IMAGE OF Λ-ADIC GALOIS REPRESENTATIONS MODULO p

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Abstract. Let $p \geq 5$ be a prime. If an irreducible component of the spectrum of the 'big' ordinary Hecke algebra does not have complex multiplication, under mild assumptions, we prove that the image of its mod p Galois representation contains an open subgroup of $SL_2(\mathbb{F}_p[[T]])$ for the canonical "weight" variable T. This fact appears to be deep, as it is almost equivalent to the vanishing of the μ-invariant of the Kubota–Leopoldt p-adic L-function and the anticyclotomic Katz p-adic L-function. Another key ingredient of the proof is the anticylotomic main conjecture proven by Rubin/Mazur–Tilouine.

Fix a prime $p \geq 5$, field embeddings $\mathbb{C} \overset{i_\infty}{\hookleftarrow} \overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p$ and a positive integer N prime to p. We sometimes identify $\mathbb{C}_p$ and $\mathbb{C}$ by a fixed field isomorphism compatible with the above embeddings. Consider the space of modular forms $\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi)$ with $(p \nmid N, r \geq 0)$ and cusp forms $\mathcal{S}_{k+1}(\Gamma_0(Np^{r+1}), \chi)$. Let the ring $\mathbb{Z}[\chi] \subset \mathbb{C}$ and $\mathbb{Z}_p[\chi] \subset \overline{\mathbb{Q}}_p$ be generated by the values χ over $\mathbb{Z}$ and $\mathbb{Z}_p$, respectively. The Hecke algebra $H = H_{k+1}(\Gamma_0(Np^{r+1}), \chi; \mathbb{Z}[\chi])$ over $\mathbb{Z}[\chi]$ is

$$H = \mathbb{Z}[\chi][T(n)|n = 1, 2, \cdots] \subset \mathrm{End}(\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi)).$$

For any $\mathbb{Z}[\chi]$-algebra $A \subset \mathbb{C}$, $H_{k+1}(\Gamma_0(Np^{r+1}), \chi; A) = H \otimes_{\mathbb{Z}[\chi]} A$ is actually the A-subalgebra of $\mathrm{End}(\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi))$ generated over A by the $T(n)$'s. Simply we put

$$H_{k+1,\chi} = H_{k+1,\chi/A} = H_{k+1}(\Gamma_0(Np^{r+1}), \chi; A) := H \otimes_{\mathbb{Z}[\chi]} A$$

for a p-adically closed subring $A \subset \mathbb{C}_p$ containing $\mathbb{Z}_p[\chi]$ (with uniformizer ϖ). Fix a complete discrete valuation ring W finite over $\mathbb{Z}_p$ inside $\overline{\mathbb{Q}}_p$ (though nothing changes even if W is just a complete discrete valuation ring over $\mathbb{Z}_p$ inside $\mathbb{C}_p$). Our $T(p)$ is often written as $U(p)$ as the level is divisible by p. The ordinary part $\mathbf{H}_{k+1,\chi/W} \subset H_{k+1,\chi/W}$ is then the maximal ring direct summand on which $U(p)$ is invertible. We write e for the idempotent of $\mathbf{H}_{k+1,\chi/W}$; so, e is the p-adic limit in $H_{k+1,\chi/W}$ of $U(p)^{n!}$ as $n \to \infty$. Via the fixed isomorphism $\mathbb{C}_p \cong \mathbb{C}$, the idempotent e not only acts on the space of modular forms with coefficients in W but also on the classical space $\mathcal{M}_{k+1}(\Gamma_0(Np^{r+1}), \chi)$. We write the image of the idempotent as $\mathcal{M}^{ord}_{k+1}$ for modular forms and $\mathcal{S}^{ord}_{k+1}$ for cusp forms. Let $\chi_1 = \chi_N \times$ the tame p-part of χ. Then, by [H86a] and [H86b], we have a unique 'big' Hecke algebra $\mathbf{H} = \mathbf{H}_{\chi_1/W}$ such that

(H1) $\mathbf{H}$ is free of finite rank over $\Lambda := W[[T]]$ equipped with $T(n), U(m) \in \mathbf{H}$ with $(n, Np) = 1, m|Np$,

(H2) if $k \geq 1$, regarding a character $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ as a W-algebra homomorphism ε_k of Λ onto $W[\varepsilon]$ sending $(1 + T)$ to $\gamma^k\varepsilon(\gamma)$ for $\gamma = 1 + p \in \mathbb{Z}_p$,

$$\mathbf{H} \otimes_{\Lambda,\varepsilon_k} W[\varepsilon] \cong \mathbf{H}_{k+1,\varepsilon\chi_k/W[\varepsilon]} \text{ for } \chi_k := \chi_1\omega^{1-k},$$

sending $T(n) \otimes 1$ to $T(n)$ and $U(m) \otimes 1$ to $U(m)$, where ω is the Teichmüller character.

Here $W[\varepsilon] \subset \overline{\mathbb{Q}}_p$ is the W-subalgebra generated by the values of ε. The corresponding objects for cusp forms are denoted by the corresponding lower case characters; so,

$$h = \mathbb{Z}[\chi][T(n)|0 < n \in \mathbb{Z}] \subset \mathrm{End}(\mathcal{S}_{k+1}(\Gamma_0(Np^{r+1}), \chi)),$$

$h_{k+1,\chi/A} = h_{k+1}(\Gamma_0(Np^{r+1}), \chi; A) := h \otimes_{\mathbb{Z}[\chi]} A$, the ordinary part $\mathbf{h}_{k+1,\chi/W} \subset h_{k+1,\chi/W}$ and the big cuspidal Hecke algebra $\mathbf{h}_{\chi_1/W}$. Replacing modular forms by cusp forms (and upper case symbols by

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lower case symbols), we can construct the "big" cuspidal Hecke algebra $\mathbf{h} = \mathbf{h}_{\chi_1}$. Then, similarly to the case of modular forms, we have the following characterization of the cuspidal 'big' Hecke algebra $\mathbf{h} = \mathbf{h}_{\chi_1/W}$:

(1) $\mathbf{h}$ is free of finite rank over Λ equipped with $T(n), U(m) \in \mathbf{h}$ with $(n, Np) = 1, m|Np$,
(2) $\mathbf{h} \otimes_{\Lambda,\varepsilon_k} W[\varepsilon] \cong \mathbf{h}_{k+1,\varepsilon\chi_k/W[\varepsilon]}$ for $k \geq 1$ (and ε as above).

We often identify Λ with the completed group algebra $W[[\Gamma]]$ of $\Gamma = 1 + p\mathbb{Z}_p = \gamma^{\mathbb{Z}_p}$ by $(1+T) \leftrightarrow \gamma$.

We have a surjective Λ-algebra homomorphism $\mathbf{H} \twoheadrightarrow \mathbf{h}$ sending $T(n)$ to $T(n)$ (and $U(m)$ to $U(m)$). Each (reduced) irreducible component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{H})$ has a continuous 2-dimensional semi-simple representation $\rho_{\mathbb{I}}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in the quotient field $Q(\mathbb{I})$ of $\mathbb{I}$ (see [H86b]). This representation preserves an $\mathbb{I}$-lattice $L \subset Q(\mathbb{I})^2$ (i.e., an $\mathbb{I}$-submodule of $Q(\mathbb{I})^2$ of finite type which spans $Q(\mathbb{I})^2$ over $Q(\mathbb{I})$), and as a map of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into the profinite group $\mathrm{Aut}_{\mathbb{I}}(L)$, it is continuous. Write $a(l)$ for the image of $T(l)$ $(l \nmid Np)$ in $\mathbb{I}$ and $a(p)$ for the image of $U(p)$. The representation $\rho_{\mathbb{I}}$ restricted to the p-decomposition group D_p is reducible. We write $\rho_{\mathbb{I}}^{ss}$ for its semi-simplification over D_p. As is well known now (e.g., [GME] §4.3), $\rho_{\mathbb{I}}$ satisfies

(Gal) $\quad \mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) = a(l)\ \ (l \nmid Np)$, $\rho_{\mathbb{I}}^{ss}([\gamma^s, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} (1+T)^s & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho_{\mathbb{I}}^{ss}([p, \mathbb{Q}_p]) \sim \left(\begin{smallmatrix} * & 0 \\ 0 & a(p) \end{smallmatrix}\right)$,

where $\gamma^s = (1+p)^s \in \mathbb{Z}_p^\times$ for $s \in \mathbb{Z}_p$ and $[x, \mathbb{Q}_p]$ is the local Artin symbol. Since $\sigma \mapsto \mathrm{Tr}(\rho_{\mathbb{I}}(\sigma))$ has values in $\mathbb{I}$, for each prime ideal P of $\mathbb{I}$, the reduction modulo P of $\mathrm{Tr}(\rho_{\mathbb{I}})$ gives rise to a pseudo-representation (of Wiles) with values in $\mathbb{I}/P$, which produces a semi-simple representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$ with $\mathrm{Tr}(\rho_P) = (\mathrm{Tr}(\rho_{\mathbb{I}}) \mod P)$, where $\kappa(P)$ is the quotient field of $\mathbb{I}/P$ (the residue field of P). When P is the maximal ideal of $\mathbb{I}$, we simply write $\overline{\rho}$ for ρ_P.

We call a prime ideal $P \subset \mathbb{I}$ a *prime divisor* if $\mathrm{Spec}(\mathbb{I}/P)$ has codimension 1 in $\mathrm{Spec}(\mathbb{I})$. If a prime divisor P of $\mathrm{Spec}(\mathbb{I})$ contains $(1 + T - \varepsilon\chi_k(\gamma)\gamma^k)$ with $k \geq 1$, by (2) we have a Hecke eigenform $f_P \in \mathcal{M}_{k+1}(\Gamma_0(Np^{r(P)+1}), \varepsilon\chi_k)$ such that its eigenvalue for $T(q)$ is given by $a_P(q) := P(a(q)) \in \overline{\mathbb{Q}}_p$ for all primes q. Here we regard P as a W-algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$. A prime divisor $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $P(1 + T - \varepsilon\chi_k(\gamma)\gamma^k) = 0$ for $k \geq 1$ is called an *arithmetic point* (or prime). We write $\varepsilon_P = \varepsilon$ and $k(P) = k \geq 1$ for an arithmetic P. Thus $\mathbb{I}$ gives rise to an analytic family $\mathcal{F}_{\mathbb{I}} = \{f_P | \text{arithmetic } P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)\}$ of slope 0 Hecke eigenforms. A component $\mathbb{I}$ (or the associated family) is called *cuspidal* if $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h})$. A cuspidal component $\mathbb{I}$ is called a *CM component* if there exists a nontrivial character $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{I}^\times$ such that $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \xi$. If a cuspidal $\mathbb{I}$ is not a CM component, we call it a non CM component.

Pick and fix a non CM component $\mathbb{I}$ of prime-to-p level N. Then we have

(R) $\overline{\rho}|_{D_p} \cong \left(\begin{smallmatrix} \overline{\epsilon} & * \\ 0 & \overline{\delta} \end{smallmatrix}\right)$ with $\overline{\delta}$ unramified.

Consider the following conditions:

(s) There exists $\sigma \in D_p$ such that $\overline{\rho}(\sigma)$ has two eigenvalues $\alpha, \beta \in \mathbb{F}_p$ with $\alpha^2 \neq \beta^2$;
(v) $\rho_{\mathbb{I}}(D_p)$ contains a unipotent element $g \in GL_2(\mathbb{I})$ with $g \not\equiv 1 \mod \mathfrak{m}_{\mathbb{I}}$.

We consider an involution $*$ of $\mathbb{F}_p[[T]]$ over $\mathbb{F}_p$ such that $\phi(T)^* = \phi((1+T)^{-1} - 1)$. Write $\mathbb{F}_p[[T]]^*$ for the subring fixed by $*$; so, $\mathbb{F}_p[[T]]$ is a quadratic extension of $\mathbb{F}_p[[T]]^*$. Similarly we write $\mathbb{F}((T))^*$ for the subfield of $\mathbb{F}_p((T))$ fixed by $*$. Put $\Phi(N) = N^2 \prod_{l|N}(1 - \frac{1}{l^2})$ for an integer $N > 1$ and its prime factors l.

Theorem. *Take a non CM cuspidal component $\mathbb{I}$ of cube-free prime-to-p level N, and let $P \in \mathrm{Spec}(\mathbb{I})$ be a prime divisor above $(p) \subset \mathbb{Z}_p[[T]]$. Suppose $p \nmid \Phi(N)$. If either* (s) *or* (v) *is satisfied, then the image of ρ_P contains an open subgroup of $SL_2(\mathbb{F}_p[[T]])$. If neither* (s) *nor* (v) *holds, there exists an inner form $S_{/\mathbb{F}_p[[T]]^*}$ of $SL(2)_{/\mathbb{F}_p[[T]]^*}$ such that ρ_P contains an open subgroup of $S(\mathbb{F}_p((T))^*)$.*

We will prove a slightly stronger result as Theorem 6.2. Here is an outline of the proof. For a prime divisor P above $(p) \subset \mathbb{Z}_p[[T]]$, let $\overline{\kappa}(P)$ be an algebraic closure of $\kappa(P)$. The Zariski closure of the image $\mathrm{Im}(\rho_P) \cap SL(2)$ in $SL(2)_{/\overline{\kappa}(P)}$ is an algebraic subgroup G_P of $SL(2)_{/\overline{\kappa}(P)}$ defined over $\kappa(P)$. Let G_P° be the connected component of G_P. Then G_P° is either a Borel subgroup, a torus or a unipotent group. If G_P° is in a Borel subgroup or is a torus, we prove that P has to be either an Eisenstein ideal or the family has congruence modulo P with a CM component $\mathbb{I}'$ having CM by an imaginary quadratic field M. In the Eisenstein case, by a result of Mazur–Wiles [MW] and Ohta [O3] combined with the mod p admissible representation theory of Vigneras [V] (see Corollary 2.2), P divides the Iwasawa power series of a Kubota–Leopoldt p-adic L-function (under $p \nmid \varphi(N) = N\prod_{l|N}(1-l^{-1})$). This is impossible as the Kubota–Leopoldt p-adic L-function has trivial μ-invariant [FeW]. In the CM case, P divides $L_p(Ad(\rho_{\mathbb{I}'})) = h \cdot L_p(\Psi^-_{\mathbb{I}'})$ (congruence criterion) for the class number h of M (see Proposition 5.6), where $L_p(\Psi^-_{\mathbb{I}'})$ is the anticyclotomic p-adic Hecke L-function constructed by de Shalit, Yager and Katz (see [K], [HT] and [H07]). By [Fi] and [H10], the anticyclotomic p-adic Hecke L-function has trivial μ-invariant (under $p \nmid \Phi(N)$); so, if $p \nmid h$, this proves the theorem. If $p|h$, by computation of the congruence power series, we prove that the congruence between CM components exhausts the p-part of the congruence power series (Theorem 5.1), and thereby, we conclude that G_P is $SL(2)$, and (Gal) implies, by a result of Pink [P], that ρ_P to have the open image property.

This type of results, asserting that $\mathrm{Im}(\rho_P)$ contains an open subgroup of $SL_2(\mathbb{Z}_p)$ for arithmetic points P of a non CM family was proven in an Inventiones paper by Ribet [Ri] and [Ri1] long ago. Thus fullness of the Galois image of special prime divisors (arithmetic or residual characteristic p) characterizes non-CM components (the importance of such characterization was brought up in the author's first lecture in the Galois trimester 2010 at Poincaré Institute in Paris, which is summarized in the last section of [H11b]). In a forthcoming paper [H13], for non CM components $\mathbb{I}$, we come close to the determination of $\mathrm{Im}(\rho_{\mathbb{I}}) \cap SL_2(\Lambda)$.

Conjecture. *Let $\mathcal{F}_{\mathbb{I}}$ be a non CM cuspidal parallel weight Hilbert modular family (in* [H88b]*) of prime-to-p level $\mathfrak{N}$ for a totally real field F. Suppose $p \geq 5$, and let P be a prime divisor of $\mathbb{I}$ over $(p) \subset \mathbb{Z}_p[[T]]$. Then we have*

(1) *The mod P Galois representation ρ_P is irreducible over* $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.
(2) *Suppose $p \nmid \Phi_F(\mathfrak{N}) = N(\mathfrak{N})^2\prod_{\mathfrak{l}|\mathfrak{N}}(1-\frac{1}{N(\mathfrak{l})^2})$ and that $\mathfrak{N}$ is prime to p and cube free. If either $\dim_F F[\mu_p] > 2$ or the strict class number of F is odd, the image of ρ_P contains a subgroup isomorphic to an open subgroup of $SL_2(\mathbb{F}_p[[T]])$, where* $\det(\rho_{\mathbb{I}}([\gamma_F^s, \mathbb{Q}_p])) = (1+T)^s$ *for a generator γ_F of $\gamma^{\mathbb{Z}_p} \cap N_{F/\mathbb{Q}}(O_p^\times)$.*

If $\dim_F F[\mu_p] = 2$ and F has a CM quadratic extension unramified everywhere, the μ-invariant of the anticyclotomic p-adic Hecke L-function could be positive [H10] (M1–3); so, irreducibility would be at most we could expect under such circumstance. The above conjecture is almost equivalent to vanishing of the μ-invariant of the Deligne–Ribet p-adic L–function and of the (multi-variable) Katz p-adic L projected to the anticyclotomic parallel (single) weight variable.

