathbb{I}}[[X]]$ interpolating $L(1+m, Ad(\rho_{\mathbb{I}}))$ (i.e., $\mathfrak{P}(L)(\gamma^m - 1) \doteqdot L(1+m, Ad(\rho_{\mathfrak{P}}) \otimes \omega^{-m})$ essentially). Eric Urban proved the divisibility $L | \mathrm{char}(\mathrm{Sel}_{\mathbb{Q}}(\rho_{\mathbb{I}} \otimes \kappa)^\vee)$ for the universal character $\kappa : \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to W[[X]]^\times$ deforming the identity character in many cases (applying Eisenstein techniques to $GSp(4)$ of Ribet–Greenberg–Wiles; see [U06]). Note here that the two variable adjoint L-function has exceptional zero along $s = 1$ as $Ad(\rho_{\mathbb{I}})$ has p-Frobenius eigenvalue 1. In other words, we have $X|L$ in $\widetilde{\mathbb{I}}[[X]]$ (an exceptional zero), and in this case, up to a simple nonzero constant, we have $(L/X)|_{X=0} \doteqdot \frac{da(p)}{dT} L_p$ essentially when the equality holds (see [H11]). Thus we have $\frac{dL}{dX}|_{s=1} = \mathcal{L}L_p$ for L_p in the above corollary and an $\mathcal{L}$-invariant $\mathcal{L} \doteqdot \frac{dL}{dX}|_{s=1} \in Q(\mathbb{I})$. Because of this exceptional zero, the divisibility: $L|\operatorname{char}(\operatorname{Sel}_{\mathbb{Q}}(\rho_{\mathbb{I}} \otimes \kappa)^\vee)$ proved by Urban combined Corollary 4.5 does not immediately imply the equality $(L) = \operatorname{char}(\operatorname{Sel}_{\mathbb{Q}}(\rho_{\mathbb{I}} \otimes \kappa)^\vee)$ without computing $\mathcal{L}$. The $\mathcal{L}$-invariant specialized to an elliptic curve with multiplicative reduction has been computed by Greenberg–Tilouine–Rosso [Ro15] to be equal to $\frac{\log_p(q)}{\operatorname{ord}_p(q)}$ for the Tate period q of the elliptic curve. This $\mathcal{L}$-invariant formula in [Ro15] combined with Urban's divisibility tells us the equality (assuming that $\operatorname{Spec}(\mathbb{I})$ has an arithmetic point giving rise to an elliptic curve with multiplicative reduction).

The p-adic L-function L_p is defined up to units in $\widetilde{\mathbb{I}}$. If $\mathbb{I}$ has complex multiplication (that is, $\rho_{\widetilde{\mathbb{I}}}$ is an induced representation of a character $\varphi : \operatorname{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ of $\operatorname{Gal}(\overline{\mathbb{Q}}/M)$ for an imaginary quadratic field M), we can choose L_p to be equal to $L(0, \left(\frac{M/\mathbb{Q}}{}\right))L_p(\varphi^-)$ for the Katz p-adic L-function $L_p(\varphi^-)$ for the anti-cyclotomic projection $\varphi^-(\sigma)\varphi(\sigma)\varphi(c\sigma c)^{-1}$ under some assumptions; see [H06] and [H09]. Even if $Ad(\operatorname{Ind}_M^{\mathbb{Q}} \varphi) \cong \left(\frac{M/\mathbb{Q}}{}\right) \oplus \operatorname{Ind}_M^{\mathbb{Q}} \varphi^-$ is easy to show, the identification of L_p with $L(0, \left(\frac{M/\mathbb{Q}}{}\right))L_p(\varphi^-)$ is a highly non-trivial endeavor which eventually proved the anti-cyclotomic CM main conjecture and the full CM main conjacture in many cases (see [T89], [MT90], [HiT94], [H06], [H09] and [Hs14]).

5. Analytic and topological methods

We compute the size $|C_0(\lambda; W)|$ for an algebra homomorphism $\lambda : \mathbb{T}/P\mathbb{T} \to W$ associated to a Hecke eigenform f by the adjoint L-value.

Here is a technical heuristic explaining some reason why at the very beginning of his work [H81a] the author speculated that the adjoint L-value would be most accessible to a non-abelian generalization of the class number formula in Theorem 3.4. The proof of the formula by Dirichlet–Kummer–Dedekind proceeds as follows: first, one relates the class number $h = |\operatorname{Sel}_{\mathbb{Q}}(\chi)|$ in Theorem 3.4 to the residue of the Dedekind zeta function ζ_F of F, that is, the L–function of the self dual Galois representation

$$\mathbf{1} \oplus \chi = \operatorname{Ind}_F^{\mathbb{Q}} \mathbf{1};$$

second, one uses the fact $\zeta_F(s) = L(s, \chi)\zeta(s)$ following the above decomposition for the Riemann zeta function $\zeta(s) = L(s, \mathbf{1})$ and the residue formula $\operatorname{Res}_{s=1}\zeta(s) = 1$ to finish the proof of the identity.

If one has good experience of calculating the value or the residue of a well defined complex meromorphic function, one would agree that a residue tends to be more accessible than the value of the function (the foremost proto-typical example is the residue of the Riemann zeta function at $s = 1$).

Note here that, unless F is either $\mathbb{Q}$ or an imaginary quadratic field, the value $L(1, \chi)$ is not critical. Thus the transcendental factor Ω_F (in Theorem 3.4) involves the regulator in addition to the period (a power of $2\pi i$).

Perhaps, the most simple (and natural) way to create a self dual representation containing the trivial representation $\mathbf{1}$ is to form the tensor product of a given n-dimensional Galois representation φ (of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) with its contragredient $\widetilde{\varphi}$: $\varphi \otimes \widetilde{\varphi}$. We define an $n^2 - 1$ dimensional representation $Ad(\varphi)$ so that $\varphi \otimes \widetilde{\varphi} \cong \mathbf{1} \oplus Ad(\varphi)$. When $n = 2$, $s = 1$ is critical with respect to $Ad(\varphi)$) if $\det \varphi(c) = -1$ for complex conjugation c. Then we expect that $\frac{L(1,Ad(\varphi))}{\text{a period}}$ should be somehow related to the size of $\mathrm{Sel}(Ad(\varphi))$ in the most favorable cases, for the Selmer group $\mathrm{Sel}(Ad(\varphi))$ of $Ad(\varphi)$, in a fashion analogous to (3.5). Even if $s = 1$ is not critical for $Ad(\varphi)$, as in the case of F, there seems to be a good way to define a natural transcendental factor of $L(1, Ad(\varphi))$ only using the data from the Hecke side (see [H99] and [BR14]) if φ is automorphic. Therefore, the transcendental factor automorphically defined in these papers via Whittaker model should contain a period associated with $Ad(\varphi)$ and a Beilinson regulator geometrically defined for $Ad(\varphi)$ (as long as φ is geometric in a reasonable sense) if one believes in the standard conjecture. It would be a challenging problem for us to factor the automorphic transcendental factor (given in these articles) in an automorphically natural way into the product of a period and the regulator of $Ad(M)$ when we know that φ is associated to a motive M.

In this section, the reader will see this heuristic is actually realized by a simple classical computation at least when φ is associated to an elliptic Hecke eigenform.

Again for simplicity, we assume $p \geq 5$ and $p \nmid \varphi(N)$ (in the disguise of $p \nmid \varphi(C)$) in this section.