Hereafter, to make our work simple, we take W sufficiently large so that each irreducible component of $\mathbf{H}$ (for a fixed N and χ_1) is geometrically irreducible over the quotient field of W; i.e., inside an algebraic closure of the quotient field of Λ, $\mathbb{I} \cap \overline{\mathbb{Q}}_p = W$ for all $\mathbb{I}$.

Contents

1. Reduced Hecke algebras

We want to prove that $\mathbf{h}$ is reduced with no nontrivial nilradical if N is cube-free. Let $K = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_l)\}$. We start with

Proposition 1.1. *Let π be a holomorphic automorphic representation of $GL_2(\mathbb{A})$ of weight $k+1$ with central character χ. Write $C(\pi)$ for the conductor of π in the sense of* [AAG] Theorem 4.24. *Fix a prime l, and write π_l for its l-component. For a new vector $f \in \pi$, write $f|T(l) = a \cdot f$, and* *define α, β to be the two roots of $X^2 - aX + \chi(l)l^k = 0$ if π_l is spherical unramified at l. Then the following is the list of all Hecke eigenvectors in π fixed by K whose eigenvalues for $T(q)$ with $q \neq l$ coincide with those for f:*

1. *If $l \nmid C(\pi)$, in addition to f, we have f_α, f_β, f_0 such that $f_x|U(l) = x \cdot f_x$ (here $f_\alpha = f_\beta$ if $\alpha = \beta$), where the minimal level of f_α, f_β, f_0 are, respectively, $C(\pi)l, C(\pi)l$ and $C(\pi)l^2$;*
2. *If $l|C(\pi)$ with $f|U(l) = a \cdot f$ for $a \neq 0$, we have $f_a = f, f_0$ under the same convention as above, where the minimal level of f_a, f_0 are, respectively, $C(\pi)$ and $C(\pi)l$;*
3. *If $l^2|C(\pi)$ with $f|U(l) = 0$, $f = f_0$.*

The above vector f_x is determined by x up to constant multiple.

This follows from the theory of new/old vectors in π (of Casselman), and one can find a proof in [H89] Corollary 2.2, for example (see also the proof of the following corollary how the theory is used to show this type of results). Put $K_r = \{g \in GL_2(\mathbb{Z}_l) | g \equiv u \mod l^r \text{ for } u \in K\}$. Consider the $\mathbb{C}$-subalgebra H of $\mathrm{End}(H^0(K_{r+\nu}, \pi_l))$ generated by $U(l)$ for π_l with l-conductor l^ν. If $r \geq 3$, H has a component H_0 (acting non-trivially on f_0) in which $U(l)$ is nilpotent. Since the set of eigenvalues of $U(l)$ is $\{\alpha, \beta, 0\}$, we can decompose $H = H_\alpha \oplus H_\beta \oplus H_0$ as a ring direct sum (as long as $\alpha \neq \beta$) so that $H_x = H/(U(l) - x)^M H$ for sufficiently large M ($x \in \{\alpha, \beta, 0\}$). Then, we have

Corollary 1.2. *Let the notation be as above, and write the l part of $C(\pi)$ as l^ν. Then we have the following description:*

1. *If $\nu = 0$ and $\alpha \neq \beta$, then $H_\alpha = \mathbb{C}$, $H_\beta = \mathbb{C}$ and $H_0 \cong \mathbb{C}[U]/(U^{r-2})$ if $r \geq 2$;*
2. *If $\nu = 0$ and $\alpha = \beta$, then $H_\alpha = \mathbb{C}$ and $H_0 \cong \mathbb{C}[U]/(U^{r-1})$ if $r \geq 1$;*
3. *If $\nu > 0$ with $f|U(l) = a \cdot f$ for $a \neq 0$, then $H_a = \mathbb{C}$ and $H_0 \cong \mathbb{C}[U]/(U^{r-1})$ if $r \geq 1$;*
4. *If $\nu > 0$ with $f|U(l) = 0$, then $H_0 \cong \mathbb{C}[U]/(U^r)$ for $r \geq 0$,*

where U is an indeterminate and $U(l)$ is sent to U by the above isomorphism. If N is cube free, the Hecke algebra $H_2(\Gamma_0(N), \chi; \mathbb{C})$ is semi-simple. Similarly, if N is cube free, $\mathbf{h}_{2,\chi}$ of level Np^j is reduced (i.e., $\mathbf{h}_{2,\chi} \otimes_{\mathbb{Z}} \mathbb{Q}$ is semi-simple) for any character χ of $(\mathbb{Z}/Np^j\mathbb{Z})^\times$ for any $j \geq 1$.

Proof. We may assume that $r \geq 3$ if $\nu = 0$ and $r > 0$ in the other cases (as the left-over cases follow just from the proposition). Writing the action $f(z) \mapsto f(lz)$ as $[l]$. Then $U(l) \circ [l] = \mathrm{id}$, and hence $(f_0|[l]^i)|U(l) = f_0|[l]^{i-1}$. By the theory of new/old forms, $H^0(K_{r+\nu}, \pi_l)$ has a basis given by $\{f, f|[l], \ldots, f|[l]^r\}$. This shows that $H^0(K_{r+\nu}, \pi_l)$ is generated by $f_x|[l]^i$ for x running over the possible eigenvalues above. Note that $f_\alpha = f - \beta f|[l]$, $f_\beta = f - \alpha f|[l]$ and $f_0 = f_\alpha - \alpha f_\alpha|[l]$ in the spherical case, and $f_0 = f - af|[l]$ in the case where $\nu > 0$ with non-zero eigenvalue a for $U(l)$. Then we can decompose $H^0(K_{r+\nu}, \pi_l) = \bigoplus_x V_x$ for the generalized eigenspaces V_x with eigenvalue x for $U(l)$. Then V_0 is generated by $f_0|[l]^i$ for $0 \leq i \leq r - m$ with the number m of non-zero eigenvalues x for $U(l)$, and $V_x = \mathbb{C}f_x$ for $x \neq 0$. On V_0, $U(l)$ is nilpotent and $f_0|[l]^i|U(l) = f_0|[l]^{i-1}$. This shows the first four assertions. Assume $k = 1$. By Coleman–Edixhoven [CE], the case where $\alpha = \beta$ and $\nu = 0$ never occurs if N is cube-free (and $k = 1$); so, $U(l)$ for $l|N$ is not nilpotent if N is cube-free by the first four assertions. Since the ordinary part is the image of the p-adic idempotent $e = \lim_{n\to\infty} U(p)^{n!}$, the projector e obviously kills the nilpotence coming from $U(p)$ for any level Np^j. Then the last assertion follows from this. □

Corollary 1.3. *If N is cube-free, the Hecke algebras $\mathbf{h}$ and $\mathbf{H}$ are reduced.*

This is one of the principal reasons why we assumed N to be cube-free.

Proof. Since the result for $\mathbf{H}$ follows from the obvious reducedness of the Eisenstein components (see the following section) and the reducedness of $\mathbf{h}$, we give a proof only for $\mathbf{h}$. If N is cube-free, by Corollary 1.2, the ordinary part $\mathbf{h}_{2,\varepsilon\chi_2}$ must be reduced. Since prime divisors

$$P_\varepsilon = (1 + T - \chi(\gamma)\varepsilon(\gamma)\gamma) \cap \Lambda$$

indexed by characters $\varepsilon : \mathbb{Z}_p^\times \to \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ form a Zariski dense subset of $\mathrm{Spec}(\Lambda)$, the diagonal map $\mathbf{h} \to \prod_\varepsilon \mathbf{h}_{2,\varepsilon\chi_2}$ given by $h \mapsto (h \mod P_\varepsilon)_\varepsilon$ is an embedding; so, $\mathbf{h}$ is reduced. □

2. Eisenstein components

Let $\mathcal{M}(N,\chi_1;\Lambda)$ (resp. $\mathcal{S}(N,\chi_1;\Lambda)$) be the space of p-ordinary Λ-adic modular forms (resp. p-ordinary Λ-adic cusp forms). Thus $\mathcal{M}(N,\chi_1;\Lambda)$ (resp. $\mathcal{S}(N,\chi_1;\Lambda)$) is the collection of all formal q-expansion $F(q)=\sum_{n=0}^{\infty}a(n,F)(T)q^n\in\Lambda[[q]]$ such that $f_P=\sum_{n=0}^{\infty}(a(n,F) \mod P)q^n$ gives rise to a modular form in $\mathcal{M}^{ord}_{k+1}(\Gamma_0(Np^{r(P)+1}),\varepsilon_P\chi_{k(P)})$ (resp. $\mathcal{S}^{ord}_{k+1}(\Gamma_0(Np^{r(P)+1}),\varepsilon_P\chi_{k(P)})$) for all arithmetic points P, where $p^{r(P)}$ is the order of ε_P. Again $F\mapsto f_P$ induces an isomorphism

$$\mathcal{M}(N,\chi_1;\Lambda)\otimes_\Lambda\Lambda/P\cong\mathcal{M}^{ord}_{k+1}(\Gamma_0(Np^{r(P)+1}),\chi_k\varepsilon_P;W[\varepsilon_P])$$

for all arithmetic points (see [GME] Theorem 3.2.15 and Corollary 3.2.18 or [LFE] §7.3). The Λ-module $\mathcal{M}(N,\chi_1;\Lambda)$ (resp. $\mathcal{S}(N,\chi_1;\Lambda)$) is naturally a faithful module over $\mathbf{H}$ (resp. $\mathbf{h}$), and the above specialization map is linear with respect to the Hecke operator action.

Let Q be the quotient field of Λ, and take an algebraic closure $\overline{Q}$ of Q. We extend scalars to a finite extension $\mathbb{I}/\Lambda$ inside $\overline{Q}$ to define $\mathcal{S}(N,\chi_1;\mathbb{I})=\mathcal{S}(N,\chi_1;\Lambda)\otimes_\Lambda\mathbb{I}$ and $\mathcal{M}(N,\chi_1;\mathbb{I})=\mathcal{M}(N,\chi_1;\Lambda)\otimes_\Lambda\mathbb{I}$. Associating the family $\{f_P\}_{P\in\mathrm{Spec}(\mathbb{I})}$ to a form $F\in\mathcal{M}(N,\chi_1;\mathbb{I})$, we may regard these as spaces of "analytic families of slope 0 of modular forms" with coefficients in $\mathbb{I}$ (we also call then the space of $\mathbb{I}$-adic p-ordinary cusp forms and the space of $\mathbb{I}$-adic p-ordinary modular forms, respectively). See [LFE] Chapter 7, [GME] Chapter 3 and [H86a] for these facts.

Put $\mathcal{S}(N,\chi_1;Q)=\mathcal{S}(N,\chi_1;\Lambda)\otimes_\Lambda Q$ and $\mathcal{M}(N,\chi_1;Q)=\mathcal{M}(N,\chi_1;\Lambda)\otimes_\Lambda Q$. Then there is a canonical decomposition

$$\mathcal{M}(N,\chi_1;Q)=\mathcal{S}(N,\chi_1;Q)\oplus\mathcal{E}(N,\chi_1;Q) \tag{2.1}$$

as modules over $\mathbf{H}$. The space $\mathcal{E}(N,\chi_1;Q)$ is spanned by Λ-adic Eisenstein series. Assuming that N is cube free, we make explicit the Eisenstein series (which are eigenforms): For any character $\psi:(\mathbb{Z}/M_1\mathbb{Z})^\times\to W^\times,\theta:(\mathbb{Z}/M_2\mathbb{Z})^\times\to W^\times$ with $\psi\theta=\chi_1$, $M_1M_2|Np$, $p|M_2$ and $p\nmid M_1$, there exists a unique Λ-adic Eisenstein series $E(\theta,\psi)$ in $\mathcal{M}(N,\chi_1;Q)$ defined by its q-expansion:

$$E(\theta,\psi):=a(\theta,\psi)(T)+\sum_{n=1}^{\infty}\left(\sum_{0<d|n}\theta(d)\psi(\frac{n}{d})\langle d\rangle(T)\right)q^n,$$

where $\langle d\rangle(T)=(1+T)^{\log_p(d)/\log_p(\gamma)}$, $a(\theta,\psi)=0$ if ψ is non-trivial, and otherwise, writing $\mathbf{1}_M$ for the trivial character modulo M, $a(\theta,\mathbf{1}_{M_1})=\frac{1}{2}G(T)\in Q$ with

$$G(\gamma^k-1)=(1-\theta_{k+1}(p)p^k)L^{(M_1)}(-k,\theta_{k+1})\quad\text{for all } 0\le k\in\mathbb{Z}.$$

Here we agree to put $\theta(d)=0$ if d has a nontrivial common factor with M_2 and that $\psi(d)=0$ if d has a nontrivial common factor with M_1, and also θ_k is the character of $(\mathbb{Z}/M_2p\mathbb{Z})^\times$ given by $\theta_k=\theta\omega^{1-k}$. The existence of the above Eisenstein series is proven under $M_1M_2|Np$ (cf. [H86a] Theorem 7.2, see also [O3]). By counting the number of pairs (θ,ψ), we can prove that they span the Hecke stable complementary space $\mathcal{E}$ of $\mathcal{S}$ if N is cube free (e.g., [H86b] §5 and [O3] §1.4). This proves reducedness of the Hecke algebra of $\mathcal{E}(N,\chi_1;Q)$ (and diagonalizability of Hecke operators on $\mathcal{E}(N,\chi_1;Q)$) if N is cube-free, and hence we have finished the proof of Corollary 1.3.

We want to prove the splitting of the Eisenstein part and the cuspidal part as in (2.1) over the localization $\Lambda_{(p)}$ in place of Q via modulo p representation theory of $GL_2(\mathbb{Q}_q)$ by Vigneras. We start with preparing some notation. Let $C_r=C_r(N)$ be the set of all cusps of $X_r:=X_1(Np^{r+1})(\mathbb{C})$, and consider the formal linear span $W[C_r]$ of C_r over W. Write simply $\Gamma_r:=\Gamma_1(Np^{r+1})$. Since the Hecke correspondence $T_{r,s}(\alpha)$ associated to the double coset $\Gamma_s\alpha\Gamma_r$ for $\alpha\in GL_2(\mathbb{Q})$ (with $\det(\alpha)>0$) sends C_r into C_s for $r,s\ge 0$, the Hecke correspondences act on $W[C_r]$. In particular, $W[C_r]$ is equipped with the action of $T(l)$ and $U(q)$ $(q|Np)$ and $\langle z\rangle=[\Gamma_r\sigma_z\Gamma_r]$ $(z\in(\mathbb{Z}/Np^r\mathbb{Z})^\times)$ for $\sigma_z\in SL_2(\mathbb{Z})$ with $\sigma_z\equiv\left(\begin{smallmatrix}*&0\\0&z\end{smallmatrix}\right) \mod Np^r$. The coset $[q]=[\Gamma(Np^r)\left(\begin{smallmatrix}q&0\\0&1\end{smallmatrix}\right)\Gamma_1(Np^r/q)]$ for a prime $q|N$ gives rise to a linear map $W[C_r(N/q)]\to W[C_r(N)]$.