5.1. **Analyticity of adjoint L-functions.** We summarize here known fact on analyticity and arithmeticity of the adjoint L-function $L(s, Ad(\lambda)) = L(s, Ad(\rho_\lambda))$ for a $\mathbb{Z}[\chi]$-algebra homomorphism λ of $h_k(\Gamma_0(C), \chi; \mathbb{Z}[\chi])$ into $\overline{\mathbb{Q}}$ and the compatible system ρ_λ of Galois representations attached to ρ_λ. We always assume that $k \geq 1$ as before. Recall here that h_k is a Hecke algebra of S_{k+1}; so, this condition means weight ≥ 2.

By new form theory, we may assume that λ is primitive of exact level C. For simplicity, we assume that χ *is primitive of conductor* C throughout this section (this assumption implies that λ is primitive of exact level C). Then writing the (reciprocal) Hecke polynomial at a prime ℓ as

$$L_\ell(X) = 1 - \lambda(T(\ell))X + \chi(\ell)\ell^k X^2 = (1 - \alpha_\ell X)(1 - \beta_\ell X),$$

we have the following Euler product convergent absolutely if $\mathrm{Re}(s) > 1$:

$$L(s, Ad(\lambda)) = \prod_\ell \left\{(1 - \frac{\alpha_\ell}{\beta_\ell}\ell^{-s})(1 - \ell^{-s})(1 - \frac{\beta_\ell}{\alpha_\ell}\ell^{-s})\right\}^{-1}.$$

The meromorphic continuation and functional equation of this L-function was proven by Shimura in 1975 [Sh75]. The earlier method of Shimura in [Sh75] is generalized, using the language of Langlands' theory, by Gelbart and Jacquet [GJ78]. Taking the primitive cusp form f such that $T(n)f = \lambda(T(n))f$ for all n, let π be the automorphic representation of $GL_2(\mathbb{A})$ spanned by f and its right translations. We write $L(s, Ad(\pi))$ for the L-function of the adjoint lift $Ad(\pi)$ to $GL(3)$ [GJ78]. This L-function coincides with $L(s, Ad(\lambda))$ and has a meromorphic continuation to the whole complex s-plane and satisfies a functional equation of the form $1 \leftrightarrow 1 - s$ whose Γ-factor is given by

$$\Gamma(s, Ad(\lambda)) = \Gamma_\mathbb{C}(s + k)\Gamma_\mathbb{R}(s + 1),$$

where $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$.

The L-function is known to be entire, and the adjoint lift of Gelbart-Jacquet is a cusp from if ρ_λ is not an induced representation of a Galois character (note that $L(s, Ad(\lambda) \otimes \left(\frac{F/\mathbb{Q}}{}\right))$ has a pole at $s = 1$ if $\rho_\lambda = \mathrm{Ind}_F^\mathbb{Q} \varphi$ for a quadratic field F).

To see this, suppose that ρ_λ is an induced representation $\mathrm{Ind}_{\mathbb{Q}(\sqrt{D})}^{\mathbb{Q}} \varphi$ for a Galois character $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D})) \to \overline{\mathbb{Q}}_p^\times$ (associated to a Hecke character). Then we have $Ad(\rho_\lambda) \cong \chi \oplus \mathrm{Ind}_{\mathbb{Q}(\sqrt{D})}^{\mathbb{Q}}(\varphi\varphi_\sigma^{-1})$, where $\chi = \left(\frac{D}{}\right)$ is the Legendre symbol, and $\varphi_\sigma(g) = \varphi(\sigma g \sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inducing a non-trivial automorphism on $\mathbb{Q}(\sqrt{D})$. Since λ is cuspidal, ρ_λ is irreducible, and hence $\varphi\varphi_\sigma^{-1} \neq 1$. Thus $L(s, Ad(\lambda)) = L(s,\chi)L(s,\varphi\varphi_\sigma^{-1})$ is still an entire function, but $L(s, Ad(\lambda)\otimes\chi)$ has a simple pole at $s = 1$.

After summarizing what we have said, we shall give a sketch of a proof of the meromorphic continuation of $L(s, Ad(\lambda))$ and its analyticity around $s = 1$ following [LFE] Chapter 9:

Theorem 5.1 (G. Shimura)**.** *Let χ be a primitive character modulo C. Let $\lambda : h_k(\Gamma_0(C),\chi;\mathbb{Z}[\chi]) \to \mathbb{C}$ be a $\mathbb{Z}[\chi]$-algebra homomorphism for $k \geq 1$. Then*

$$\Gamma(s, Ad(\lambda))L(s, Ad(\lambda))$$

has an analytic continuation to the whole complex s-plane and

$$\Gamma(1, Ad(\lambda))L(1, Ad(\lambda)) = 2^{k+1}C^{-1}\int_{\Gamma_0(C)\backslash\mathfrak{H}} |f|^2 y^{k-1}dxdy,$$

where $f = \sum_{n=1}^\infty \lambda(T(n))q^n \in S_{k+1}(\Gamma_0(C),\chi)$ and $z = x + iy \in \mathfrak{H}$. If $C = 1$, we have the following functional equation:

$$\Gamma(s, Ad(\lambda))L(s, Ad(\lambda)) = \Gamma(1-s, Ad(\lambda))L(1-s, Ad(\lambda)).$$

Proof. We consider $L(s-k, \rho_\lambda \otimes \widetilde{\rho}_\lambda)$ for the Galois representation associated to λ. Since $\rho_\lambda \otimes \widetilde{\rho}_\lambda = \mathbf{1} \oplus Ad(\rho_\lambda)$, we have

$$L(s, \rho_\lambda \otimes \widetilde{\rho}_\lambda) = L(s, Ad(\lambda))\zeta(s) \tag{5.1}$$

for the Riemann zeta function $\zeta(s)$. Then, the Rankin-convolution method tells us (cf. [LFE] Theorem 9.4.1) that

$$\left(2^{2-s}\prod_{p|C}(1-\frac{1}{p^{s-k}})\right)\Gamma_{\mathbb{C}}(s)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda) = \int_{\Gamma_0(C)\backslash\mathfrak{H}} |f|^2 E'_{0,C}(s-k,\mathbf{1})y^{-2}dxdy,$$

where $E'_0(s,\mathbf{1})$ is the Eisenstein series of level C for the trivial character $\mathbf{1}$ defined in [LFE] page 297. Since the Eisenstein series is slowly increasing and f is rapidly decreasing, the integral converges absolutely on the whole complex s-plane outside the singularity of the Eisenstein series. The Eisenstein series has a simple pole at $s = 1$ with constant residue: $\pi\prod_{p|C}(1-\frac{1}{p})$, which yields

$$\begin{aligned}\mathrm{Res}_{s=k+1}\left(\left(2^{2-s}\prod_{p|C}(1-\frac{1}{p^{s-k}})\right)\Gamma_{\mathbb{C}}(s)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda)\right)&\\ = \pi\prod_{p|C}(1-\frac{1}{p})\int_{\Gamma_0(C)\backslash\mathfrak{H}}|f|^2y^{-2}dxdy.&\end{aligned}$$

This combined with (5.1) yields the residue formula and analytic continuation of $L(s, Ad(\lambda))$ over the region of $\mathrm{Re}(s) \geq 1$. Since $\Gamma_{\mathbb{C}}(s)E'_{0,C}(s,\mathbf{1})$ satisfies a functional equation of the form $s \mapsto 1-s$ (see [LFE] Theorem 9.3.1), we have the meromorphic continuation of $\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k)L(s-k,\rho_\lambda\otimes\widetilde{\rho}_\lambda)$. Dividing the above zeta function by

$\Gamma_{\mathbb{R}}(s-k)\zeta(s-k)$, we get the L-function $\Gamma(s-k, Ad(\lambda))L(s-k, Ad(\lambda))$, and hence meromorphic continuation of $\Gamma(s, Ad(\lambda))L(s, Ad(\lambda))$ to the whole s-plane and its holomorphy around $s = 1$.