In [O3] §2.1, these operators are computed explicitly, choosing a standard representative set $A_{Np^r}/\sim$ (see below for A_{Np^r}) for the cusps $C_r(N):=\Gamma_1(Np^r)\backslash\mathbf{P}^1(\mathbb{Q})\subset X_r(\mathbb{C})$. In Ohta's article, the action of $T(l)$ $(l\nmid Np)$ in this paper is denoted by $T^*(l)$, and $U(q)$ $(q|Np)$ is denoted by $T^*(q)$. The covering map $X_s\twoheadrightarrow X_r$ for $s>r$ induces a projection $\pi_{s,r}:C_s\twoheadrightarrow C_r$, and we define $W[[C_\infty(N)]]:=\varprojlim_r W[C_r(N)]$. Since Hecke operators $T(l)$, $T(l,l)$ (in [IAT] Chapter 3),

$U(q)$ and $[q]$ are compatible with the projection $\pi_{s,r}$, these operators act on $W[[C_\infty(N)]]$. We let the group $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ act on $W[[C_\infty]]$ by the character $l \mapsto l \cdot T(l,l)$ for primes l diagonally embedded in $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ Then $T(l), T(l,l), U(q) \in \mathrm{End}_\Lambda(W[[C_\infty(N)]])$ and $[q] : W[[C_\infty(N)]] \to W[[C_\infty(N/q)]]$ are Λ-linear maps. The p-adic projector $e = \lim_{n\to\infty} U(p)^{n!}$ is well defined on $W[C_r]$ and hence on $W[[C_\infty]]$. Thus $e \cdot W[[C_\infty]]$ is a Λ-module on which Hecke operators acts Λ-linearly. As proved in [O1] Proposition 4.3.14, $e \cdot W[[C_\infty]]$ is free of finite rank over Λ (and the rank is given explicitly there). Ohta's choice of the action of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ is a one time twist of our action by the p-adic cyclotomic character; so, his definition of $E(\theta,\psi)$ appears different from ours, but our definition is equivalent to that of [O3] under this twist. Supposing that $p \nmid \varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$, we can decompose

$$e \cdot W[[C_\infty]] = \bigoplus_\psi e \cdot W[[C_\infty]][\psi],$$

where $e \cdot W[[C_\infty]][\psi]$ is the ψ-eigenspace of a character $\psi : (\mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^\times) \to W^\times$ regarding $(\mu_{p-1} \times (\mathbb{Z}/N\mathbb{Z})^\times) \subset \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. In [O3] (2.4.6), Ohta obtained a canonical exact sequence of Hecke equivariant maps

$$0 \to \mathcal{S}(N,\chi_1;\Lambda) \to \mathcal{M}(N,\chi_1;\Lambda) \xrightarrow{\mathrm{Res}} e \cdot W[[C_\infty]][\chi_1] \to 0, \tag{2.2}$$

where the last map Res is canonical and called the residue map in [O3]. Thus as Hecke modules, $e \cdot W[[C_\infty]][\chi_1] \otimes_\Lambda Q \cong \mathcal{E}(N,\chi_1;Q)$.

We extend the definition of $e \cdot W[[C_\infty]]$ slightly. Take a prime $q \nmid pN$ and consider the $\mathbb{C}$-points of the elliptic Shimura curve $\mathbf{X}(N;q^j) = GL_2(\mathbb{Q})\backslash(GL_2(\mathbb{A}^{(\infty)}) \times (\mathbb{C} - \mathbb{R}))/\Delta(Nq^j)$ and its connected component $X(N;q^j) = SL_2(\mathbb{Q})\backslash(SL_2(\mathbb{A}^{(\infty)}) \times \mathfrak{H})/\Delta(Nq^j) \cap SL_2(\mathbb{A}^{(\infty)})$ for $\mathfrak{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, where $\Delta(Nq^j) = \widehat{\Gamma}_1(N) \cap \widehat{\Gamma}(q^j)$ for

$$\begin{aligned} \widehat{\Gamma}(q^j) &= \left\{ x \in GL_2(\widehat{\mathbb{Z}}) \middle| x \equiv 1 \mod q^j M_2(\widehat{\mathbb{Z}}) \right\}, \\ \widehat{\Gamma}_1(N) &= \left\{ x \in GL_2(\widehat{\mathbb{Z}}) \middle| x \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \mod N M_2(\widehat{\mathbb{Z}}) \right\}. \end{aligned} \tag{2.3}$$

Note that $\mathbf{X}(N;q^j)$ is isomorphic to a disjoint union of copies of $X(N;q^j)$ indexed by $(\mathbb{Z}/q^j\mathbb{Z})^\times$. We write $\mathbf{C}(N;q^j)$ (resp. $C(N;q^j)$) for the set of cusps of $\mathbf{X}(N;q^j)$ (resp. $X(N;q^j)$). Then we have $C(Np^r;q^j) \cong \{(A_N/\sim \times A_{q^j})\}/\{\pm 1\}$ (cf. [IAT] Lemma 1.42), where

$$A_N = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2 \middle| x(\mathbb{Z}/N\mathbb{Z}) + y(\mathbb{Z}/N\mathbb{Z}) = \mathbb{Z}/N\mathbb{Z} \right\}$$

with $\binom{x}{y} \sim \binom{x'}{y'} \Leftrightarrow y = y'$ and $x \equiv x' \mod y(\mathbb{Z}/N\mathbb{Z})$. If $\{\pm 1\}$ acts freely on A_N, we have

$$C(N;q^j) \cong ((A_N/\sim) \times A_{q^j})/\{\pm 1\} \cong ((A_N/\sim)/\{\pm 1\}) \times A_{q^j}.$$

Replacing the auxiliary level N by Np^r for sufficiently large r (noting $p \geq 5$), we may assume that $\{\pm 1\}$ acts freely on A_{Np^r}. Thus for $r \gg 0$, we have

$$\mathbf{C}(Np^r;q^j) \cong \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} ((A_{Np^r}/\sim) \times A_{q^j})/\{\pm 1\} \cong \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} ((A_{Np^r}/\sim)/\{\pm 1\}) \times A_{q^j}.$$

As before (see [O3] §2.1), $GL_2(\mathbb{Z}_q)$ acts on A_{q^j} by natural left multiplication on column vectors. Then $u \in GL_2(\mathbb{Z}_q)$ acts on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$ via this multiplication but also permuting indices in $(\mathbb{Z}/q^j\mathbb{Z})^\times$ via multiplication by $\det(u)$. The set $\mathbf{C}(Np^r;q^j)$ of cusps inherits the $GL_2(\mathbb{Z}_q)$-action from the curve $\mathbf{X}(Np^r;q^j)$, and this action is compatible with the action (including permutation of the components) on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$. Consider $W[[\mathbf{C}(Np^\infty;q^j)]] = \varprojlim_r W[\mathbf{C}(Np^r;q^j)]$, which is naturally a Λ-module in the same manner as for $W[[C_\infty(N)]]$ through the action of $\mathbb{Z}_p^\times$ on $(A_{Np^r}/\sim)$. Then we define $V_q = \varinjlim_j W[[\mathbf{C}(Np^\infty;q^j)]]$, where we regard $W[[\mathbf{C}(Np^\infty;q^j)]] = W[[C_\infty(N)]][A_{q^j}]$ as a space of $W[[C_\infty]]$-valued functions on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$, and by the pull-back of the projection $\bigsqcup_{(\mathbb{Z}/q^{j+1}\mathbb{Z})^\times} A_{q^{j+1}} \twoheadrightarrow \bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$, we have taken the *inductive* limit. The idempotent e is well defined on V_q, and $e \cdot V_q = \varinjlim_r e \cdot W[[\mathbf{C}(Np^\infty;q^j)]]$. The group $GL_2(\mathbb{Q}_q)$ acts on V_q by the correspondence action, and the $GL_2(\mathbb{Q}_q)$-action induces the action of the maximal open compact subgroup $GL_2(\mathbb{Z}_q)$ already described on $\bigsqcup_{(\mathbb{Z}/q^j\mathbb{Z})^\times} A_{q^j}$. Plainly V_q is a smooth representation of $GL_2(\mathbb{Q}_p)$ with coefficients in Λ. At each finite q-level, $e\cdot W[[\mathbf{C}(Np^\infty;q^j)]]$ is free of finite rank over Λ as proved by Ohta in [O1] §4.3. Though the curve $X_1(Np^r)$ is dealt with in [O1] §4.3, the argument for $\mathbf{X}(Np^r;q^j)$ is the same, or actually for a suitable choice of $g\in GL_2(\mathbb{Q}_q)$ (such that $g^{-1}\widehat{\Gamma}(q^j)_q g\supset\widehat{\Gamma}_1(q^{2j})_q$ for the principal congruence subgroup $\widehat{\Gamma}(q^j)_q\subset GL_2(\mathbb{Z}_q)$), the right multiplication by g induces a covering $X_1(Np^rq^{2j})\xrightarrow{g}\mathbf{X}(Np^r;q^j)^\circ$ for any geometrically connected component $\mathbf{X}(Np^r;q^j)^\circ$; so, Ohta's result actually implies this finiteness. We have by definition $H^0(\widehat{\Gamma}(q^j)_q, e\cdot V_q)=e\cdot W[[\mathbf{C}(Np^\infty;q^j)]]$, which is free of finite rank over Λ. Thus $e\cdot V_q\otimes_\Lambda Q$ (for the quotient field Q of Λ) is a finitely generated admissible smooth representation of $GL_2(\mathbb{Q}_q)$, and $e\cdot V_q$ is a Λ-lattice stable under the $GL_2(\mathbb{Q}_q)$-action.

Recall the Euler function $\varphi(N)=|(\mathbb{Z}/N\mathbb{Z})^\times|$. We would like to prove

Proposition 2.1. *Suppose $p\nmid\varphi(N)$ and that N is cube free. After tensoring the localization $\Lambda_{(p)}$ at the prime $(p)=p\mathbb{Z}_p[[T]]$, Ohta's exact sequence (2.2) is canonically split as a sequence of $\mathbf{H}$-modules.*

Proof. Taking a pair of characters (θ,ψ) defined modulo M_1 and M_2 respectively, $\mathcal{E}(N,\chi_1;Q)$ is a direct sum of Hecke eigenspaces spanned by $E(\theta,\psi)$. It is easy to see that systems of the Hecke eigenvalues of $E(\theta,\psi)$ are distinct modulo the unique prime divisor P of Λ over $(p)\subset\mathbb{Z}_p[[T]]$. Thus $e\cdot W[[C_\infty]][\chi_1]_{(p)}=\bigoplus_{(\theta,\psi)}\Lambda_{(p)}e(\theta,\psi)$ for an eigen basis $e(\theta,\psi)$ with the same eigenvalues as $E(\theta,\psi)$. Writing $C(\theta)$ for the conductor of θ, if $C(\theta)C(\psi)=N$ or Np and $(\theta,\psi)\neq(\omega^{-1},\mathbf{1}_1)$, by [O3] Theorem 2.4.10, up to units in $\Lambda_{(p)}$,

$$\mathrm{Res}(E(\theta,\psi))=A(T;\theta,\psi)e(\theta,\psi),\tag{2.4}$$

where taking $G(T;\xi)\in\Lambda$ so that $G(\gamma^s-1;\xi)=L_p(-s,\xi\omega)$ $(\gamma=1+p)$ for the Kubota–Leopoldt p-adic L-function $L_p(s,\xi)$ with a primitive even Dirichlet character ξ, we have

$$A(T;\theta,\psi)=G(T;\theta\psi^{-1}\omega)\prod_{l|N,l\nmid C(\theta\psi^{-1})}\{\omega(l)l^{-1}(\langle l\rangle(T)-\theta\psi^{-1}\omega(l)l^{-1})\}.$$

In the exceptional case $(\theta,\psi)=(\omega^{-1},\mathbf{1}_1)$ (which is equivalent to the case of $(\omega^{-2},\mathbf{1}_1)$ in Ohta's paper), as is well known, the Eisenstein ideal is trivial.

Note here that

$$\text{if } (\theta,\psi)\neq(\omega^{-1},\mathbf{1}_1),\ p\nmid A(T;\theta,\psi)\text{ in }W[[T]]$$

by the vanishing of μ-invariant of the Kubota–Leopoldt p-adic L-function (see [FeW]). Thus $A(T;\theta,\psi)\in\Lambda_{(p)}^\times$. So, if $N=1$, we can define a Hecke equivariant section over $\Lambda_{(p)}$ by

$$e\cdot W[[C_\infty]][\chi_1]\to\mathcal{M}(N,\chi_1;\Lambda)\ \text{ by }\ e(\theta,\psi)\mapsto E(\theta,\psi)\ \text{ for }(\theta,\psi)\neq(\omega^{-1},\mathbf{1}_1),$$

and $e(\omega^{-1},\mathbf{1}_1)\mapsto((1+T)-\gamma^{-1})E(\omega^{-1},\mathbf{1}_1)$ otherwise.

We proceed by induction on the number of prime factors of N. Suppose we have a section

$$e\cdot W[[C_\infty(N/q)]]_{(p)}\to\mathcal{M}(N/q,\chi_1;\Lambda_{(p)})$$

for $\mathcal{M}(N/q,\chi_1;\Lambda_{(p)})=\mathcal{M}(N/q,\chi_1;\Lambda)\otimes_\Lambda\Lambda_{(p)}$. Take (θ,ψ) with $M_1M_2|N/q$. We claim that the space V spanned by $e(\theta,\psi)|[q]$ and $e(\theta,\psi)$ in $W[[C_\infty(N)]]_{(p)}$ has rank 2 over $\Lambda_{(p)}$, and moreover, it is a direct summand of $e\cdot W[[C_\infty(N)]]_{(p)}$.

To prove this claim, we use the admissible representation $(e\cdot V_q)\otimes_\Lambda Q$ of $GL_2(\mathbb{Q}_q)$ defined for the prime to q-part $N^{(q)}$ of N (in place of N) which is explained just before stating the proposition. Then $e(\theta,\psi)$ generates a principal series representation $\pi_q\subset e\cdot V_q\otimes_\Lambda Q$ isomorphic to $\pi(\theta_q,\widehat{\psi}_q)$ over Q, where $\widehat{\psi}_q:\mathbb{Q}_q^\times\to\Lambda^\times$ is the unramified character sending the prime q to $\psi(q)\langle q\rangle$, and θ_q is just $\theta|_{\mathbb{Q}_q^\times}$ regarding θ as an idele character. Then by Vigneras' modulo p representation theory of admissible representations (see [V] Theorem 3), an old-new congruence at q occurs only when the ratio $(\widehat{\psi}_q/\theta_q)(q)$ is congruent to $q^{\pm1}$ modulo $\mathfrak{m}_{\Lambda_{(p)}}$ (for the maximal ideal $\mathfrak{m}_{\Lambda_{(p)}}=(\varpi)$ of $\Lambda_{(p)}$). Note that $\Lambda_{(p)}$ is a discrete valuation subring of Q of mixed characteristic $(0,p)$. Since

$$(\widehat{\psi}_q/\theta_q)(q)\not\equiv q^{\pm1}\mod\mathfrak{m}_{\Lambda_{(p)}},$$

$\pi(\theta_q, \widehat{\psi}_q) \mod \mathfrak{m}_{\Lambda_{(p)}}$ is irreducible. Thus the two vectors $e(\theta, \psi)$ and $e(\theta, \psi)|[q]$ modulo $\mathfrak{m}_{\Lambda_{(p)}}$ in $\overline{\pi}_q := (\pi_q \mod \mathfrak{m}_{\Lambda_{(p)}})$ are linearly independent. This shows the above claim.

To get the section of level N, first we assume that q is prime to N/q. Letting (θ_1, ψ) be the pair with θ_1 which is θ regarded as a character modulo $M_1 q$, we have $e(\theta_1, \psi) = e(\theta, \psi) - \theta_q(q)e(\theta, \psi)|[q]$ up to units in $\Lambda_{(p)}$ by the argument in the previous section. Similarly $e(\theta, \psi_1) = e(\theta, \psi) - \widehat{\psi}_q(q)e(\theta, \psi)|[q]$ for ψ_1 which is ψ regarded as a character modulo $M_1 q$. Then

$$\mathrm{Res}(E(\theta_1, \psi)) = \mathrm{Res}(E(\theta, \psi) - \theta_q(q)E(\theta, \psi)|[q]) = \mathrm{Res}(E(\theta, \psi)) - \theta_q(q)\mathrm{Res}(E(\theta, \psi))|[q],$$
$$\mathrm{Res}(E(\theta, \psi_1)) = \mathrm{Res}(E(\theta, \psi) - \widehat{\psi}_q(q)E(\theta, \psi)|[q]) = \mathrm{Res}(E(\theta, \psi)) - \widehat{\psi}_q(q)\mathrm{Res}(E(\theta, \psi))|[q].$$

Thus the section of level N/q extends to the level N.

Note that N is cube-free. Thus the remaining case is when $q^2|N$. If $C(\theta)$ and $C(\psi)$ are both prime to q, by the irreducibility of $\overline{\pi}_q := (\pi_q \mod \mathfrak{m}_{\Lambda_{(p)}})$,

$$e(\theta, \psi), e(\theta, \psi)|[q] \text{ and } e(\theta, \psi)|[q]^2$$

span a three-dimensional subspace in $\overline{\pi}_q$. Thus we have $e(\theta_1, \psi_1) = e(\theta, \psi_1) - \theta(q)e(\theta, \psi_1)$ which does not vanish in $\overline{\pi}_q$. Then $e(\theta_1, \psi_1) \mapsto E(\theta_1, \psi_1)$ gives a section on (θ_1, ψ_1)-eigenspace. If $q|C(\theta)$ but $q \nmid C(\psi)$, we define $e(\theta_1, \psi) = e(\theta, \psi_1) - \theta(q)e(\theta, \psi_1)$, and if $q \nmid C(\theta)$ but $q|C(\psi)$, $e(\theta, \psi_1) = e(\theta, \psi_1) - \widehat{\psi}(q)e(\theta, \psi_1)$, and the same argument works well. If $q|C(\theta)$ and $q|C(\psi)$ but one of the characters is imprimitive at another prime q', we apply our argument to q' in place of q, and we get the section. The case where $N|C(\theta)C(\psi)$ is covered by Ohta's result explained at the beginning of the proof. This finishes the proof. □

Let $\mathbf{E}$ be the image of $\mathbf{H}$ in $\mathrm{End}_\Lambda(\mathcal{E}(N, \chi_1; \Lambda))$ and define $\mathcal{C} = \mathbf{h} \otimes_{\mathbf{H}} \mathbf{E}$ (the Eisenstein congruence module; see (5.2) in the text and [MFG] §5.3.3).

Corollary 2.2. *Suppose N is cube-free and $p \nmid \varphi(N)$. Then $\mathcal{C} \otimes_\Lambda \Lambda_{(p)} = 0$. In other words, we have $\mathbf{H}_{(p)} = \mathbf{h}_{(p)} \oplus \mathbf{E}_{(p)}$, and there is no congruence between a Λ-adic Eisenstein series and an $\mathbb{I}$-adic cuspidal Hecke eigenform modulo a prime divisor above $(p) \subset \mathbb{Z}_p[[T]]$.*

Proof. We have a pairing $\mathbf{H} \times \mathcal{M}(N, \chi_1; \Lambda)$ given by $(h, f) = a(1, f|h)$. Define

$$\widetilde{\mathcal{M}}(N, \chi_1; \Lambda) = \{f \in \mathcal{M}(N, \chi_1; Q) | a(n, f) \in \Lambda \text{ for all } n > 0\}.$$

As is well known ([H86a] §2), this pairing $\mathbf{H} \times \widetilde{\mathcal{M}}(N, \chi_1; \Lambda)$ is perfect: $\mathrm{Hom}_\Lambda(\mathbf{H}, \Lambda) \cong \widetilde{\mathcal{M}}(N, \chi_1; \Lambda)$ and $\mathrm{Hom}_\Lambda(\widetilde{\mathcal{M}}(N, \chi_1; \Lambda), \Lambda) \cong \mathbf{H}$ as Λ-modules by $(\sum_n a(n, F)q^n \mapsto a(n, F)) \mapsto T(n)$. However, by definition, $\widetilde{\mathcal{M}}(N, \chi_1; \Lambda)/\mathcal{M}(N, \chi_1; \Lambda) \hookrightarrow Q/\Lambda$ by $f \mapsto a(0, f)$, and the inclusion $f \mapsto a(0, f)$ is Γ-equivariant. The group Γ acts on the constant term by $\gamma \mapsto \gamma^{-1}$ (as by our choice of the action, weight 1 corresponds to the trivial action). This shows that after localizing at (p), the pairing is perfect over $\Lambda_{(p)}$. The Λ-perfectness of the pairing on $\mathbf{h} \times \mathcal{S}(N, \chi_1; \Lambda)$ holds in the same way as in the case of $\mathbf{H}$. Note that $\Lambda_{(p)}$ is a discrete valuation ring. Since $\mathbf{H}_{(p)} \twoheadrightarrow \mathbf{h}_{(p)}$ is the $\Lambda_{(p)}$-dual of $\mathcal{S}(N, \chi_1; \Lambda_{(p)}) \hookrightarrow \mathcal{M}(N, \chi_1; \Lambda_{(p)})$, we get $\mathbf{H}_{(p)} = \mathbf{h}_{(p)} \oplus \mathbf{E}_{(p)}$, which implies the vanishing of $\mathcal{C}_{(p)}$. □

3. CM components

We start with a simple lemma:

Lemma 3.1. *For a given integer N and imaginary quadratic fields M and M', if the set of primes prime to N inert in M is equal to the set of primes prime to N inert in M', we have $M = M'$.*

Proof. Write $M = \mathbb{Q}(\sqrt{-D})$ and $M' = \mathbb{Q}(\sqrt{-d})$ for the two quadratic fields with discriminant D and d. The two characters $\left(\frac{-D}{}\right)$ and $\left(\frac{-d}{}\right)$ have equal values over the set of primes outside NDd; so, they are equal as characters modulo the least common multiple of D and d; so, having the equal conductor $D = d$ as desired. □

Let $\mathrm{Spec}(\mathbb{I})$ be an irreducible component of $\mathrm{Spec}(\mathbf{h})$. Recall that the component $\mathbb{I}$ is called a *CM component* (or $\mathbb{I}$ has *complex multiplication*) if there exists a nontrivial character $\xi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{I}^\times$ such that $\rho_\mathbb{I} \cong \rho_\mathbb{I} \otimes \xi$. If $\mathbb{I}$ is a CM component, we call the corresponding family $\mathcal{F}$ a CM family (or $\mathcal{F}$ has complex multiplication). Suppose that $\mathcal{F}$ is a CM family associated to $\mathbb{I}$ with $\rho_\mathbb{I} \cong \rho_\mathbb{I} \otimes \xi$. Taking the determinant of this defining identity, we find that ξ must have order 2 if $\mathbb{I}$ is a CM component, which cuts out a quadratic field M, i.e., $\xi = \left(\frac{M/\mathbb{Q}}{}\right)$. As we will see below, we have $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi$ for a Galois character Ψ. By ordinarity, p splits in M as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with Ψ unramified at $\overline{\mathfrak{p}}$. Thus Ψ (unramified at $\overline{\mathfrak{p}}$) restricted to the $\mathfrak{p}$-inertia subgroup has infinite order, which by class field theory implies that M can have only finitely many units; i.e., M is imaginary. Let D be the discriminant of M, and consider $M \subset \overline{\mathbb{Q}}$. Thus $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ induces a prime factor $\mathfrak{p}$ of p in M.

Let $\mathfrak{O}$ be the integer ring of M, and put $\widehat{\mathfrak{O}}^{(p)} = \prod_{l \neq p}(\mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{Z}_l)$. For any continuous character $\psi : M^\times \backslash M_{\mathbb{A}}^\times \to R^\times$ unramified outside Np with values in a p-profinite ring R, the restriction $\psi : (\widehat{\mathfrak{O}}^{(p)})^\times \to R^\times$ has to be a finite order character, as ψ is ramified only at finitely many primes and $R^\times$ is an almost p-profinite group. Thus we have an integral ideal $\mathfrak{C}(\psi)$ maximal among ideals $\mathfrak{a}$ prime to p with $(1 + \mathfrak{a}\widehat{\mathfrak{O}}^{(p)}) \cap (\widehat{\mathfrak{O}}^{(p)})^\times \subset \mathrm{Ker}(\psi)$. We call $\mathfrak{C}(\psi)$ the *prime-to-p conductor* of ψ. For a rational prime l, the l-primary part $\mathfrak{C}_l(\psi)$ of $\mathfrak{C}(\psi)$ is called the l-conductor of ψ.

Proposition 3.2. *Let $\widetilde{\mathbb{I}}$ be the integral closure of $\mathbb{I}$ in its quotient field. The following five conditions are equivalent:*

(CM1) *$\mathcal{F}$ is a CM family with $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{}\right)$;*

(CM2) *The prime p splits in M, and we have $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$ for a character $\Psi_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ with prime-to-p conductor $\mathfrak{C} = \mathfrak{C}(\Psi_{\mathbb{I}})$ unramified outside $\mathfrak{C}\mathfrak{p}$. We have $D \cdot N(\mathfrak{C})|N$ for the discriminant D of M;*

(CM3) *For all arithmetic points P of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, f_P is a binary Hecke eigen theta series of the norm form of $M/\mathbb{Q}$ with prime-to-p conductor $N(\mathfrak{C})D$ (as a Hecke eigenform);*

(CM4) *For an arithmetic point P of $\mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, f_P is a binary Hecke eigen theta series of the norm form of $M/\mathbb{Q}$ with prime-to-p conductor $N(\mathfrak{C})D$ (as a Hecke eigenform);*

(CM5) *For an arithmetic prime P, ρ_P is an induced representation of a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$ with prime-to-p conductor $\mathfrak{C}$.*

Here we say that a Hecke eigenform f has conductor $C(f)$ if the automorphic representation generated by f has conductor $C(f)$ (in the sense of Casselman; see [AAG] Theorem 4.24); so, f itself could be an old form. The prime-to-p part C of this conductor $C(f)$ is the Artin prime-to-p conductor of the 2-dimensional p-adic Galois representation of f (see [GME] Theorem 5.1.8). We put $\mathfrak{C}(\mathbb{I}) = \mathfrak{C}(\Psi_{\mathbb{I}})$ and $C = C(\mathbb{I}) = N(\mathfrak{C}(\mathbb{I}))D$ for $\Psi_{\mathbb{I}}$ as in (CM2), where $N(\mathfrak{a})$ is the norm of a fractional ideal $\mathfrak{a}$ of N. We call a binary Hecke eigen theta series of the norm form of an imaginary quadratic field a *CM theta series*. We say that $\mathbb{I}$ has CM by M if the above equivalent conditions are satisfied by a specific imaginary quadratic field M.

Proof. Since this fact is well known, we only give a sketch of the proof. The condition (CM1) is equivalent to (CM2): $\rho_{\mathbb{I}} \cong \mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$ for a character $\Psi_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ unramified outside Np (e.g., [MFG] Lemma 2.15). Since the characteristic polynomial of $\rho_{\mathbb{I}}(\sigma)$ has coefficients in $\mathbb{I}$, its eigenvalues fall in $\widetilde{\mathbb{I}}$; so, the character $\Psi_{\mathbb{I}}$ must have values in $\widetilde{\mathbb{I}}^\times$ (see, [H86c] Corollary 4.2). Since $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$ ramifies at prime factors l of $D \cdot N(\mathfrak{C})$ for the prime-to-p conductor $\mathfrak{C}$ of $\Psi_{\mathbb{I}}$, we have $l|D \cdot N(\mathfrak{C}) \Rightarrow l|N$ for the prime-to-p conductor $\mathfrak{C}$ of $\Psi_{\mathbb{I}}$.

Here is how to see $D \cdot N(\mathfrak{C})|N$. Following E. Artin, we can define the prime-to-p conductor of $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$, which is equal to $D \cdot N(\mathfrak{C})$ (see [GME] Theorem 5.1.8). The conductor only depends on the restriction of $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$ to the inertia group I_l at ramified primes $l \neq p$ of $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$. Since $\mathbb{I}$ is p-profinite, as we have seen by class field theory, $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}|_{I_l}$ has finite image. Since the multiplicative groups $1 + P \subset \widetilde{\mathbb{I}}$ and $1 + P \cdot M_2(\widetilde{\mathbb{I}}) \subset GL_2(\widetilde{\mathbb{I}})$ are torsion-free for a prime divisor P with characteristic 0 residue field $\kappa(P)$, the prime-to-p conductors of ρ_P and $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}}$ are equal. Taking P to be arithmetic, by the solution of the local Langlands conjecture, the prime-to-p conductor of the associated modular form is equal to the prime-to-p conductor of ρ_P (see the paragraph after [GME] Theorem 5.1.9 how to deduce this fact from the local Langlands conjecture); so, $D \cdot N(\mathfrak{C})|N$.

Regard $\mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p) = \mathrm{Hom}_{W\text{-alg}}(\widetilde{\mathbb{I}}, \overline{\mathbb{Q}}_p)$. Then by (Gal), $\Psi_P = P \circ \Psi_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \overline{\mathbb{Q}}_p^\times$ for an arithmetic $P \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ is a locally algebraic p-adic character, which is the p-adic avatar of a

Hecke character $\lambda_P : M^\times_{\mathbb{A}}/M^\times \to \mathbb{C}^\times$ of type A_0 of a quadratic field $M/\mathbb{Q}$. Then, f_P is the theta series with q-expansion $\sum_{\mathfrak{a}} \lambda_P(\mathfrak{a})q^{N(\mathfrak{a})}$, where $\mathfrak{a}$ runs over all integral ideals of M. If p is inert or ramified in M, such theta series is not ordinary (with positive or infinite slope). Thus p must split in M. By $k(P) \geq 2$, M has to be an imaginary quadratic field (as holomorphic binary theta series of real quadratic fields are limited to weight 1; cf., [MFM] §4.8). As already remarked, the prime-to-p conductors of $\Psi_{\mathbb{I}}$ and Ψ_P are equal for any $P \in \mathrm{Spec}(\overline{\mathbb{Q}}_p)$, and similarly by the theory of new/old forms, the prime-to-p conductors of a Λ-adic eigenform and its specialization are equal (see [H86a] Theorem 3.5). This shows (CM1)⇒(CM2)⇒(CM3)⇒(CM4).

If (CM3) is satisfied, we have an identity

$$\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) \equiv a(l) \equiv \left(\frac{M/\mathbb{Q}}{}\right) a(l) \equiv \mathrm{Tr}((\rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{}\right))(Frob_l)) \mod P$$

for all arithmetic points P and all primes l outside Np. Density of arithmetic points in $\mathrm{Spec}(\mathbb{I})$ implies

$$\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) = a(l) = \left(\frac{M/\mathbb{Q}}{}\right) a(l) = \mathrm{Tr}((\rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{}\right))(Frob_l))$$

for all primes l outside Np. By Chebotarev density, we have $\mathrm{Tr}(\rho_{\mathbb{I}}) = \mathrm{Tr}((\rho_{\mathbb{I}} \otimes \left(\frac{M/\mathbb{Q}}{}\right)))$, and we get (CM1) from (CM3) as $\rho_{\mathbb{I}}$ is semi-simple.

If an irreducible component $\mathrm{Spec}(\mathbb{I})$ contains an arithmetic point P with theta series f_P of $M/\mathbb{Q}$ as above, either $\mathbb{I}$ is a CM component or otherwise P is in the intersection in $\mathrm{Spec}(\mathbf{h})$ of a component $\mathrm{Spec}(\mathbb{I})$ not having CM by M and another component having CM by M (as all families with CM by M are made up of theta series of M by the construction of CM components in [H86a] §7). The latter case cannot happen as two distinct components never cross at an arithmetic point in $\mathrm{Spec}(\mathbf{h})$; in other words, the reduced part of the localization $\mathbf{h}_P$ is étale over Λ_P for any arithmetic point $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$ (see [HMI] Proposition 3.78). Thus (CM4) implies (CM3), and the converse is trivial. The equivalence of (CM4) and (CM5) can be proven similarly as in the case of the equivalence of (CM1) and (CM3). □

Proposition 3.3. *Let A be a p-profinite local integral domain with quotient field Q. Let M and L be two distinct imaginary quadratic fields in which p splits, and suppose that we have continuous characters $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to A^\times$ and $\phi : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to A^\times$ with absolutely irreducible $\mathrm{Ind}_M^{\mathbb{Q}} \varphi$ over Q such that $\mathrm{Ind}_M^{\mathbb{Q}} \varphi \cong \mathrm{Ind}_L^{\mathbb{Q}} \phi$. If the representations $\mathrm{Ind}_M^{\mathbb{Q}} \varphi \cong \mathrm{Ind}_L^{\mathbb{Q}} \phi$ are ordinary at p, then φ and ϕ are both unramified at p and of finite order, and the anticyclotomic projection φ^- defined by $\varphi^-(\tau) = \varphi(\tau)\varphi(c\tau c)^{-1}$ for complex conjugation c is non-trivial and factors through $\mathrm{Gal}(ML/M)$.*

Here the word "ordinary" means that the representation restricted to a decomposition group at p is isomorphic to an upper triangular representation with an unramified one dimensional quotient. In our case, the restriction of, say, $\mathrm{Ind}_M^{\mathbb{Q}} \varphi$ to a decomposition group at p is the direct sum $\varphi \oplus \varphi^\sigma$ or $\mathbf{1} \oplus \left(\frac{M/\mathbb{Q}}{}\right)$ for the identity character $\mathbf{1}$, where $\varphi^\sigma(\tau) = \varphi(\sigma\tau\sigma^{-1})$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inducing a nontrivial automorphism on M. This implies that φ is unramified at one prime in M over p.

Proof. For any prime l outside Np inert in L and split in M (such primes have positive density by Lemma 3.1),

$$0 = \mathrm{Tr}(\mathrm{Ind}_L^{\mathbb{Q}} \phi(Frob_l)) = \mathrm{Tr}(\mathrm{Ind}_M^{\mathbb{Q}} \varphi(Frob_l)) = \varphi(\mathfrak{l}) + \varphi(\mathfrak{l}^\sigma)$$

for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inducing a generator of $\mathrm{Gal}(M/\mathbb{Q})$. Thus defining $\varphi^-(\tau) = \varphi(\tau)\varphi(\sigma\tau\sigma^{-1})^{-1}$, we have $\varphi^-(Frob_l) = -1$ if l is inert in L and split in M. In particular, for any other primes q outside Np inert in M and split in L, $\varphi^-(Frob_l) = -1 = \varphi^-(Frob_q)$. Since $Frob_l Frob_q^{-1}$ fixes ML, by moving q, Chebotarev density tells us that φ^- factors through $\mathrm{Gal}(ML/M)$. Irreducibility of $\mathrm{Ind}_M^{\mathbb{Q}} \varphi$ tells us $\varphi^- \neq \mathbf{1}$ (Mackey's theorem; e.g., [LRF] §7.4). Thus $\varphi = \left(\frac{ML/M}{}\right) \varphi^\sigma$ for $\varphi^\sigma(\tau) = \varphi(\sigma\tau\sigma^{-1})$. By the remark preceding this proof, we may assume that φ is unramified at one prime factor $\mathfrak{p}^\sigma$ of p, and this identity shows that φ is unramified also at $\mathfrak{p}$, factoring through the finite ray class group $Cl_M(\mathfrak{c})$ of M modulo $\mathfrak{c}$ for the prime-to-p conductor $\mathfrak{c}$ of φ. In the similar manner, we conclude that ϕ is unramified at p of finite order. □

If N is cube-free, $\mathbf{h}$ is reduced (Corollary 1.3). Let $\mathrm{Spec}(\mathbf{h}^M_{cm})$ be the minimal closed subscheme of $\mathrm{Spec}(\mathbf{h})$ containing all irreducible components having CM by a fixed imaginary quadratic field M. Without assuming reducedness of $\mathbf{h}$, by minimality, $\mathrm{Spec}(\mathbf{h}^M_{cm})$ is a reduced scheme.