When $C = 1$, the functional equation of the Eisenstein series is particularly simple:

$$\Gamma_{\mathbb{C}}(s)E_{0,1}(s, \mathbf{1}) = 2^{1-2s}\Gamma_{\mathbb{C}}(1-s)E_{0,1}(1-s, \mathbf{1}),$$

which combined with the functional equation of the Riemann zeta function (e.g. [LFE] Theorem 2.3.2 and Corollary 8.6.1) yields the functional equation of the adjoint L-function $L(s, Ad(\lambda))$. □

5.2. **Integrality of adjoint L-values.** By the explicit form of the Gamma factor, $\Gamma(s, Ad(\lambda))$ is finite at $s = 0, 1$, and hence $L(1, Ad(\lambda))$ is a critical value in the sense of Deligne and Shimura, as long as $L(s, Ad(\lambda))$ is finite at these points. Thus we expect the L-value divided by a period of the λ-eigenform to be algebraic. This fact was first shown by Sturm (see [St80] and [St89]) by using Shimura's integral expression (in [Sh75]). Here we shall describe the integrality of the value, following [H81a] and [H88a]. This approach is different from Sturm. Then we shall relate in the following subsection, as an application of the "$R = T$" theorem Theorem 4.3, the size of the module $\mathrm{Sel}(Ad(\rho_\lambda))$ and the p-primary part of the critical value $\frac{\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}$. Since our argument can be substantially simplified if $p \nmid |(\mathbb{Z}/C\mathbb{Z})^\times| = \varphi(C)$, we simply assume this condition hereafter.

Consider the defining inclusion $I : SL_2(\mathbb{Z}) \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^2) = GL_2(\mathbb{C})$. Let us take the nth symmetric tensor representation $I^{sym\otimes n}$ whose module twisted by the action of χ, we write as $L(n, \chi; \mathbb{C})$. Recall the Eichler-Shimura isomorphism,

$$\delta : S_{k+1}(\Gamma_0(C), \chi) \oplus \overline{S}_{k+1}(\Gamma_0(C), \chi) \cong H^1_{cusp}(\Gamma_0(C), L(n, \chi; \mathbb{C})), \tag{5.2}$$

where $k = n + 1$, $\overline{S}_{k+1}(\Gamma_0(C), \chi)$ is the space of anti-holomorphic cusp forms of weight $k + 1$ of "Neben" type character χ, and

$$H^1_{cusp}(\Gamma_0(C), L(n, \chi; \mathbb{C})) \subset H^1(\Gamma_0(C), L(n, \chi; \mathbb{C}))$$

is the *cuspidal* cohomology groups defined in [IAT] Chapter 8 (see also [LFE] Chapter 6 under the formulation close to this chapter; in these books H^1_{cusp} is written actually as H^1_P and is called the *parabolic* cohomology group).

The periods $\Omega(\pm, \lambda; A)$ measure the difference of two rational structure coming from algebro-geometric space $S_{k+1,\chi/\mathbb{Z}[\chi]}$ and topologically defined

$$H^1_{cusp}(\Gamma_0(C), L(n, \chi; \mathbb{Z}[\chi])) \cong H^1_{cusp}(X_0(C), \mathcal{L}(n, \chi; \mathbb{Z}[\chi]))$$

(for the $\mathbb{Z}[\chi]$-rational symmetric tensors $L(n, \chi; \mathbb{Z}[\chi])$ and the associated sheaf $\mathcal{L}(n, \chi; \mathbb{Z}[\chi])$ on the modular curve $X_0(C)$) connected by Eishler-Shimura comparison map.

Since the isomorphism classes over $\mathbb{Q}$ of $I^{sym\otimes n}$ can have several classes over $\mathbb{Z}$, we need to have an explicit construction of the $\Gamma_0(C)$-module $L(n, \chi; \mathbb{Z}[\chi])$. To do the construction, let A be a $\mathbb{Z}[\chi]$-algebra. Here is a more concrete definition of $SL_2(\mathbb{Z})$-module as the space of homogeneous polynomial in (X, Y) of degree n with coefficients in A. We let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \cap GL_2(\mathbb{Q})$ act on $P(X, Y) \in L(n, \chi; A)$ by

$$(\gamma P)(X, Y) = \chi(d)P((X, Y)^t\gamma^\iota),$$

where $\gamma^\iota = (\det\gamma)\gamma^{-1}$. The *cuspidal cohomology group* $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))$ is defined in [IAT] Chapter 8 and [LFE] Chapter 6 as the image of compactly supported cohomology group of the sheaf associated to $L(n,\chi;A)$, whose definition we recall later in this subsection.

The Eichler-Shimura map in (5.2) δ is specified in [LFE] as follows: We put

$$\omega(f) = \begin{cases} f(z)(X-zY)^n dz & \text{if } f \in S_k(\Gamma_0(C),\chi), \\ f(z)(X-\overline{z}Y)^n d\overline{z} & \text{if } f \in \overline{S}_k(\Gamma_0(C),\chi). \end{cases}$$

Then we associate to f the cohomology class of the 1-cocycle $\gamma \mapsto \int_z^{\gamma(z)} \omega(f)$ of $\Gamma_0(C)$ for a fixed point z on the upper half complex plane. The map δ does not depend on the choice of z.

Let us prepare preliminary facts for the definition of cuspidal cohomology groups. Let $\Gamma = \Gamma_C = \Gamma(3) \cap \Gamma_0(C)$ for $\Gamma(3) = \{\gamma \in SL_2(\mathbb{Z}) | \gamma \equiv 1 \mod 3\}$. The good point of Γ_C is that it acts on $\mathfrak{H}$ freely without fixed point. To see this, let Γ_z be the stabilizer of $z \in \mathfrak{H}$ in Γ. Since the stabilizer of z in $SL_2(\mathbb{R})$ is a maximal compact subgroup C_z of $SL_2(\mathbb{R})$, $\Gamma_z = \Gamma \cap C_z$ is compact-discrete and hence is finite. Thus if Γ is torsion-free, it acts freely on $\mathfrak{H}$. Pick a torsion-element $\gamma \in \Gamma$. Then two eigenvalues ζ and $\overline{\zeta}$ of γ are roots of unity complex conjugate each other. Since Γ cannot contain -1, $\zeta \notin \mathbb{R}$. Thus if $\gamma \neq 1$, we have $-2 < \mathrm{Tr}(\gamma) = \zeta + \overline{\zeta} < 2$. Since $\gamma \equiv 1 \mod 3$, $\mathrm{Tr}(\gamma) \equiv 2 \mod 3$, which implies $\mathrm{Tr}(\gamma) = -1$. Thus γ satisfies $\gamma^2 + \gamma + 1 = 0$ and hence $\gamma^3 = 1$. Thus $\mathbb{Z}[\gamma] \cong \mathbb{Z}[\omega]$ for a primitive cubic root ω. Since 3 ramifies in $\mathbb{Z}[\omega]$, $\mathbb{Z}[\omega]/3\mathbb{Z}[\omega]$ has a unique maximal ideal $\mathfrak{m}$ with $\mathfrak{m}^2 = 0$. The ideal $\mathfrak{m}$ is principal and is generated by ω. Thus the matrix ($\gamma - 1 \mod 3$) corresponds ($\omega - 1 \mod 3$), which is non-zero nilpotent. This $\gamma - 1 \mod 3$ is non-zero nilpotent, showing $\gamma \notin \Gamma(3)$, a contradiction.