Corollary 3.4. *Let M and L be distinct imaginary quadratic fields in which p splits. If $P \in \mathrm{Spec}(\mathbf{h}^M_{cm}) \cap \mathrm{Spec}(\mathbf{h}^L_{cm})$ is a prime divisor, $P \cap \mathbb{Z}_p[[T]]$ contains T.*

Proof. For a reduced ring A, write $Q(A)$ for the total quotient ring. We have a Galois representation $\rho_? : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(Q(\mathbf{h}^?_{cm}))$ such that $\rho_?(Frob_l)$ is given by the image of $T(l)$ in $\mathbf{h}^?_{cm}$ for all primes l outside Np (with $? = L, M$). Then we find a finite extension $\widetilde{Q}(\mathbf{h}^?_{cm})$ of $Q(\mathbf{h}^?_{cm})$ and two continuous characters $\Psi_? : \mathrm{Gal}(\overline{\mathbb{Q}}/?) \to \widetilde{Q}(\mathbf{h}^?_{cm})^\times)$ such that $\rho_? \cong \mathrm{Ind}^{\mathbb{Q}}_? \Psi_?$ over $\widetilde{Q}(\mathbf{h}^?_{cm})$. By compactness of the Galois group, the subring $\widetilde{\mathbf{h}}^?_{cm}$ of $\widetilde{Q}(\mathbf{h}^?_{cm})$ generated by the values of $\Psi_?$ is finite over $\mathbf{h}^?_{cm}$. Thus we may replace P and $\mathbf{h}^?_{cm}$ by prime $P_?$ of $\widetilde{\mathbf{h}}^?_{cm}$ over P such that $\rho_M \mod P_M \cong \rho_L \mod P_L$ over a finite integral domain A over $\mathbf{h}^M_{cm}/P = \mathbf{h}^L_{cm}/P$. Apply the above proposition to this A and $\varphi = \Psi_M \mod P_M$ and $\phi = \Psi_L \mod P_L$. Then φ is unramified at p. By (CM2), p splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in M, and we have $\mathfrak{O}_{\mathfrak{p}} = \mathbb{Z}_p$. Since $M_{\mathfrak{p}} \cong \mathbb{Q}_p$, we conclude $\Psi_M([u, M_{\mathfrak{p}}]) = (1+T)^{\log_p(u)/\log_p(\gamma)} = 1$ for $u \in \mathbb{Z}_p^\times = \mathfrak{O}_{\mathfrak{p}}^\times$ by (Gal), where $[u, M_{\mathfrak{p}}]$ is the local Artin symbol at $\mathfrak{p}$. Therefore we find $T \in P_M$ and similarly $T \in P_L$. □

Let $\mathrm{Spec}(\mathbb{I})$ be a CM irreducible component in $\mathrm{Spec}(\mathbf{h}^M_{cm})$ and $\mathbb{T}$ be a local ring of $\mathbf{h}$ with $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$; i.e., $\mathrm{Spec}(\mathbb{T})$ is the connected component of $\mathrm{Spec}(\mathbf{h})$ containing $\mathrm{Spec}(\mathbb{I})$. Consider the following condition for any CM component $\mathbb{I}'$ of $\mathbb{T}$:

(H) If $\Psi_{\mathbb{I}'}$ ramifies at an inert or a ramified prime $\mathfrak{l}$ outside p of M, $\Psi_{\mathbb{I}'} \mod \mathfrak{m}_{\mathbb{I}'}$ ramifies at $\mathfrak{l}$.

Since $D \cdot N_{M/\mathbb{Q}}(\mathfrak{C}(\mathbb{I}'))|N$ for the discriminant D of M, (H) holds if N is square-free. By the following lemma, it also holds if $p \nmid \Phi(N)$ (for the function $\Phi(N) = N^2 \prod_{l|N}(1 - l^{-2})$).

Lemma 3.5. *The condition* (H) *holds if $p \nmid N(\mathfrak{l}) - 1$ for the norm $N(\mathfrak{l}) = |\mathfrak{O}/\mathfrak{l}|$.*

Proof. The image of the inertia group at $\mathfrak{l}$ in the Galois group of the maximal abelian extension of M over M is isomorphic to $\mathfrak{O}_{\mathfrak{l}}^\times$ by class field theory. Thus a character $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to R^\times$ (for a p-profinite local ring R) ramifies at $\mathfrak{l}$ if and only if its reduction modulo the maximal ideal ramifies at $\mathfrak{l}$ as long as $|(\mathfrak{O}/\mathfrak{l})^\times| = N(\mathfrak{l}) - 1$ is prime to p. □

Recall the Euler function $\varphi(N) = N \prod_{l|N}(1 - \frac{1}{l}) = |(\mathbb{Z}/N\mathbb{Z})^\times|$.

Corollary 3.6. *Let $\mathrm{Spec}(\mathbb{T})$ be a connected component of $\mathrm{Spec}(\mathbf{h})$, and let P be a prime divisor of $\mathbb{T}$ above $(p) \subset \mathbb{Z}_p[[T]]$. If either* (H) *with $p \nmid \varphi(N)$ holds or $p \nmid \Phi(N)$, for any CM component $\mathbb{I}$ of $\mathbb{T}$ with $\mathrm{Spec}(\mathbb{I}_P) \subset \mathrm{Spec}(\mathbb{T}_P)$, the prime-to-$p$ conductor $\mathfrak{C}(\mathbb{I})$ of $\Psi_{\mathbb{I}}$ as in* (CM2) *is determined by $\mathbb{T}_P$. In other words, any CM component $\mathbb{I}$ of $\mathbb{T}$ with $\mathrm{Spec}(\mathbb{I}_P) \subset \mathrm{Spec}(\mathbb{T}_P)$ has CM by a unique imaginary quadratic field M and has prime-to-p conductor depending only on $\mathbb{T}_P$. In particular, the prime-to-p conductor of ρ_P is equal to the prime-to-p conductor of $\rho_{\mathbb{I}} = \mathrm{Ind}^{\mathbb{Q}}_M \Psi_{\mathbb{I}}$.*

Proof. By Corollary 3.4, the CM components of $\mathrm{Spec}(\mathbb{T}_P)$ can have complex multiplication by a unique imaginary quadratic field M. Then if two components $\mathbb{I}_P$ and $\mathbb{I}'_P$ of $\mathrm{Spec}(\mathbb{T}_P)$ with CM by M have different prime-to-p conductor $\mathfrak{C}(\mathbb{I}_P)$ and $\mathfrak{C}(\mathbb{I}'_P)$, for a prime $\mathfrak{l}$ appearing in the reduced expression of $\mathfrak{C}(\mathbb{I})/\mathfrak{C}(\mathbb{I}')$, $\Psi_{\mathbb{I}} \equiv \Psi_{\mathbb{I}'} \mod P$ implies $\Psi_{\mathbb{I}}/\Psi_{\mathbb{I}'} \mod P$ is unramified at $\mathfrak{l}$ but $\Psi_{\mathbb{I}}/\Psi_{\mathbb{I}'}$ ramifies at $\mathfrak{l}$. Thus, by the proof of the above lemma, p must be a factor of the order of $(\mathfrak{O}/\mathfrak{l})^\times$. As $N(\mathfrak{C}(\mathbb{I}))D$ and $N(\mathfrak{C}(\mathbb{I}'))D$ divide N, if either (H) with $p \nmid \varphi(N)$ holds or $p \nmid \Phi(N)$, we must have $\mathfrak{C}(\mathbb{I}) = \mathfrak{C}(\mathbb{I}')$. This proof shows that the prime-to-p-conductor $\mathfrak{C}(\mathbb{I})$ is equal to the prime-to-p conductor of $\Psi_{\mathbb{I}} \mod P$. Since the prime-to-p conductor of $\rho_{\mathbb{I}}$ is given by $N(\mathfrak{C}(\mathbb{I}))D$ (see [GME] Theorem 5.1.8), this proves the last assertion. □

For a fractional ideal $\mathfrak{a}$ of M, taking a generator α of $\mathfrak{a}^h$ for sufficiently large $0 < h \in \mathbb{Z}$, we define $\log_p(\mathfrak{a}) = \frac{1}{h}\log_p(\alpha)$ for the Iwasawa p-adic logarithm $\log_p$. The value $\log_p(\mathfrak{a})$ is determined independently of the choice of α. Now p-adically interpolating the eigenvalues of members of a CM family over its CM component, as was done in [H86a] §7, [LFE] §7.6 (2a) and [H86c] §4, we get an $\mathbb{I}$-adic Hecke eigenform which interpolates CM theta series new at primes outside p. We call an

$\mathbb{I}$-adic Hecke eigenform F new of prime-to-p conductor $C(F)$ if it interpolates Hecke eigenforms of prime-to-p conductor $C(F)$. Then incorporating also CM old forms as in Proposition 1.1, we get

Proposition 3.7. *Let* $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h}^M_{cm})$ *be a CM component associated to the character* $\Psi = \Psi_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ *as in* (CM2) *unramified outside* $\mathfrak{C}\mathfrak{p}$ *for the prime-to-p conductor* $\mathfrak{C} = \mathfrak{C}(\Psi)$ *of* Ψ*. Let* C_l *be the l-primary part of* $C = N(\mathfrak{C})D$ *for the discriminant* D *of* M*. Regard* Ψ *as a character of the fractional ideal group of* M *made of fractional ideals prime to* $\mathfrak{C}\mathfrak{p}$ *and write* $\Psi(\mathfrak{a}) = \Psi([\mathfrak{a}, M])$ *for the value at a fractional ideal* $\mathfrak{a}$ *prime to* $\mathfrak{C}\mathfrak{p}$*, and put* $\Psi(\mathfrak{a}) = 0$ *if* $\mathfrak{a}$ *is a fractional ideal not prime to* $\mathfrak{C}\mathfrak{p}$*. Here* $[\mathfrak{a}, M]$ *is the global Artin symbol. Write c for complex conjugation. Let* $N_l(\mathbb{I})$ *be the l-part of the minimal prime-to-p level* $N(\mathbb{I})$ *of* $\mathbb{I}$ *(i.e.,* $\mathbf{h} \to \mathbb{I}$ *factors through the big Hecke algebra of prime-to-p level* $N(\mathbb{I})$*). Then* $C|N(\mathbb{I})$*, and for primes l outside N, we have*

(O1) $a(l) = \Psi(\mathfrak{l}) + \Psi(\mathfrak{l}^c)$ *if* $(l) = \mathfrak{l}\mathfrak{l}^c$ *with* $\mathfrak{l} \neq \mathfrak{l}^c$*, where*

$$\Psi(\mathfrak{l}) = \zeta(1+T)^{\log_p(\mathfrak{l})/\log_p(\gamma)} \text{ and } \Psi(\mathfrak{l}^c) = \zeta'(1+T)^{\log_p(\mathfrak{l}^c)/\log_p(\gamma)}$$

for roots of unity ζ, ζ'*. In particular,* $\Psi(\mathfrak{l}) \neq \Psi(\mathfrak{l}^c)$*.*

(O2) $a(l) = 0$ *if* (l) *is inert in* $M/\mathbb{Q}$*.*

We have the following possibility of $a(l)$ *for a prime* $l|N$*:*

(N1) *If l is prime to* $\mathfrak{C}$ *and* $(l) = \mathfrak{l}\mathfrak{l}^c$ *with* $\mathfrak{l} \neq \mathfrak{l}^c$*, we have* $a(l) = \Psi(\mathfrak{l})$ *or* $\Psi(\mathfrak{l}^c)$ *where*

$$\Psi(\mathfrak{l}) = \zeta(1+T)^{\log_p(\mathfrak{l})/\log_p(\gamma)} \text{ and } \Psi(\mathfrak{l}^c) = \zeta'(1+T)^{\log_p(\mathfrak{l}^c)/\log_p(\gamma)}$$

for roots of unity ζ, ζ'*. In particular,* $\Psi(\mathfrak{l}) \neq \Psi(\mathfrak{l}^c)$*,* $C_l = 1$ *and* $N_l(\mathbb{I}) = l$*.*

(N2) *If* $(l) = \mathfrak{l}\mathfrak{l}^c$ *with* $\mathfrak{l} \neq \mathfrak{l}^c$ *and* $\mathfrak{l}^c|\mathfrak{C}$ *with* $\mathfrak{l} \nmid \mathfrak{C}$*, we have* $a(l) = \Psi(\mathfrak{l})$ *and* $N_l(\mathbb{I}) = C_l$*, where for a root of unity* ζ*,* $\Psi(\mathfrak{l}) = \zeta(1+T)^{\log_p(\mathfrak{l})/\log_p(\gamma)}$*.*

(N3) $a(l) = \pm\sqrt{\Psi(l)}$ *when l is inert and* $C_l = 1$ *with* $N_l(\mathbb{I}) = l$*, where, for a root of unity* ζ*,* $\Psi(l) = \zeta(1+T)^{\log_p(l)/\log_p(\gamma)}$*.*

(N4) *If* $(l) = \mathfrak{l}^2$ *is ramified outside* $\mathfrak{C}$ *with* $N_l(\mathbb{I}) = C_l$*, we have* $a(l) = \Psi(\mathfrak{l})$*, where*

$$\Psi(\mathfrak{l}) = \zeta(1+T)^{\log_p(\mathfrak{l})/\log_p(\gamma)}$$

for a root of unity ζ*.*

(N5) *We have* $a(l) = 0$ *in the following cases:*
- (a) $(l) = \mathfrak{l}^2$ *is ramified in M with* $\mathfrak{l}|\mathfrak{C}$*,*
- (b) $N_l(\mathbb{I}) = l^2$ *and* $C_l = 1$*,*
- (c) $(l) = \mathfrak{l}\mathfrak{l}^c$ *with* $\mathfrak{l} \neq \mathfrak{l}^c$ *and* $(l)|\mathfrak{C}$*,*
- (d) $(l) = \mathfrak{l}\mathfrak{l}^c$ *with* $\mathfrak{l} \neq \mathfrak{l}^c$*,* $\mathfrak{l}|\mathfrak{C}$ *and* $\mathfrak{l}^c \nmid \mathfrak{C}$*,* $N_l(\mathbb{I}) = l \cdot C_l$*,*
- (e) *l is inert in M and* $l|C_l$*.*

Recall the quotient field Q of $W[[T]]$ and its algebraic closure $\overline{Q}$. In the above proposition, strictly speaking, $Q(\mathbb{I})$ is embedded in $\overline{Q}$ and any of the roots of a polynomial in $Q(\mathbb{I})[X]$ (including square roots of an element in $Q(\mathbb{I})$) and $(1+T)^s$ with $s \in \mathbb{Q}_p$ are taken in $\overline{Q}$.

Proof. Regard first Ψ as an ideal character. Then for a fractional ideal $\mathfrak{a}$ of M prime to $\mathfrak{C}\mathfrak{p}$, it is well known that $\Psi(\mathfrak{a})$ has the shape $(1+T)^{\log_p(\mathfrak{a})/\log_p(\gamma)}$ up to roots of unity (see [H86a] §7 or [LFE] §7.6), since $\Psi \mod (T)$ is of finite order. From this, the explicit formula of the value in (O1–2) and (N1–4) follows.