By the above argument, the fundamental group of $Y = \Gamma_C \backslash \mathfrak{H}$ is isomorphic to Γ_C. Then we may consider the locally constant sheaf $\mathcal{L}(n,\chi;A)$ of sections associated to the following covering:

$$\mathcal{X} = \Gamma_C \backslash (\mathfrak{H} \times L(n,\chi;A)) \twoheadrightarrow Y \quad \text{via } (z,P) \mapsto z.$$

Since Γ_C acts on $\mathfrak{H}$ without fixed point, the space $\mathcal{X}$ is locally isomorphic to Y, and hence $\mathcal{L}(n,\chi;A)$ is a well defined locally constant sheaf. In this setting, there is a canonical isomorphism (see [LFE] Appendix Theorem 1 and Proposition 4):

$$H^1(\Gamma_C, L(n,\chi;A)) \cong H^1(Y, \mathcal{L}(n,\chi;A)).$$

Note that $\Gamma_0(C)/\Gamma_C$ is a finite group whose order is a factor of 24. Thus as long as 6 is invertible in A, we have

$$H^0(\Gamma_0(C)/\Gamma_C, H^1(\Gamma_C, L(n,\chi;A))) = H^1(\Gamma_0(C), L(n,\chi;A)). \tag{5.3}$$

As long as 6 is invertible in A, all perfectness of Poincaré duality for smooth quotient $\Gamma_C \backslash \mathfrak{H}$ descends over A to $H^1(\Gamma_0(C), L(n,\chi;A))$; so, we pretend as if $X_0(C)$ is smooth hereafter, as we always assume that 6 is invertible in A.

For simplicity, we write Γ for $\Gamma_0(C)$ and $Y = Y_0(C) := \Gamma_0(C)\backslash\mathfrak{H}$. Let $\mathfrak{S} = \Gamma\backslash\mathbf{P}^1(\mathbb{Q}) \cong \Gamma\backslash SL_2(\mathbb{Z})/\Gamma_\infty$ for $\Gamma_\infty = \{\gamma \in SL_2(\mathbb{Z}) | \gamma(\infty) = \infty\}$. Thus $\mathfrak{S}$ is the set of cusps of Y. We can take a neighborhood of ∞ in Y isomorphic to the cylinder $\mathbb{C}/\mathbb{Z}$. Since we have a neighborhood of each cusp isomorphic to a given neighborhood of ∞, we can take an open neighborhood of each cusp of Y isomorphic to the cylinder. We then compactify Y adding a circle at every cusp. We write $\overline{Y}$ for the compactified space. Then

$$\partial\overline{Y} = \bigsqcup_{\mathfrak{S}} S^1,$$

and

$$H^q(\partial\overline{Y}, \mathcal{L}(n,\chi;A)) \cong \bigoplus_{s\in\mathfrak{S}} H^q(\Gamma_s, L(n,\chi;A)),$$

where Γ_s is the stabilizer in Γ of a cusp $s \in \mathbf{P}^1(\mathbb{Q})$ representing an element in $\mathfrak{S}$. Since $\Gamma_s \cong \mathbb{Z}$, $H^q(\partial\overline{Y}, \mathcal{L}(n,\chi;A)) = 0$ if $q > 1$.

We have a commutative diagram whose horizontal arrows are given by the restriction maps:

$$\begin{array}{ccc} H^1(Y,\mathcal{L}(n,\chi;A)) & \xrightarrow{res} & H^1(\partial\overline{Y},\mathcal{L}(n,\chi;A)) \\ \wr\downarrow & & \wr\downarrow \\ H^1(\Gamma, L(n,\chi;A)) & \xrightarrow{res} & \bigoplus_{s\in\mathfrak{S}} H^1(\Gamma_s, L(n,\chi;A)). \end{array}$$

We then define H^1_{cusp} by the kernel of the restriction map.

We have the boundary exact sequence (cf. [LFE] Appendix Corollary 2):

$$\begin{aligned} 0 \to H^0(Y,\mathcal{L}(n,\chi;A)) &\to H^0(\partial\overline{Y},\mathcal{L}(n,\chi;A)) \to H^1_c(Y,\mathcal{L}(n,\chi;A)) \\ &\xrightarrow{\pi} H^1(Y,\mathcal{L}(n,\chi;A)) \to H^1(\partial\overline{Y},\mathcal{L}(n,\chi;A)) \to H^2_c(Y,\mathcal{L}(n,\chi;A)) \to 0. \end{aligned}$$

Here H^1_c is the sheaf cohomology group with compact support, and the map π send each compactly supported cohomology class to its usual cohomology class. Thus H^1_{cusp} is equal to the image of π, made of cohomology classes rapidly decreasing towards cusps (when $A = \mathbb{C}$). We also have (cf. [LFE] Chapter 6 and Appendix)

$$(5.4)\quad H^2_c(Y,\mathcal{L}(n,\chi;A)) \cong L(n,\chi;A)/\sum_{\gamma\in\Gamma}(\gamma-1)L(n,\chi;A) \text{ (so, } H^2_c(Y,A) = A),$$

$$H^0_c(Y,\mathcal{L}(n,\chi;A)) = 0 \quad\text{and}\quad H^0(Y,\mathcal{L}(n,\chi;A)) = H^0(\Gamma, L(n,\chi;A)).$$

When $A = \mathbb{C}$, the isomorphism $H^2_c(Y,\mathbb{C}) \cong \mathbb{C}$ is given by $[\omega] \mapsto \int_Y \omega$, where ω is a compactly supported 1-form representing the cohomology class $[\omega]$ (de Rham theory; cf. [LFE] Appendix Proposition 6).

Suppose that $n!$ is invertible in A. Then the $\binom{n}{j}^{-1} \in A$ for binomial symbols $\binom{n}{j}$. We can then define a pairing $[\ ,\ ] : L(n,\chi;A) \times L(n,\chi^{-1};A) \to A$ by

$$(5.5)\qquad [\sum_j a_j X^{n-j}Y^j, \sum_j b_j X^{n-j}Y^j] = \sum_{j=0}^{n} (-1)^j \binom{n}{j}^{-1} a_j b_{n-j}.$$

By definition, $[(X - zY)^n, (X - \overline{z}Y)^n] = (z - \overline{z})^n$. It is an easy exercise to check that $[\gamma P, \gamma Q] = \det\gamma^n[P,Q]$ for $\gamma \in GL_2(A)$. Thus we have a Γ-homomorphism $L(n,\chi;A) \otimes_A L(n,\chi^{-1};A) \to A$, and we get the cup product pairing

$$[\ ,\ ] : H^1_c(Y,\mathcal{L}(n,\chi;A)) \times H^1(Y,\mathcal{L}(n,\chi^{-1};A)) \longrightarrow H^2_c(Y,A) \cong A.$$

This pairing induces the cuspidal pairing

$$(5.6)\qquad [\ ,\ ] : H^1_{cusp}(Y,\mathcal{L}(n,\chi;A)) \times H^1_{cusp}(Y,\mathcal{L}(n,\chi^{-1};A)) \longrightarrow A.$$

By (5.3), we identify $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))$ as a subspace of $H^1_{cusp}(Y, \mathcal{L}(n,\chi;A))$ and write $[\ ,\ ]$ for the pairing induced on $H^1_{cusp}(\Gamma_0(C), \mathcal{L}(n,\chi;A))$ by the above pairing of $H^1_{cusp}(Y, \mathcal{L}(n,\chi;A))$.