Regard now Ψ as an idele character $\Psi : M^\times \backslash M^\times_{\mathbb{A}} \to \widetilde{\mathbb{I}}^\times$. Since Ψ is unramified outside $\mathfrak{C}\mathfrak{p}$ and the Sylow p-subgroup of $\mathfrak{O}^\times_{\mathfrak{C}} = \varprojlim_n (\mathfrak{O}/\mathfrak{C}^n)^\times = \prod_{\mathfrak{l}|\mathfrak{C}} \mathfrak{O}^\times_{\mathfrak{l}}$ is finite, almost p-profiniteness of $\widetilde{\mathbb{I}}^\times$ tells us that $\Psi|_{\mathfrak{O}^\times_{\mathfrak{C}}}$ has finite order. For any $P \in \mathrm{Spec}(\widetilde{\mathbb{I}})(\overline{\mathbb{Q}}_p)$ regarded as a prime ideal, P/P^n is a torsion-free additive group; so, $1 + P \subset \widetilde{\mathbb{I}}^\times$ is a torsion-free multiplicative subgroup. Now regard P as an algebra homomorphism $P : \widetilde{\mathbb{I}} \to \overline{\mathbb{Q}}_p$. Then, on any finite subgroup μ of $\widetilde{\mathbb{I}}$, P (regarded as an algebra homomorphism $P : \mathbb{I} \to \overline{\mathbb{Q}}_p$) induces an injection $P : \mu \to \overline{\mathbb{Q}}_p^\times$ because of torsion-freeness of $1 + P \subset \widetilde{\mathbb{I}}^\times$. Thus for $\Psi_P = P \circ \Psi$,

$$\mathrm{Ker}(\Psi_P : \mathfrak{O}^\times_{\mathfrak{C}} \to \overline{\mathbb{Q}}_p^\times) = \mathrm{Ker}(\Psi : \mathfrak{O}^\times_{\mathfrak{C}} \to \widetilde{\mathbb{I}}^\times),$$

and hence $\mathfrak{C}(\Psi_P)=\mathfrak{C}(\Psi)$. If P is arithmetic and Ψ_P is the p-adic avatar of $\lambda_P : M^\times\backslash M^\times_{\mathbb{A}} \to \mathbb{C}^\times$, the theta series $\theta(\lambda_P)=\sum_{\mathfrak{a}\subset\mathfrak{O},\mathfrak{a}+\mathfrak{p}=\mathfrak{O}}\lambda_P(\mathfrak{a})q^{N(\mathfrak{a})}$ (cf. [MFM] §4.8) is a p-stabilized cusp form new outside p whose prime-to-p conductor is C (see [MFM] §4.8–§4.9). Here $\lambda_P(\mathfrak{a})=\Psi_P([\mathfrak{a},M])$ for the Artin symbol $[\mathfrak{a},M]$ as long as $\mathfrak{a}$ is prime to $\mathfrak{p}\mathfrak{C}(\Psi)$, and $\lambda_P(\mathfrak{a})=0$ if $\mathfrak{a}+\mathfrak{p}\mathfrak{C}(\Psi)\subsetneq\mathfrak{O}$. Similarly we define $\Psi(\mathfrak{a})=\Psi([\mathfrak{a},M])$ for $\mathfrak{a}$ prime to $\mathfrak{p}\mathfrak{C}(\Psi)$ and otherwise $\Psi(\mathfrak{a})=0$. Then $\mathcal{F}=\{\theta(\lambda_P)\}_P$ is an analytic family corresponding to an $\widetilde{\mathbb{I}}$-adic Hecke eigenform $\Theta=\sum_{\mathfrak{a}\subset\mathfrak{O},\mathfrak{a}+\mathfrak{p}=\mathfrak{O}}\Psi(\mathfrak{a})q^{N(\mathfrak{a})}$. Since $\Theta|T(l)=a(l)\Theta$ for

$$a(l)=\begin{cases}\Psi(\mathfrak{l})+\Psi(\mathfrak{l}^c) & \text{if } (l)=\mathfrak{l}\mathfrak{l}^c \text{ with } \mathfrak{l}\neq\mathfrak{l}^c,\\ \Psi(\mathfrak{l}) & \text{if } (l)=\mathfrak{l} \text{ or } (l)=\mathfrak{l}^2,\\ 0 & \text{otherwise}\end{cases}$$

for prime ideals $\mathfrak{l}$ in M, we get a projection $\pi:\mathbf{h}\twoheadrightarrow\widetilde{\mathbb{I}}$ by sending $h\in\mathbf{h}$ to the eigenvalue $\pi(h)$ of Θ: $\Theta|h=\pi(h)\Theta$. This projection gives rise to a primitive irreducible component $\mathbb{I}$ corresponding to Θ. Any other component giving rise to the same Galois representation $\mathrm{Ind}_M^{\mathbb{Q}}\Psi$ is a family of old Hecke eigenforms made out of C-new eigenforms in $\mathcal{F}_{\mathbb{I}}$. Therefore the old form family has the same Hecke eigenvalues for $T(l)$ for l outside N. This proves (O1–2) and (N1–5) for the family of cusp forms new at $l|N$.

We now study family of old forms coming from the above family $\mathcal{F}=\{\theta(\lambda_P)\}_P$ of C-new forms. Write π_P for the automorphic representation generated by the p-stabilized Hecke eigenform $\theta(\lambda_P)$ new outside Cp. For each primes $l|N/C$, choose a root α of $\det(X-\rho_{\mathbb{I}}(Frob_l))=0$ if $l\nmid C$, and put $\alpha=0$ otherwise. Extend scalars $\mathbb{I}$ to $\mathbb{I}[\alpha]$. Then $\alpha_P=P(\alpha)$ gives rise to a root of the polynomial $\det(X-\rho_P(Frob_l))$ if $\alpha\neq0$. Then we can think of the Hecke eigenform f_{α_P} with $f_{\alpha_P}|U(l)=\alpha_P f_{\alpha_P}$ in π_P as in Proposition 1.1. This family $\mathcal{F}_\alpha=\{f_{\alpha_P}|P:\text{arithmetic}\}$ has associated Galois representation identical to $\rho_{\mathbb{I}}$ but is made up of old forms f_{α_P}. By the proof of Corollary 1.2, this family $\mathcal{F}_\alpha$ corresponds to the $\mathbb{I}[\alpha]$-adic Hecke eigenform $\Theta_\alpha(q)=\Theta(q)-\beta\Theta(q^l)$ where β is another root of $\det(X-\rho_{\mathbb{I}}(Frob_l))=0$ if $l\nmid C$ and we take $\beta=\alpha$ otherwise. Then (N1–5) will be proven once we can determine α as specified in (N1–5). If l splits or ramifies in M and $\alpha\neq0$, we have $\alpha=\Psi(\mathfrak{l})$ for a prime factor $\mathfrak{l}|l$ as $\rho_{\mathbb{I}}=\mathrm{Ind}_M^{\mathbb{Q}}\Psi$. If l is inert in M and $\alpha\neq0$, we have $\alpha=\pm\sqrt{\Psi(l)}$. Then the description of $N_l(\mathbb{I})$ in terms of C follows from Proposition 1.1 (and its corollary). □

Corollary 3.8. *Let $\mathbb{I}$ and $\mathbb{I}'$ be two distinct CM components in $\mathrm{Spec}(\mathbf{h}^M_{cm})$ embedded in $\overline{Q}$ as Λ-algebras and write $a(l)$ and $a'(l)$ for the image of $T(l)$ in $\mathbb{I}$ and $\mathbb{I}'$ respectively. If $a(l)=a'(l)$ for almost all primes l, the intersection $\mathrm{Spec}(\mathbb{I})\cap\mathrm{Spec}(\mathbb{I}')$ in $\mathrm{Spec}(\mathbf{h})$ does not contain any prime divisors P over $(p)=p\mathbb{Z}_p[[T]]$.*

Proof. Let $\Gamma=1+p\mathbb{Z}_p\subset\mathbb{Z}_p^\times$. Then we can identify, for any complete local p-profinite ring A, $A[[T]]$ with the completed group ring $A[[\Gamma]]=\varprojlim_n A[\Gamma/\Gamma^{p^n}]$ by $1+T\leftrightarrow\gamma\in\Gamma$. Then the group element of $\gamma^s\in\Gamma$ in $A[\Gamma]\subset A[[\Gamma]]$ corresponds to $(1+T)^s=(1+T)^{\log_p(\gamma^s)/\log_p(\gamma)}$. In particular, $\{(1+T)^{\log_p(u)/\log_p(\gamma)}|u\in\Gamma^{p^{-n}}\}$ for some $n\geq0$ is linearly independent over A. Take $A=\mathbb{F}$ for a finite field $\mathbb{F}$ of characteristic p. Then if $P\in\mathrm{Spec}(\mathbb{I})$ and $P'\in\mathrm{Spec}(\mathbb{I}')$ is over $(p)=p\mathbb{Z}_p[[[T]]$, we may assume by Proposition 3.7 that $\mathbb{I}/P$ and $\mathbb{I}'/P'$ are isomorphically embedded into $\mathbb{F}[[T']]$ for T' with $(1+T')^{p^n}=(1+T)$ for some $n\geq0$. Since $a(l)=a'(l)$ for almost all primes l, by the above corollary, we can find a prime $q|N$ such that $a(q)\neq a'(q)$ with $a(q)=\zeta(1+T)^s$ and $a'(q)=\zeta'(1+T)^{s'}$ $(\zeta,\zeta'\in\mathbb{F})$ with $s\neq s'$ if both ζ and ζ' are nonzero. Thus P cannot be equal to P' in $\mathrm{Spec}(\mathbf{h})$, and the result follows. □

Consider the ray class group $Cl_M(\mathfrak{C}\mathfrak{p}^r)$ of M modulo $\mathfrak{C}\mathfrak{p}^r$, and put $Z=\varprojlim_r Cl_M(\mathfrak{C}\mathfrak{p}^r)$. Assume that $\mathfrak{C}$ is prime to p. Let Z_p be the Sylow p-part of Z. Then we have $Z=Z^{(p)}\times Z_p$ with finite group $Z^{(p)}$ of order prime to p. We write Δ for the maximal finite subgroup of Z_p. Let $\mathrm{Spec}(\mathbb{I})$ be a CM irreducible component and $\mathrm{Spec}(\mathbb{T})$ be the connected component of $\mathrm{Spec}(\mathbf{h})$ containing $\mathrm{Spec}(\mathbb{I})$. Let $\mathfrak{C}$ be the prime-to-p conductor of the associated character $\Psi_{\mathbb{I}}$. The character $\Psi_{\mathbb{I}}$ restricted to $\mathbb{Z}_p[\Delta]$ has image $W_{\mathbb{I}}\subset\mathbb{I}$ whose normalization is a discrete valuation ring finite flat over $\mathbb{Z}_p$; so, $\mathbb{I}\supset W_{\mathbb{I}}[[T]]$ and $W\supset W_{\mathbb{I}}$ (as we take W large so that $\mathbb{I}$ is geometrically irreducible over the quotient field of W).

We define $\mathrm{Spec}(\mathbb{T}_{cm})$ to be the minimal closed subscheme in $\mathrm{Spec}(\mathbb{T})$ containing all CM components of $\mathrm{Spec}(\mathbb{T})$.

Proposition 3.9. *Let the notation be as above, and let $\mathbb{I}$ be a CM component $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T}_{cm})$ having CM by an imaginary quadratic field M. Put $\mathfrak{C} = \mathfrak{C}(\mathbb{I})$. Assume*

(1) *the prime-to-p level N of $\mathbf{h}$ is cube free;*
(2) *either* (H) *with $p \nmid \varphi(N)$ or $p \nmid \Phi(N)$.*

If P is a prime divisor of $\mathbf{h}$ over the principal prime ideal $(p) \subset \mathbb{Z}_p[[T]]$ in $\mathrm{Spec}(\mathbb{I})$, the localization $\mathbb{I}_{cm,P}$ is a local complete intersection canonically isomorphic to the localization of $W[[Z_p]]$ at a prime divisor over $(p) = p\mathbb{Z}_p[[T]]$.

Proof. Since N is cube free, $\mathbf{h}$ is reduced (see Corollary 1.3). Thus $\mathrm{Spec}(\mathbb{T}_{cm})$ is reduced. We only need to look into a local ring $\mathbb{T}_P$ containing $\mathrm{Spec}(\mathbb{I}_P)$. By Corollary 3.6, after localization at P, each local ring of $\mathbf{h}_P$ contains CM components of a single imaginary quadratic field M and the prime-to-p conductor of the components is independent of the component $\mathbb{I}$. Thus we may ignore other components of $\mathrm{Spec}(\mathbb{T})$ coming from another imaginary quadratic field and of different level. Thus we assume that $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbf{h}^M_{cm})$. By Corollary 3.6, a CM component $\mathrm{Spec}(\mathbb{I})$ with nontrivial intersection with $\mathbb{T}_P$ comes from a character $\Psi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$ whose prime-to-p conductor $\mathfrak{C}$ is determined by $\mathbb{T}_P$. The character Ψ ramifies at $\mathfrak{p}$ but is unramified at $\overline{\mathfrak{p}}$. Thus we may regard Ψ as a character of Z. Since for primes l inert in M, $a(l)$ is either 0 or $\sqrt{\Psi(l)}$ (which is in $\mathbb{Z}_p[[T]]$), the CM component $\mathbb{I}$ is generated by $a(l)$ of split or ramified primes, and such $a(l)$ is given by

$$a(l) = \begin{cases} \Psi(\mathfrak{l}) + \Psi(\mathfrak{l}^\sigma) & \text{if } (l) = \mathfrak{l}\bar{\mathfrak{l}}, \\ \Psi(\mathfrak{l}) & \text{if } (l) = \mathfrak{l}^2. \end{cases}$$

Here we put $\Psi(\mathfrak{l}) = 0$ if $\mathfrak{l} | \mathfrak{C}\mathfrak{p}$.

Write $\mathfrak{O}$ for the integer ring of M and $Z_0 \subset Z$ for the natural image of $\mathfrak{O}_\mathfrak{p}^\times$ in Z. Thus Z/Z_0 is isomorphic to the class group $Cl_M(1)$. For each ideal $\mathfrak{a}$ prime to $\mathfrak{C}\mathfrak{p}$, we write $[\mathfrak{a}]$ for its class in Z. In [H86c] §4, we have constructed a continuous $\mathbb{Z}_p[[T]]$-algebra homomorphism $\Theta : \mathbf{h} \to W[[Z]]$ such that

$$\Theta(T(l)) = \begin{cases} [\mathfrak{l}] + [\bar{\mathfrak{l}}] & \text{if } (l) = \mathfrak{l}\bar{\mathfrak{l}} \text{ with } \mathfrak{l} \neq \bar{\mathfrak{l}} \text{ for primes } l \text{ outside } Np, \\ [\mathfrak{l}] & \text{if } (l) = \mathfrak{l}^2 \text{ and } \mathfrak{l} \nmid \mathfrak{C}, \end{cases} \tag{3.1}$$

and by [H86c] Proposition 4.1, Z_0 acts (by translation) trivially on $\mathrm{Coker}(\Theta)$. In other words, T kills $\mathrm{Coker}(\Theta)$; so, after localization at P, Θ is surjective. Since Θ factors through $\mathbb{T}$ as we remarked, after localization at P, Θ induces an isomorphism of the desired local ring $\mathbb{I}_{cm,P}$ with the local ring of $W[[Z]]$ localized at P, which is canonically isomorphic to the localization of $W[[Z_p]]$ at the image of P. Since a completed group algebra is a local complete intersection (e.g., [H86c] §1), its localization is also a local complete intersection. □

4. Irreducibility and Gorenstein property

Let $\mathbb{T}$ be a local ring of $\mathbf{h}$. Then the reduced spectrum $\mathrm{Spec}(\mathbb{T}^{red})$ is a union of its irreducible components: $\mathrm{Spec}(\mathbb{T}^{red}) = \bigcup_{\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})} \mathrm{Spec}(\mathbb{I})$. Let $\rho_\mathbb{T} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(Q(\mathbb{T}^{red}))$ be the Galois representation associated to this component, that is, $\rho_\mathbb{T} := \prod_{\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})} \rho_\mathbb{I}$. Let P be a prime in $\mathrm{Spec}(\mathbb{T})$; so, $P \in \mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$, and we have the associated semi-simple Galois representation $\rho_P : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\kappa(P))$ for the residue field $\kappa(P) = Q(\mathbb{T}/P)$ of P as described in the introduction. We prove a version of the result given in [MW1] Proposition 2 in §9:

Theorem 4.1. *If ρ_P is absolutely irreducible and $\rho_P|_{I_p} \cong \left(\begin{smallmatrix} \delta & * \\ 0 & 1 \end{smallmatrix}\right)$ with $\delta \neq 1$ for the inertia group $I_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ at p, then the localization $\mathbb{T}_P$ is a Gorenstein ring.*

To prove this, we apply Mazur's argument giving Lemma 15.1 of [M]: irreducibility ⇒ Gorenstein-ness. We prove

$$\mathrm{Hom}_{\Lambda_P}(\mathbb{T}_P, \Lambda_P) \cong \mathbb{T}_P$$

as $\mathbb{T}_P$ modules (which implies Gorenstein-ness of $\mathbb{T}_P$).

We prepare some notation and a lemma to prove the theorem. Let $J_1(Np^r)$ be the jacobian of the modular curve over $\mathbb{Q}$. We consider its p-adic Tate module $T_pJ_1(Np^r)$ and its projective limit $\varprojlim_r T_pJ_1(Np^r)$ via Albanese functoriality. The limit is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. The ordinary part J of $\varprojlim_r T_pJ_1(Np^r)$ (that is the image of $e = \lim_{n\to\infty} U(p)^{n!}$ of the limit) still carries the Galois action. By Diamond operators, $(\mathbb{Z}/N\mathbb{Z})^\times \times \mu_{p-1} \subset (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{Z}_p^\times$ acts on J. We can take the maximal quotient L of $J \otimes_{\mathbb{Z}_p} W$ on which $(\mathbb{Z}/N\mathbb{Z})^\times \times \mu_{p-1}$ acts by χ_2 for χ_k in (H2) in the introduction. The Galois module L is naturally an $\mathbf{h}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module. The Galois representation of $\mathbf{h}$ is realized on L as $L \otimes_{\mathbb{Z}_p[[T]]} Q$ is free of rank 2 over $\mathbf{h} \otimes_{\mathbb{Z}_p[[T]]} Q$ (see [H86b]).

First suppose that the p-part $\chi_{2,p}$ of χ_2 is non-trivial. Then the χ_2-part of the Barsotti-Tate group of $J_1(Np^r)[p^\infty] \otimes_{\mathbb{Z}_p} W$ extends uniquely to a Barsotti-Tate group over the valuation ring $A_r = \mathbb{Z}_p[\mu_{p^r}]^{\mathrm{Ker}(\chi_{2,p})}$ (see [AME] Chapter 14). Thus regarding the Pontryagin dual of L as the injective limit of the generic fiber of these Barsotti–Tate groups over A_∞, we have a connected-étale exact sequence: $L^{mult} \hookrightarrow L \twoheadrightarrow L^{et}$. As Ohta remarked in [O] §4, we have a perfect Λ-linear pairing

$$(\cdot,\cdot) : L \times L \to \Lambda$$

such that $(hx, y) = (x, hy)$ for $h \in \mathbf{h}$, L^{mult} is isotropic and is dual to L^{et}. Moreover the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by $(\sigma(x), \sigma(y)) = \kappa(\sigma)(x, y)$, where $\kappa(\sigma) = \chi_2(\sigma)(1+T)^{\log_p(\mathcal{N}(\sigma))/\log_p(\gamma)}$ for the p-adic cyclotomic character $\mathcal{N}$. As seen in [H86b] Theorem 9.3, we have

$$L^{mult} \cong \mathbf{h} \text{ and } L^{et} \cong \mathrm{Hom}_\Lambda(\mathbf{h}, \Lambda)$$

as $\mathbf{h}$-modules. This comes out as follows: Serre associated a differential of first kind on a curve $X_{/\overline{\mathbb{F}}_p}$ to each p-torsion point of the jacobian of X. Take X to be the Igusa curve C of level $\Gamma_1(Np)$ containing the infinity cups. Thus each point in $J[p](\overline{\mathbb{F}}_p)$ for the jacobian J of C has a q-expansion (of the differential associated to the point), and hence a Hecke equivariant $\mathbb{F}_p$-linear embedding: $J[p](\overline{\mathbb{F}}_p) \hookrightarrow H^0(C, \Omega_{C/\overline{\mathbb{F}}_p})$. By the q-expansion principle, Hecke eigenvalues have multiplicity one in the χ_2-eigenspace of $e \cdot J[p](\overline{\mathbb{F}}_p)$ (under the diamond operator action of $(\mathbb{Z}/p\mathbb{Z})^\times$), which is in turn isomorphic to the χ_2-eigenspace of the étale quotient of $e \cdot J_1(Np)[p](W(\overline{\mathbb{F}}_p))$ (for the Witt vector ring $W(\overline{\mathbb{F}}_p)$). Indeed, $U(p)$ acts on C by the relative p-power Frobenius map and on the other Igusa curve containing the 0 cusp by the dual of the Frobenius by a result of Wiles (e.g., [GME] Theorem 4.2.6); so, the projector e projects down the χ_2-eigenspace of $J_1(Np)[p](\overline{\mathbb{F}}_p)$ to the χ_2-eigenspace of $J[p](\overline{\mathbb{F}}_p)$. This fact is exploited in the proof of Lemma 14.7 of [M] (see also [H81] §5). From this, we get $L^{et}/\mathfrak{m}L^{et} \cong \mathrm{Hom}_W(\mathbf{h}/\mathfrak{m}\mathbf{h}, W/\mathfrak{m}_W)$ as $\mathbf{h}$-modules, where $\mathfrak{m} \subset \Lambda$ and $\mathfrak{m}_W \subset W$ are maximal ideals. By Cartier duality, we get $L^{mult}/\mathfrak{m}L^{mult} \cong \mathbf{h}/\mathfrak{m}\mathbf{h}$. So, by Nakayama's lemma, we have a surjective $\mathbf{h}$-linear map $\mathbf{h} \twoheadrightarrow L^{mult}$. Since $\mathbf{h}$ acts faithfully on L^{mult} (by definition), we have $\mathbf{h} \cong L^{mult}$ and again by duality, we get $L^{et} \cong \mathrm{Hom}_\Lambda(\mathbf{h}, \Lambda)$.

If $\chi_{2,p}$ is trivial, the connected component of the Barsotti-Tate group still extends (as Barsotti–Tate subgroup of the Néron model of $J_1(Np^r)$). Thus we have $L^{mult} \subset L$. Then we can define $L^{et} := L/L^{mult}$. We use this definition of L^{et} and L^{mult} in the case where $\chi_{2,p} = 1$. Without any assumption on $\chi_{2,p}$ (including the case where $\chi_{2,p} \neq 1$), Ohta gave a new proof of

$$\mathbf{h} \cong L^{mult} \quad\text{and}\quad L^{et} \cong \mathrm{Hom}_\Lambda(\mathbf{h}, \Lambda)$$

in [O2] Corollary (in the introduction), via Tate's $\mathbb{Z}_p$-integral p-adic Hodge theory. Ohta's isomorphism is therefore canonical. Thus we have

Lemma 4.2. *We have a canonical connected-étale exact sequence $L^{mult} \hookrightarrow L \twoheadrightarrow L^{et}$ with Λ-free $L^{mult} \subset L$ and Λ-free $L^{et} = L/L^{mult}$. Moreover we have canonical isomorphisms of $\mathbf{h}$-modules:*

$$L^{mult} \cong \mathbf{h} \textit{ and } L^{et} \cong \mathrm{Hom}_\Lambda(\mathbf{h}, \Lambda).$$

Proof of Theorem 4.1. We follow the proof of [M] Lemma 15.1 and Corollary 15.2. Take a prime $P \in \mathrm{Spec}(\mathbb{T}) \subset \mathrm{Spec}(\mathbf{h})$ as in the theorem and put $V = L_P/PL_P$ as Galois module. Then, by [O2] Theorem (where actually the Galois module $V' := V \otimes \det(\rho_P)^{-1}$ is studied), $V^{mult} := L^{mult}/PL^{mult}$ (isomorphic to V'^{I_p} just as vector spaces) is the eigen subspace of L on which the inertia group acts by the nontrivial character ϵ. By the above lemma, V^{mult} is one dimensional over $\kappa(P)$. If V is two dimensional, we have $\dim(L_P^{et}/PL_P^{et}) = 1$, and hence by Nakayama's lemma $L_P^{et} \cong \mathbb{T}_P = \mathbf{h}_P$. Since $L^{et} \cong \mathrm{Hom}_\Lambda(\mathbf{h}, \Lambda)$, this shows

$$\mathbb{T}_P = \mathbf{h}_P \cong \mathrm{Hom}_{\Lambda_P}(\mathbf{h}_P, \Lambda_P) \cong \mathrm{Hom}_{\Lambda_P}(\mathbb{T}_P, \Lambda_P)$$

as desired.

To show $\dim V = 2$, let $\Phi_l(X) = \det(X - \rho_P(Frob_l)) \in \mathbb{I}[X]$ for primes l outside Np. Since L is killed by $\Phi_l(Frob_l)$, by the irreducibility of $\rho_\mathbb{I}$, V is killed by $\Phi_l(Frob_l)$. Irreducible subquotients of V are all isomorphic to ρ_P. Thus the semi-simplification V^{ss} is isomorphic to ρ_P^m for $m > 0$. The subspace $V^{mult} := L_P^{mult}/PL_P^{mult} \subset V$ is the unique 1-dimensional subspace on which I_p acts by ϵ. Then I_p acts trivially on $L_P^{et}/PL_P^{et} = V/V^{mult}$. Since multiplicity of ϵ on V^{ss} is m, we have $m = 1$ and hence $\dim L_P/PL_P = 2$, which finishes the proof. □

Remark 4.1. Let the notation be as in Theorem 4.1. Write $\rho_P|_{D_p} = \left(\begin{smallmatrix} \epsilon & * \\ 0 & \delta \end{smallmatrix}\right)$ so that the p-decomposition group D_p acts on L^{mult} by ϵ. If $\epsilon \neq \delta$, by the same argument proving the theorem (using the action of D_p in place of I_p), we can prove that $\mathbb{T}_P$ is a Gorenstein ring and $L_P \cong \mathbf{h}_P^2$ as $\mathbf{h}$-modules.

5. Congruence modules

Assume that N is cube free. Let $\mathbb{I}$ be a CM irreducible component in $\mathbf{h}_{cm}^M$ and $\mathbb{T}$ be a local ring of $\mathbf{h}$ with $\mathrm{Spec}(\mathbb{I}) \subset \mathrm{Spec}(\mathbb{T})$. Write $\mathfrak{C} = \mathfrak{C}(\mathbb{I})$ for the prime-to-p conductor of the associated character $\Psi_\mathbb{I} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \widetilde{\mathbb{I}}^\times$. Write $\mathrm{Spec}(\mathbb{T}_{cm}^M) = \mathrm{Spec}(\mathbf{h}_{cm}^M) \cap \mathrm{Spec}(\mathbb{T})$; so, $\mathrm{Spec}(\mathbb{T}_{cm}^M)$ is minimal closed subschemes in $\mathrm{Spec}(\mathbb{T})$ containing all components with CM by M. Pick a prime divisor P in $\mathrm{Spec}(\mathbb{T}_{cm}^M)$ over (p). Since $\Psi_\mathbb{I} \mod P$ restricted to $I_\mathfrak{p}$ has infinite order and is unramified at $\overline{\mathfrak{p}}$, ρ_P is absolutely irreducible (so, $\mathbb{T}_P$ is Gorenstein by Theorem 4.1). By Proposition 3.9, we have $\mathbb{T}_{cm,P}^M = \mathbb{I}_{cm,P}$, and $\mathbb{T}_{cm,P}^M$ is a local complete intersection (and hence Gorenstein). For the torsion-free part $\Gamma_M := C_p/\Delta$ of C_p, $\mathbb{I} = W[[\Gamma_M]]$; so, $\mathbb{I}$ is a regular ring. We have therefore the projection maps

$$\mathbb{T}_P \twoheadrightarrow \mathbb{T}_{cm,P}^M \twoheadrightarrow \mathbb{I}_P$$

where all rings involved are Gorenstein rings. Now we suppose the condition (H) stated in a slightly different way:

(H) If an inert or a ramified prime $\mathfrak{l}'$ of M divides $\mathfrak{C}(\mathbb{I}')$ for any component $\mathbb{I}'$ of $\mathbb{T}_{cm}^M$, $\Psi_{\mathbb{I}'} \mod \mathfrak{m}_{\mathbb{I}'}$ is ramified at $\mathfrak{l}$.

Theorem 5.1. *Suppose* (H) *and that N is cube free with $p \nmid l - 1$ for all prime factors l of N. Let $P \in \mathrm{Spec}(\mathbb{T}_{cm})$ be a prime divisor over $(p) \subset \mathbb{Z}_p[[T]]$. Then, we have $\mathbb{T}_P = \mathbb{T}_{cm,P}^M$. In other words, CM components of* $\mathrm{Spec}(\mathbb{T})$ *do not have congruence modulo a prime divisor over* $(p) \subset \mathbb{Z}_p[[T]]$ *with non CM components.*

We prepare some notation, three lemmas and a proposition for the proof of the theorem. For simplicity, we write $\Lambda_{(p)}$ for the localization of Λ at (p) and the sequence $\mathbb{T}_P \twoheadrightarrow \mathbb{T}_{cm,P}^M \twoheadrightarrow \mathbb{I}_P$ as $R \xrightarrow{\theta} S \xrightarrow{\mu} A$ and we put $\lambda = \mu \circ \theta : R \to A$. Since N is cube-free, $\mathbf{h}$ is reduced by Corollary 1.3. Then $Q(R)$ is a product of fields, and we have $Q(R) = Q_S \oplus Q(S)$ for the complementary semi-simple algebra Q_S. Let R_S be the projection of R in Q_S. Then we have the following (unique) decomposition

(1) $\mathrm{Spec}(R) = \mathrm{Spec}(R_S) \cup \mathrm{Spec}(S)$ made of closed subschemes inducing $R \hookrightarrow (R_S \oplus S)$ with Λ-torsion module $C_0(\theta, S) := (R_S \oplus S)/R$.

Similarly, we have $Q(S) = Q_A \oplus Q(A)$ and $Q(R) = Q'_A \oplus Q(A)$ as algebra direct sums. Write S_A (resp. R_A) for the projected image of S (resp. R) in Q_A (resp. Q'_A). Then we have

(2) $\mathrm{Spec}(S) = \mathrm{Spec}(S_A) \cup \mathrm{Spec}(A)$ made of closed subschemes inducing $S \hookrightarrow (S_A \oplus A)$ with Λ-torsion module $C_0(\mu, A) := (S_A \oplus A)/S$.

(3) $\mathrm{Spec}(R) = \mathrm{Spec}(R_A) \cup \mathrm{Spec}(A)$ made of closed subschemes inducing $R \hookrightarrow (R_A \oplus A)$ with Λ-torsion module $C_0(\lambda, A) := (R_A \oplus A)/R$.

Again by Corollary 1.3, S is a reduced algebra, and by Theorem 4.1 and Proposition 3.9, we have

$$\mathrm{Hom}_{\Lambda_{(p)}}(R, \Lambda_{(p)}) \cong R,\ \mathrm{Hom}_{\Lambda_{(p)}}(S, \Lambda_{(p)}) \cong S \text{ and } \mathrm{Hom}_{\Lambda_{(p)}}(A, \Lambda_{(p)}) \cong A \text{ as } R\text{-modules.} \tag{5.1}$$

Write $\pi_S : R \twoheadrightarrow R_S$ and $\theta : R \to S$ for the two projections and $(\cdot,\cdot)_R : R \times R \to \Lambda_{(p)}$ and $(\cdot,\cdot)_S : S \times S \to \Lambda_{(p)}$ for the pairing inducing the above self-duality isomorphisms. We recall Lemma 1.6 of [H86c]:

Lemma 5.2. *The S-ideal $\mathrm{Ker}(\pi_S : R \to R_S)$ is principal and S-free of rank 1.*

Proof. Let $\mathfrak{a} = \mathrm{Ker}(\theta : R \to S)$ and $\mathfrak{b} = \mathrm{Ker}(\pi_S : R \to R_S)$. Since $\mathbb{T}$ is the direct summand of $\mathbf{h}$, R is $\Lambda_{(p)}$-free of finite rank. By Proposition 3.9, S is $\Lambda_{(p)}$-free; so, $\mathfrak{a}$ is $\Lambda_{(p)}$-free, and by duality, we have an exact sequence $0 \to S^* \xrightarrow{\theta^*} R^* \to \mathfrak{a}^* \to 0$. Note that $\mathfrak{a}^*$ is naturally an R_S-module which is free of finite rank over $\Lambda_{(p)}$. Thus identifying $S = S^*$ and $R^* = R$ by (5.1), we have $\theta^*(S^*) = (Q(S) \oplus 0) \cap R = \mathfrak{b}$ inside $Q(R)$, and hence θ^* induces $S = S^* \cong \mathfrak{b}$ as S-modules. □

By [H88a] Lemma 6.3 (or [MFG] §5.3.3), we get the following isomorphisms:

$$C_0(\lambda; A) = R_A \otimes_R A,\ C_0(\theta; S) = R_S \otimes_R S \text{ and } C_0(\mu; A) = S_A \otimes_S A \tag{5.2}$$

as R-modules. For the reader's convenience, we give a proof of the following fact first proved in [H88a] as Theorem 6.6:

Lemma 5.3. *We have the following exact sequence of R-modules:*

$$0 \to C_0(\mu; A) \to C_0(\lambda; A) \to C_0(\theta; S) \otimes_S A \to 0.$$

Proof. Write $M^* = \mathrm{Hom}_{\Lambda_{(p)}}(M, \Lambda_{(p)})$ as an R-module for a R-module M. Note that

$$\mathrm{Ker}(\theta) = R_S \cap S \subset R_S \oplus S,\ \mathrm{Ker}(\lambda) = R_A \cap A \subset R_A \oplus A \text{ and } \mathrm{Ker}(\mu) = S_A \cap A \subset S_A \oplus A.$$