There are three natural operators acting on the cohomology group (cf. [LFE] 6.3): one is the action of Hecke operators $T(n)$ on $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))$, and the second is an involution τ induced by the action of $\tau = \left(\begin{smallmatrix} 0 & -1 \\ C & 0 \end{smallmatrix}\right)$, and the third is is an action of complex conjugation c given by $c\omega(z) = \mathbf{e}\omega(-\overline{z})$ for $\mathbf{e} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and a differential form ω. In particular, δ and c commute with $T(n)$. We write $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))[\pm]$ for the $\pm$-eigenspace of c. Then it is known ([IAT] or [LFE] (11) in Section 6.3) that $H^1_{cusp}(\Gamma_0(C), L(n,\chi;\mathbb{Q}(\lambda)))[\pm]$ is $h_\kappa(C,\chi;\mathbb{Q}(\lambda))$-free of rank 1. Supposing that A contains the eigenvalues $\lambda(T(n))$ for all n, we write $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))[\lambda,\pm]$ for the λ-eigenspace under $T(n)$.

The action of $\tau = \left(\begin{smallmatrix} 0 & -1 \\ C & 0 \end{smallmatrix}\right)$ defines a quasi-involution on the cohomology

$$\tau : H^1_{cusp}(\Gamma_0(C), L(n,\chi;A)) \to H^1_{cusp}(\Gamma_0(C), L(n,\chi^{-1};A)),$$

which is given by $u \mapsto \{\gamma \mapsto \tau u(\tau\gamma\tau^{-1})\}$ for each homogeneous 1-cocycle u. The cocycle $u|\tau$ has values in $L(n,\chi^{-1};A)$ because conjugation by τ interchanges the diagonal entries of γ. We have $\tau^2 = (-C)^n$ and $[x|\tau, y] = [x, y|\tau]$. Then we modify the above pairing $[\ ,\ ]$ by τ and define $\langle x, y\rangle = [x, y|\tau]$ ([LFE] 6.3 (6)). As described in [IAT] Chapter 8 and [LFE] Cahpter 6, we have a natural action of Hecke operators $T(n)$ on $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))$. The operator $T(n)$ is symmetric with respect to this pairing:

$$\langle x|T(n), y\rangle = \langle x, y|T(n)\rangle. \tag{5.7}$$

We now regard $\lambda : h_{k,\chi/\mathbb{Z}[\chi]} \to \mathbb{C}$ as actually having values in $W \cap \overline{\mathbb{Q}}$ (via the fixed embedding: $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$). Put $A = W \cap \mathbb{Q}(\lambda)$. Then A is a valuation ring of $\mathbb{Q}(\lambda)$ of residual characteristic p. Thus for the image L of $H^1_{cusp}(\Gamma_0(C), L(n,\chi;A))$ in $H^1_{cusp}(\Gamma_0(C), L(n,\chi;\mathbb{Q}(\lambda)))$,

$$H^1_{cusp}(\Gamma_0(C), L(n,\chi;\mathbb{Q}(\lambda)))[\lambda,\pm] \cap L = A\xi_\pm$$

for a generator $\xi_\pm$. Then for the normalized eigenform $f \in S_\kappa(\Gamma_0(C),\chi)$ with $T(n)f = \lambda(T(n))f$, we define $\Omega(\pm,\lambda;A) \in \mathbb{C}^\times$ by

$$\delta(f) \pm c(\delta(f)) = \Omega(\pm,\lambda;A)\xi_\pm.$$

The above definition of the period $\Omega(\pm,\lambda;A)$ can be generalized to the Hilbert modular case as in [H94].

We now compute

$$\langle \Omega(+,\lambda;A)\xi_+, \Omega(-,\lambda;A)\xi_-\rangle = \Omega(+,\lambda;A)\Omega(-,\lambda;A)\langle \xi_+, \xi_-\rangle.$$

Note that $\delta(f)|\tau = W(\lambda)(-1)^n C^{(n/2)}\delta(f_c)$, where $f_c = \sum_{m=1}^\infty \overline{\lambda(T(m))}q^m$ and $f|\tau = W(\lambda)f_c$ for and $W(\lambda) \in \mathbb{C}$ with $|W(\lambda)| = 1$. By definition, we have

$$2\Omega(+,\lambda;A)\Omega(-,\lambda;A)\langle \xi_+, \xi_-\rangle = [\delta(f) + c\delta(f), (\delta(f) - c\delta(f))|\tau],$$

which is equal to, up to sign,

$$
\begin{aligned}
(5.8)\quad 4i\int_{Y_0(C)}[\delta(f)|\tau, c\delta(f)]dx\wedge dy &= 2^{k+1}i^kW(\lambda)C^{((k-1)/2}\int_{Y_0(C)}|f_c|^2y^{k-1}dxdy\\
&= 2^{k+1}i^kW(\lambda)C^{(k-1)/2}\int_{Y_0(C)}|f|^2y^{k-1}dxdy\\
&= i^kW(\lambda)C^{(k+1)/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda)),
\end{aligned}
$$

where $Y_0(C)=\Gamma_0(C)\backslash\mathfrak{H}$. This shows

Theorem 5.2. *Let χ be a character of conductor C. Let $\lambda: h_k(\Gamma_0(C),\chi;\mathbb{Z}[\chi])\to\overline{\mathbb{Q}}$ ($k\geq 1$) be a $\mathbb{Z}[\chi]$-algebra homomorphism. Then for a valuation ring A of $\mathbb{Q}(\lambda)$, we have, up to sign,*

$$\frac{i^kW(\lambda)C^{(k+1)/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}=\langle\xi_+,\xi_-\rangle\in\mathbb{Q}(\lambda).$$

Moreover we have $\langle\xi_+,\xi_-\rangle\in n!^{-1}\cdot A$ if $p\nmid\varphi(C)$ with $p>3$.

The proof of rationality of the adjoint L-values as above can be generalized to even non-critical values $L(1,Ad(\lambda)\otimes\alpha)$ for quadratic Dirichlet characters α (see [H99]).

If one insists on p-ordinarity: $\lambda(T(p))\in A^\times$ for the residual characteristic $p\geq 5$ of A, we can show that $\langle\xi_+,\xi_-\rangle\in A$. This follows from the perfectness of the duality pairing $\langle\ ,\ \rangle$ on the p-ordinary cohomology groups defined below even if $n!$ is not invertible in A (see Theorem 5.4 in the text and [H88a]).

Let W be the completion of the valuation ring A. Let $\mathbb{T}_k=\mathbb{T}/(t-\gamma^k)\mathbb{T}$ be the local ring of $h_k(\Gamma_0(C),\chi;W)$ through which λ factor through. Let 1_k be the idempotent of $\mathbb{T}_k$ in the Hecke algebra. Since the conductor of χ coincides with C, $h_k(\Gamma_0(C),\chi;W)$ is reduced (see [MFM, Theorem 4.6.8]). Thus for the quotient field K of W, the unique local ring $\mathbb{I}_K$ of $h_k(\Gamma_0(C),\chi;K)$ through which λ factors is isomorphic to K. Let 1_λ be the idempotent of $\mathbb{I}_K$ in $h_k(\Gamma_0(C),\chi;K)$. Then we have the following important corollary.

Corollary 5.3. *Let the assumption be as in* Theorem 5.2. *Assume that $p\nmid\varphi(C)$. Let A be a valuation ring of residual characteristic $p>3$. Suppose that $\langle\ ,\ \rangle$ induces a perfect duality on $1_kH^1_{cusp}(\Gamma_0(C),L(n,\chi;W))$ for $k=n+1$. Then*

$$\left|\frac{i^{k+1}W(\lambda)C^{k/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}\right|_p^{-r(W)}=|L^\lambda/L_\lambda|,$$

where $r(W)=\mathrm{rank}_{\mathbb{Z}_p}W$, $L^\lambda=1_\lambda L$ for the image L of $H^1_{cusp}(\Gamma_0(C),L(n,\chi;W))[+]$ in the cohomology $H^1_{cusp}(\Gamma_0(C),L(n,\chi;K))[+]$, and L_λ is given by the intersection $L^\lambda\cap L$ in $H^1_{cusp}(\Gamma_0(C),L(n,\chi;K))[+]$.

Writing $1_k=1_\lambda+1'_\lambda$ and defining ${}^\perp L_\lambda=1'_\lambda L$ with ${}^\perp L^\lambda={}^\perp L_\lambda\cap L$, we have ${}^\perp L_\lambda/{}^\perp L^\lambda\cong 1_kL/(L_\lambda\oplus{}^\perp L^\lambda)\cong L^\lambda/L_\lambda$ as modules over $\mathbb{T}_k$. If $L^\lambda/L_\lambda\neq 0$ (i.e., the L-value is divisible by $\mathfrak{m}_W$), by the argument proving Proposition 2.3 applied to $(L_\lambda,{}^\perp L^\lambda,1_kL)$ in place of $(\mathfrak{a},\mathfrak{b},R)$, we conclude the existence of an algebra homomorphism $\lambda':\mathbb{T}_k\to\overline{\mathbb{Q}}_p$ factoring through the complementary factor $1'_\lambda\mathbb{T}_k$ such that $\lambda\equiv\lambda' \mod \mathfrak{p}$ for the maximal ideal $\mathfrak{p}$ above $\mathfrak{m}_W$ in the integral closure of W in $\overline{\mathbb{Q}}_p$. In this way, the congruence criterion of [H81a] was proven.

Proof. By our choice, ξ_+ is the generator of L_λ. Similarly we define $M^\lambda = 1_\lambda M$ for the image M of $H^1_{cusp}(\Gamma_0(C), L(n,\chi;W))[-]$ in $H^1_{cusp}(\Gamma_0(C), L(n,\chi;K))[-]$, and $M_\lambda = M^\lambda \cap M$ in $H^1_{cusp}(\Gamma_0(C), L(n,\chi;K))[-]$. Then ξ_- is a generator of M_λ. Since the pairing is perfect, $L_\lambda \cong \mathrm{Hom}_W(M^\lambda, W)$ and $L^\lambda \cong \mathrm{Hom}_W(M_\lambda, W)$ under $\langle\ ,\ \rangle$. Then it is an easy exercise to see that $|\langle \xi_+, \xi_-\rangle|_p^{-1} = |L^\lambda/L_\lambda|$. □

As for the assumption of the perfect duality, we quote the following slightly technical result from [H81a] and [H88a, Section 3]:

Theorem 5.4. *Let the notation and assumption be as in* Theorem 5.2. *Suppose* $p > 3$. *If either* $\lambda(T(p)) \in A^\times$ *or* $\frac{1}{n!} \in A$, *then*

(1) $1_k H^1_{cusp}(\Gamma_0(C), L(n,\chi;W))$ *is* W*-free;*
(2) *the pairing* $\langle\ ,\ \rangle$ *induces a perfect duality on* $1_k H^1_{cusp}(\Gamma_0(C), L(n,\chi;W))$.

What is really proven in [H88a, Section 3] is the W-freeness and the perfect self-duality of $H^1_{cusp}(\Gamma_1(C), L(n;W))$. Thus if $C = 1$, the theorem follows from this result. If $C > 1$ and $p \nmid \varphi(C)$, we have an orthogonal decomposition (from the inflation-restriction sequence):

$$H^1_{cusp}(\Gamma_1(C), L(n;W)) \cong \bigoplus_\chi H^1_{cusp}(\Gamma_0(C), L(n,\chi;W)),$$

and thus the theorem follows from [H88a, Theorem 3.1]. If the reader scrutinizes the argument in [H88a, Section 3], replacing $(\Gamma_1(Np^r), L(n;W))$ there by $(\Gamma_0(C), L(n,\chi;W))$ here, he or she will find that the above theorem holds without assuming $p \nmid \varphi(C)$ (but we need $p > 3$ if $\Gamma_0(C)/\{\pm 1\}$ has non-trivial torsion).

5.3. **Congruence and adjoint L-values.** Here we study a non-abelian adjoint version of the analytic class number formula, which follows from the theorem of Taylor-Wiles (Theorem 4.3) and some earlier work of the author (presented in the previous subsection). Actually, long before the formula was established, Doi and the author had found an intricate relation between congruence of modular forms and the adjoint L-value (see the introduction of [DHI98]), and later via the work of Taylor–Wiles, it was formulated in a more precise form we discuss here. In [W95], Wiles applied Taylor–Wiles system argument to $1_{\mathbb{T}_Q} H^1(\Gamma_0(C) \cap \Gamma_Q, W)$ for varying Q and obtained Theorem 4.3. Here $\Gamma_Q := \Gamma_1(\prod_{q \in Q} q)$ for suitably chosen sets Q of primes outside Cp, and $\mathbb{T}_Q$ is the local ring of $h_1(\Gamma_0(C) \cap \Gamma_Q, \chi; W)$ covering $\mathbb{T}_1$ note here $k = 1$). As a by-product of the Taylor-Wiles argument, we obtained the local complete intersection property in Theorem 4.3 and in addition

$$1_1 H^1(\Gamma_0(C), W) \textit{ is a free } \mathbb{T}_1\textit{-module.}$$

As many followers of Taylor–Wiles did later, this can be applied to general k and also one can replace $1_{\mathbb{T}_Q} H^1(\Gamma_0(C) \cap \Gamma_Q, W)$ by $1_{\mathbb{T}_Q} H^1(\Gamma_0(C) \cap \Gamma_Q, L(n,\chi;W)[\pm]$. Then we obtain Theorem 4.3 as stated and further

Theorem 5.5. *Let the notation and assumption be as in* Theorem 4.3. *Then* $1_{\mathbb{T}_Q} H^1(\Gamma_0(C), L(n,\chi;W))[\pm] \cong \mathbb{T}_k$ *as* $\mathbb{T}_k$*-modules* ($k = n+1$).

The assumption here is that $\overline{\rho} = \rho_\lambda \mod \mathfrak{m}_W$ is absolutely irreducible over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{(-1)^{(p-1)/2}p}])$, $\overline{\epsilon} \neq \overline{\delta}$ and $\lambda(T(p)) \in A^\times$. Actually, by a result of Mazur, we do not need the irreducibility over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{(-1)^{(p-1)/2}p}])$ but irreducibility just over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is sufficient as explained in [MFG, §5.3.2]. However at the end, we eventually need to assume stronger assumption of irreducibility over $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\sqrt{(-1)^{(p-1)/2}p}])$ to use Theorem 3.5 to relate the L-value with $|C_1(\lambda;W)|$ and the size of the Selmer group.

To relate the size $|L^\lambda/L_\lambda|$ to the size of the congruence module $|C_0(\lambda;W)|$ for $\lambda:\mathbb{T}_k \twoheadrightarrow W$, we apply the theory in §2.2. To compare with the notation in §2.2, now we rewrite W as A forgetting about the dense subring $W\cap\overline{\mathbb{Q}}$ denoted by A in the previous subsection, and we write $R=\mathbb{T}_k$, $\lambda=\phi:R\to A=W$ and S for the image of $\mathbb{T}_k$ in X, decomposing $R\otimes_W K=K\oplus X$ as algebra direct sum. Under the notation of Corollary 5.3, we get $L^\lambda\otimes_A W/L_\lambda\otimes_A W\cong L^\lambda/L_\lambda$ as $h_k(\Gamma_0(C),\chi;A)$-modules. Since on $L^\lambda\otimes_A W/L_\lambda\otimes_A W$, the Hecke algebra $h_k(\Gamma_0(C),\chi;A)$ acts through λ, it acts through R. Thus multiplying $\mathbf{1}_k$ does not alter the identity $L^\lambda\otimes_A W/L_\lambda\otimes_A W\cong L^\lambda/L_\lambda$, and we get

$$1_k(L^\lambda\otimes_A W)/1_k(L_\lambda\otimes_A W)\cong L^\lambda/L_\lambda.$$

Fix an isomorphism of R-modules: $1_kL\cong R$ by Theorem 5.5. Then we have $1_k(L_\lambda\otimes_A W)\cong R\cap(A\oplus 0)=\mathfrak{a}$ in $R\otimes_A K$ and $1_k(L^\lambda\otimes_A W)\cong R$. Thus $1_k(L^\lambda\otimes_A W)/1_k(L_\lambda\otimes_A W)\cong A/\mathfrak{a}\cong C_0(\phi;A)$. In conclusion, we get

Theorem 5.6. *Let the assumption be as in* Theorem 4.1 *and the notation be as in* Corollary 5.3. *Then we have*

$$\left|\frac{i^{k+1}W(\lambda)C^{k/2}\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)}\right|_p^{-r(W)}=\big||C_0(\lambda;W)|\big|_p^{-1}=\big||C_1(\lambda;W)|\big|_p^{-1}.$$

The last identity follows from Theorem 4.3 and Theorem 3.5.

As already described, primes appearing in the discriminant of the Hecke algebra gives congruence among algebra homomorphisms of the Hecke algebra into $\overline{\mathbb{Q}}$, which are points in $\mathrm{Spec}(h_k)(\overline{\mathbb{Q}})$. For the even weights $k=26,22,20,18,16,12$, we have $\dim_{\mathbb{C}} S_k(SL_2(\mathbb{Z}))=1$, and the Hecke field $h_k\otimes_{\mathbb{Z}}\mathbb{Q}$ is just $\mathbb{Q}$ and hence the discriminant is 1. As is well known from the time of Hecke that $h_{24}\otimes_{\mathbb{Z}}\mathbb{Q}=\mathbb{Q}[\sqrt{144169}]$. The square root of the value in the following table is practically the adjoint L-value $L(1,Ad(f))$ for a Hecke eigenform $f\in S_k(SL_2(\mathbb{Z}))$ for the weight k in the table. Here is a table by Y. Maeda of the discriminant of the Hecke algebra of weight k for $S_k(SL_2(\mathbb{Z}))$ when $\dim S_k(SL_2(\mathbb{Z}))=2$:

Discriminant of Hecke algebras.

weight	dim	Discriminant
24	2	$2^6\cdot 3^2\cdot 144169$
28	2	$2^6\cdot 3^6\cdot 131\cdot 139$
30	2	$2^{12}\cdot 3^2\cdot 51349$
32	2	$2^6\cdot 3^2\cdot 67\cdot 273067$
34	2	$2^8\cdot 3^4\cdot 479\cdot 4919$
38	2	$2^{10}\cdot 3^2\cdot 181\cdot 349\cdot 1009$

A bigger table (computed by Maeda) can be found in [MFG, §5.3.3] and in [Ma14], and a table of the defining equations of the Hecke fields is in [Ma14]. The author believes that by computing Hecke fields in the mid 1970's, Maeda somehow reached the now famous conjecture asserting irreducibility of the Hecke algebra of $S_{2k}(SL_2(\mathbb{Z}))$ (see [Ma14] and [HiM97]).

5.4. **Adjoint non-abelian class number formula.** Let $\lambda^\circ : h_k(C,\chi;\mathbb{Z}[\chi]) \to W$ be a primitive $\mathbb{Z}[\chi]$-algebra homomorphism of conductor C with $\overline{\rho} = \rho_\lambda \mod \mathfrak{m}_W$. Suppose λ° is ordinary; so, $\chi = \psi\omega^{-k}$. Define N to be the prime to p-part of C, and write $C = Np^{r+1}$ if $p|C$ and otherwise, we put $r = 1$. Assume that conductor of ψ is divisible by N. Combining all what we have done, by Theorem 3.12, we get the following order formula of the Selmer group (compare with Theorem 3.4):

Theorem 5.7. *Suppose $p \nmid 6\varphi(C)$ in addition to the assumption of* Theorem 4.3 *for $\overline{\rho}$. Let $\lambda : h_k(Np^{r+1},\chi;W) \to W$ be the algebra homomorphism equivalent to λ° with $\lambda(U(p)) \in W^\times$. We put $A = \mathbb{Q}(\lambda^\circ) \cap W$. Take an element $\eta(\lambda) \in A$ such that $A/\eta(\lambda) \cong C_0(\lambda^\circ;A)$. Then we have*

$$\text{(CN1)} \qquad \frac{C^{(k+1)/2}W(\lambda)\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda;A)\Omega(-,\lambda;A)} = \eta(\lambda) \quad \textit{up to A-units, and}$$

$$\text{(CN2)} \qquad \Big|\frac{C^{(k+1)/2}W(\lambda)\Gamma(1,Ad(\lambda))L(1,Ad(\lambda))}{\Omega(+,\lambda^\circ;A)\Omega(-,\lambda^\circ;A)}\Big|_p^{-[W:\mathbb{Z}_p]} = \#(\mathrm{Sel}(Ad(\rho_\lambda))_{/\mathbb{Q}}).$$

The definition of the Selmer group can be also done through Fontaine's theory as was done by Bloch–Kato, and the above formula can be viewed as an example of the Tamagawa number formula of Bloch and Kato for the motive $M(Ad(\rho_\lambda))$ (see [W95] p.466, [BK90] Section 5 and [F92]). The finiteness of the Bloch–Kato Selmer groups $\mathrm{Sel}(Ad(\rho_\lambda))$ for λ of weight 2 associated to an elliptic curve (under some additional assumptions) was first proven by M. Flach [F92], and then relating Bloch–Kato Selmer groups to Greenberg Selmer groups, he showed also the finiteness of Greenberg Selmer groups. By adopting the definition of Bloch–Kato, we can define the Selmer group $\mathrm{Sel}_{crys}(Ad(\rho_\lambda))$ when ρ_λ is associated to a p-divisible Barsotti–Tate group and a crystalline modular motives at p over $\mathbb{Z}_p$, and the formula (CN2) is valid even for the non-ordinary cases (see [DFG04]).

5.5. **p-Adic adjoint L-functions.** Assume the assumptions of Theorem 4.1. Then $(\mathbb{T},\rho_{\mathbb{T}})$ is universal among deformations of $\overline{\rho}$ under the notation of Theorem 4.1. Let $\boldsymbol{\lambda} : \mathbb{T} \to \Lambda$ be a Λ-algebra homomorphism (so, $\mathrm{Spec}(\Lambda) \xrightarrow{\boldsymbol{\lambda}^*} \mathrm{Spec}(\mathbb{T})$ is an irreducible component). Write $\rho_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \circ \varrho^{ord}$. We thus have $\mathrm{Sel}(Ad(\rho_{\boldsymbol{\lambda}}))$. By Theorem 3.12 (and Proposition 2.2 (ii)), we have, as Λ-modules,

$$\text{(5.9)} \qquad \mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\boldsymbol{\lambda}}))^\vee \cong \mathrm{Ker}(\boldsymbol{\lambda})/\mathrm{Ker}(\boldsymbol{\lambda})^2 = C_1(\boldsymbol{\lambda};\Lambda).$$

Thus $\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\boldsymbol{\lambda}}))^\vee$ is a torsion Λ-module, and hence we have a characteristic power series $\Phi(T) \in \Lambda$. We would like to construct a p-adic L-function $L_p(T) \in \Lambda$ from the Hecke side such that $L_p(T) = \Phi(T)$ up to units in Λ.

Consider the algebra homomorphism $\pi_\ell : \Lambda \to W$ given by $\Phi(T) \mapsto \Phi(\gamma^\ell - 1) \in W$ $(\gamma = 1+p)$. So we have $P_\ell = \mathrm{Ker}(\pi_\ell) = (t - \gamma^\ell)$. After tensoring W via π_ℓ, we get an W-algebra homomorphism

$$\lambda_\ell : h_\ell(p,\psi\omega^{-\ell};W) \twoheadrightarrow \mathbb{T}/P_\ell\mathbb{T} \to W.$$

Thus λ_ℓ is associated to a Hecke eigenform f_ℓ. We then have $\eta_\ell \in W$ such that $C_0(\lambda_\ell;W) \cong W/\eta_\ell W$ and

$$\frac{W(\lambda_\ell^\circ)C(\lambda_\ell^\circ)^{(\ell+1)/2}\Gamma(1,Ad(\lambda_\ell))L(1,Ad(\lambda_\ell))}{\Omega(+,\lambda_\ell;A_\ell)\Omega(-,\lambda_\ell;A_\ell)} = \eta_\ell$$

up to units in W, where $A_\ell = \mathbb{Q}(\lambda_\ell) \cap W$. We require to have

$$L_p(P_\ell) = (L_p \mod P_\ell) = L_p(\gamma^\ell - 1) = \eta_\ell$$

up to W-units for all $\ell \geq 2$.

Since $\mathbb{T}$ can be embedded into $\prod_\ell \mathbf{h}_{\ell,\psi\omega^{-\ell}/W}$, the reducedness of the Hecke algebras $h_\ell^{ord}(p, \psi\omega^{-\ell}; W)$ shows that $\mathbb{T}$ is reduced. Thus for the field of fractions Q of Λ,

$$\mathbb{T} \otimes_\Lambda Q \cong \mathcal{L} \oplus X,$$

where the projection to $\mathcal{L}$ is $\boldsymbol{\lambda} \otimes \mathrm{id}$. We then define $C_j(\boldsymbol{\lambda}; \Lambda)$ as in §5.3.

Again Theorem 3.5 tells us that the characteristic power series of $C_1(\boldsymbol{\lambda}; \Lambda)$ and $C_0(\boldsymbol{\lambda}; \Lambda)$ coincide. Since $C_0(\boldsymbol{\lambda}; \Lambda) \cong \mathbb{T}/\mathbb{T} \cap (\Lambda \oplus 0)$ in $\mathbb{T} \otimes_\Lambda Q$, Λ-freeness of $\mathbb{T}$ tells us that $\mathfrak{a} := \mathbb{T} \cap (\Lambda \oplus 0)$ is principal generated by $L_p \in \Lambda$. Put $\mathbb{T}_\ell = \mathbb{T}/(t - \gamma^\ell)\mathbb{T} \subset \mathbf{h}_{\ell,\psi\omega^{-\ell}/W}$. Let $\pi_\ell : \Lambda \twoheadrightarrow \Lambda/(t - \gamma^\ell) = W$ be the projection which is realized by $\Phi(T) \mapsto \Phi(\gamma^\ell - 1)$. Since $\mathbb{T}_\ell \cong \mathbb{T} \otimes_{\Lambda,\pi_\ell} W$, we see easily from a diagram chasing that

$$C_0(\boldsymbol{\lambda}, \Lambda) \otimes_{\Lambda,\pi_\ell} W \cong C_0(\lambda_\ell; W).$$

This assures us $L_p(P_\ell) = \eta_\ell$ up to W-units. The Iwasawa module $C_0(\boldsymbol{\lambda}; \Lambda)$ was first introduced in [H86a] to study behavior of congruence between modular forms as one varies Hecke eigenforms f_ℓ associated to λ_ℓ. The fact that the characteristic power series of $C_0(\boldsymbol{\lambda}; \Lambda)$ interpolates p-adically the adjoint L-values was pointed out in [H86a] (see also [H88a]). We record here what we have proven:

Corollary 5.8. *Let the notation and the assumption be as in* Theorem 5.7. *Then there exists* $0 \neq L_p(T) \in \Lambda$ *such that*

(1) $L_p(T)$ *gives a characteristic power series of* $\mathrm{Sel}_{\mathbb{Q}}(Ad(\rho_{\boldsymbol{\lambda}}))^\vee$*;*
(2) *We have, for all* $\ell \geq 2$,

$$L_p(P_\ell) = \frac{W(\lambda_\ell^\circ)C(\lambda_\ell)^{(\ell+1)/2}\Gamma(1, Ad(\lambda_\ell))L(1, Ad(\lambda_\ell))}{\Omega(+, \lambda_\ell; A_\ell)\Omega(-, \lambda_\ell; A_\ell)}$$

up to units in W*, where* $C(\lambda_\ell)$ *is the conductor of* λ_ℓ*.*

Though we presented the above corollary assuming $\widetilde{\mathbb{I}} = \mathbb{I} = \Lambda$ for simplicity, the same method works well for $\widetilde{\mathbb{I}} \neq \Lambda$ by Corollaries 4.4 and 4.5. We leave the reader to formulate the general result.

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