From an exact sequence $0 \to \mathrm{Ker}(\theta) \to R \to S \to 0$ (noting S is $\Lambda_{(p)}$-free by Proposition 3.9), we have the following commutative diagram with exact rows (for $\mathfrak{b} = \mathrm{Ker}(\pi_S : R \to R_S)$):

$$\begin{tikzcd}
S^* \arrow[r, hook] \arrow[d, "\wr"'] & R^* \arrow[r, two heads] \arrow[d, "\wr"'] & \mathrm{Ker}(\theta)^* \arrow[d] \\
S \cong \mathfrak{b} \arrow[r, hook] & R \arrow[r, two heads] & R_S,
\end{tikzcd}$$

which shows $R_S \cong \mathrm{Ker}(\theta)^* = (R_S \cap R)^*$ as R-modules. Similarly, we get $\mathrm{Ker}(\lambda)^* \cong R_A$, $\mathrm{Ker}(\mu)^* \cong S_A$ and $(R \cap S)^* \cong S$ as R-modules, where $R \cap S = R \cap (0 \oplus S)$ is taken in $R_S \oplus S$. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ker}(\theta) \arrow[r] \arrow[d] & R \arrow[r, "\theta"] \arrow[d, "\lambda"'] & S \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma, we get an exact sequence of R-modules:

$$0 \to \mathrm{Ker}(\theta) \to \mathrm{Ker}(\lambda) \to \mathrm{Ker}(\mu) \to 0.$$

By $\Lambda_{(p)}$-freeness of A and S, all the terms of the above exact sequence are $\Lambda_{(p)}$-free. Thus the above sequence is split as a sequence of $\Lambda_{(p)}$-modules, and we have the dual exact sequence:

$$\begin{tikzcd}
\mathrm{Ker}(\mu)^* \arrow[r, hook] \arrow[d, "\wr"'] & \mathrm{Ker}(\lambda)^* \arrow[r, two heads] \arrow[d, "\wr"'] & \mathrm{Ker}(\theta)^* \arrow[d, "\wr"'] \\
S_A \arrow[r, hook] & R_A \arrow[r, two heads] & R_S.
\end{tikzcd}$$

Tensoring with A over R, from (5.2), we get an exact sequence:

$$\mathrm{Tor}^1_R(R_S, A) \to C_0(\mu; A) \to C_0(\lambda; A) \to C_0(\theta; S) \otimes_R A \to 0.$$

Thus we need to show the vanishing: $\mathrm{Tor}^1_R(R_S, A) = 0$. To see this, we recall $(S \cap R)^* \cong S$, and taking the dual, we get $(S \cap R) \cong S^* \cong S$ (where $(S \cap R) \subset (R_S \oplus S)$). Thus the exact sequence

$(R \cap S) \hookrightarrow R \twoheadrightarrow R_S$ can be rewritten as $S \hookrightarrow R \twoheadrightarrow R_S$. Tensoring A over R, we get an exact sequence

$$0 = \mathrm{Tor}^1_R(R, A) \to \mathrm{Tor}^1_R(R_S, A) \to S \otimes_R A \xrightarrow{\alpha} R \otimes_R A \to R_S \otimes_R A \to 0.$$

Since we have a commutative diagram:

$$\begin{tikzcd} S \otimes_R A \arrow[r, equal] \arrow[d, "\alpha"'] & A \arrow[d] \\ R \otimes_R A \arrow[r, equal] & A \end{tikzcd}$$

and $\mathrm{Coker}(\alpha)$ is a torsion A-module, α is a nontrivial A-linear map of the integral domain A into itself; so, α is injective, and we conclude $\mathrm{Tor}^1_R(R_S, A) = 0$ as desired. □

By (5.2), the three congruence modules $C_0(\mu; A), C_0(\lambda; A), C_0(\theta; S) \otimes_R A$ are residue rings of R; so, cyclic A-modules. Moreover they are given A modulo principal ideals, as $Ac_\lambda = A \cap R \subset (R_A \oplus A)$, $Ac_\mu = A \cap S \subset (S_A \oplus A)$ and $Sc_\theta = S \cap R \subset (R_S \oplus S)$ (e.g., [H86c] Lemma 1.6). Thus we have $C_0(\lambda; A) = A/c_\lambda A$, $C_0(\mu; A) = A/c_\mu A$ and $C_0(\theta; S) \otimes_S A = A/\overline{c}_\theta A$ for the image $\overline{c}_\theta \in A$ of $c_\theta \in S$. By the above lemma, we conclude the following result:

Corollary 5.4. *We have $\overline{c}_\theta \cdot c_\mu = c_\lambda$ up to units in A.*

We have a natural morphism $(\mathbb{Z}/(\mathfrak{C} \cap \mathbb{Z}))^\times \to Cl_M(\mathfrak{C})$ sending an ideal (n) for an integer n prime to $\mathfrak{C}$ to its class in $Cl_M(\mathfrak{C})$, and we write $h^-(\mathfrak{C})$ for the order of the cokernel of this map. Write $l(\mathfrak{l})$ for the residual characteristic of $\mathfrak{l}$. Then

$$\frac{h^-(\mathfrak{C})}{h(M) \cdot \prod_{\mathfrak{l}|\mathfrak{C}, \mathfrak{l}: \text{ inert prime}} (l(\mathfrak{l}) + 1) \prod_{\mathfrak{l}|\mathfrak{C}, \mathfrak{l}: \text{ split prime with } l(\mathfrak{l})|\mathfrak{C}} (l(\mathfrak{l}) - 1)}$$

is prime to p (if $p > 3$), where $h(M)$ is the class number of M. Write $\mathfrak{C}$ for $\mathfrak{C}(\mathbb{I})$.

Lemma 5.5. *Let the notation be as in* Theorem 5.1*. Then we have $c_\mu = h^-(\mathfrak{C})$ up to units in $\mathbb{I}_P$, where $\mathfrak{C}$ is the prime-to-p conductor of $\Psi_{\mathbb{I}}$ in* (CM2).

Proof. Since S is the p-localization of the group algebra $W[[Z_p]]$ by Proposition 3.9, this follows from Lemma 1.9 and Lemma 1.11 in [H86c]. □

Let $\Psi^-_{\mathbb{I}}(\tau) = \Psi_{\mathbb{I}}(c\tau c^{-1}\tau^{-1})$ be the anticyclotomic projection of $\Psi_{\mathbb{I}}$ and $L_p(\Psi^-_{\mathbb{I}})$ be the anticyclotomic Katz p-adic L-function as in [H06] and [H07]. We regard $L_p(\Psi^-_{\mathbb{I}}) \in \mathbb{I}$ (though strictly speaking, we may need to extend scalars to an unramified extension of $\mathbb{Z}_p$ to have $L_p(\Psi^-_{\mathbb{I}}) \in \mathbb{I}$).

Proposition 5.6. *Let the notation be as in* Theorem 5.1*, and assume that p is prime to $l - 1$ for all prime factors l of N. If $N = N(\mathfrak{C})D$, then we have $c_\lambda = h^-(\mathfrak{C})L_p(\Psi^-_{\mathbb{I}})$ up to units in $\mathbb{I}_P$ for the prime-to-p conductor $\mathfrak{C}$ of $\Psi_{\mathbb{I}}$ in* (CM2).

Proof. The fixed field $\widetilde{M}/M$ of $\mathrm{Ker}(\Psi)$ for $\Psi = \Psi_{\mathbb{I}}$ has Galois group $\mathrm{Gal}(\widetilde{M}/M) \cong \mathrm{Im}(\Psi)$. The maximal torsion-free quotient Γ_M of $\mathrm{Gal}(\widetilde{M}/M)$ is a $\mathbb{Z}_p$-free module of rank 1. Fix a decomposition $\mathrm{Gal}(\widetilde{M}/M) = \Delta \times \Gamma_M$ for the maximal finite subgroup Δ of $\mathrm{Gal}(\widetilde{M}/M)$. Let $\psi := \Psi|_\Delta$ (which has values in $W^\times$). By Proposition 3.9 (and its proof), the character Ψ induces an algebra isomorphism $\Psi_* : W[[\Gamma_M]] \cong \widetilde{\mathbb{I}}$ and $\widetilde{\mathbb{I}}_{(p)} = \mathbb{I}_{(p)}$. Then the maximal p-abelian extension $L/\widetilde{M}$ unramified outside $\mathfrak{p}$ has Galois group X which is naturally a $W[[\mathrm{Gal}(\widetilde{M}/M)]]$-module (in the standard manner of Iwasawa's theory). Let $X(\psi) = X \otimes_{W[\Delta],\psi} W$ which is the maximal quotient of X on which Δ acts by ψ. Thus $X(\psi)$ is naturally an $\widetilde{\mathbb{I}}$-module via Ψ_*, and it is known to be a torsion $\widetilde{\mathbb{I}}$-module of finite type. Let $\mathcal{F}^-(\Psi_{\mathbb{I}})$ be the Iwasawa power series in $\widetilde{\mathbb{I}}$ of $X(\psi)$; i.e., the characteristic power series of $X(\psi)$ as a torsion $\widetilde{\mathbb{I}}$-module of finite type (see [MFG] page 291 for the characteristic power series). By the proof of the main conjecture over M by K. Rubin [Ru] or the proof of its anticyclotomic version by Tilouine/Mazur ([T] and [MT]), we know $\mathcal{F}^-(\Psi^-_{\mathbb{I}}) = L^-_p(\Psi_{\mathbb{I}})$ up to units in $\widetilde{\mathbb{I}}$. By [H09] Corollary 3.8, if $p \geq 5$ and $N = N(\mathfrak{C})D$, we have proven that $h^-(\mathfrak{C})L_p(\Psi^-_{\mathbb{I}})|c_\lambda$ in $\widetilde{\mathbb{I}}_{(p)} = \mathbb{I}_{(p)}$. By [MT], we also know $c_\lambda | h^-(\mathfrak{C})\mathcal{F}^-(\Psi^-_{\mathbb{I}})$. Combining all of these, we conclude the equality of the proposition. If $\Psi_{\mathbb{I}} \not\equiv \Psi^\sigma_{\mathbb{I}} \mod \mathfrak{m}_{\mathbb{I}}$ (i.e., the residual representation of $\rho_{\mathbb{I}}$ is absolutely irreducible), actually, the above identity is proven in [H09] without using the solution of the main conjecture (and in this way, the anticyclotomic main conjecture is proven in [H09]). □

Proof of Theorem 5.1. Let $C = N(\mathfrak{C})D$, and first assume $N = C$. Note that the assertion of the theorem is equivalent to the vanishing $C_0(\theta; S) = 0$, which is in turn, by Nakayama's lemma, equivalent to $C_0(\theta; S) \otimes_R A = 0$. We study $C_0(\theta; S) \otimes_R A$. By the above two lemmas and the proposition, we find that $\overline{c}_\theta = c_\lambda/c_\mu$ (Corollary 5.4); so, $\overline{c}_\theta = L_p(\Psi_{\mathbb{I}}^-)$ up to units in A. Let p^μ ($0 \le \mu \in \mathbb{Q}$) be the exact power dividing $L_p(\Psi_{\mathbb{I}}^-)$ in A. Then $\overline{c}_\theta = 1$ (up to units in A) $\Leftrightarrow C_0(\theta; S) \otimes_R A = 0 \Leftrightarrow \mu = 0$. The vanishing of μ is proven in [H10] and [Fi] under (H) and the theorem follows if $N = N(\mathfrak{C})D$.

We now assume to have a prime $l|(N/C)$. We now proceed by induction on N. On the contrary to the desired outcome, expecting absurdity, we assume that the CM component $\mathrm{Spec}(\mathbb{I}_P)$ intersects another component $\mathrm{Spec}(\mathbb{I}'_P)$ without CM by M in $\mathrm{Spec}(\mathbb{T}_P)$ for the prime divisor P in the theorem. By Proposition 3.9 (and Corollary 3.6), if $\mathbb{I}'$ has CM, it has CM by M, and $\mathbb{I}'$ has prime-to-p level C; so, $\mathbb{I}'$ cannot have CM. Write ρ for $\rho_{\mathbb{I}}$, ρ' for $\rho_{\mathbb{I}'}$ and $\rho'_{P'}$ for $\rho' \mod P'$ (with $P' \in \mathrm{Spec}(\mathbb{I}')$). Let C' be the prime-to-p Artin conductor of ρ'. If $C = C'$, the family of $\mathbb{I}$ (resp. $\mathbb{I}'$) is induced by a unique family of prime-to-p level C made up of C-new forms giving rise to an irreducible component $\mathrm{Spec}(\mathbb{I}_0)$ (resp. $\mathrm{Spec}(\mathbb{I}'_0)$) of the spectrum of the big Hecke algebra of prime-to-p level C. Since $\mathrm{Spec}(\mathbb{I}_0)$ and $\mathrm{Spec}(\mathbb{I}'_0)$ must have an intersection at a prime divisor above (p), by the theorem for the lesser level C, this cannot happen. Thus $C \ne C'$. Let $P' \in \mathrm{Spec}(\mathbb{I}')$ be the prime divisor induced by P. Since the prime-to-p conductor C of $\rho'_{P'}$ is a factor of the prime-to-p conductor C' of ρ', we have a prime $l|(C'/C)$. Thus the family $\mathcal{F}$ associated to $\mathbb{I}$ is made up of l-old forms. Let F' (resp. F) be the $\mathbb{I}'$-adic form (resp. the $\mathbb{I}$-adic form) giving rise to the family associated to the component $\mathbb{I}'$ (resp. $\mathbb{I}$). If $l \nmid C$, each arithmetic specialization of F' is new at l with the same central character as the corresponding member of $\mathcal{F}$. If $C \cdot l^2|C'$, then $F'|U(l) = 0$ (by Corollary 1.2) and $H_0(I_l, \rho') = 0$ for the inertia group I_l at l (by the definition of Artin conductor; e.g., [GME] (5.2)). Since $\rho'_{P'} = \rho_P$, l is unramified for $\rho' \mod P'$; so, $H := \rho'(I_l)$ is a p-profinite group with non-trivial p-profinite abelian quotient H^{ab}. Then, by local class field theory, its p-profinite abelian quotient H^{ab} is a surjective image of $\mathbb{Z}_l^\times$. Since $l \ne p$, we conclude that p divides $|(\mathbb{Z}/l\mathbb{Z})^\times| = l - 1$, against our assumption. Thus $C' = C \cdot l$, and each arithmetic specialization of F' generates an automorphic representation Steinberg at l, and $\rho'|_{D_l} \cong \eta \otimes \left(\begin{smallmatrix} \mathcal{N}_l & * \\ 0 & 1 \end{smallmatrix}\right)$ for an unramified character $\eta : D_l \to \mathbb{I}'^\times$ such that $\eta(Frob_l) = \zeta\sqrt{l^{-1}\langle l\rangle}$ for a root of unity ζ (e.g., [MFM] Theorem 4.6.17 (2)), where D_l is the decomposition group at l and $\mathcal{N}_l$ is the cyclotomic character. On the other hand, $\rho_P|_{D_l}$ is irreducible if l does not split in M and is isomorphic to $\Psi|_{D_l} \oplus \Psi^c|_{D_l}$ if l splits in M. Thus $\rho_P = \rho'_{P'}$ is impossible by Proposition 3.7. Thus we conclude $l|C$.

By the definition of prime-to-p Artin conductor (cf. [GME] (5.2)), $C' = l \cdot C$ and $l|C$ implies $\dim H_0(I_l, \rho') = 0$ and $\dim H_0(I_l, \rho_P) = \dim H_0(I_l, \rho_{\mathbb{I}}) = 1$. This shows $l \nmid (C/l)$, and $l \nmid (C'/l^2)$. Thus the l-primary part N_l coincides with the l-primary part C'_l of C' as N is cube-free. Since N is cube-free with $N_l = C'_l|(C \cdot l)|N$, l cannot be inert in M (as otherwise, $l^2|C$). Writing F (resp. F') for the $\mathbb{I}$-adic form giving rise to the component $\mathbb{I}$ (resp. $\mathbb{I}'$), we see $F|U(l) = \alpha F$ for $\alpha \ne 0$, because the eigenvalue 0 for $U(l)$ can show up only at the level $C \cdot l^2$ or higher by Corollary 1.2 (or else, by $\dim H_0(I_l, \rho_P) = \dim H_0(I_l, \rho_{\mathbb{I}}) = 1$). Here we even know $\alpha \not\equiv 0 \mod P$, as we have $\alpha = \Psi(\mathfrak{l})$ as $(l) = \mathfrak{l}\overline{\mathfrak{l}}$ or $(l) = \mathfrak{l}^2$ in M (by Proposition 3.7). Since $\dim H_0(I_l, \rho') = 0$, $F'|U(l) = 0$, we cannot have $F \equiv F' \mod P$ as long as N is cube-free. Thus non CM $\mathrm{Spec}(\mathbb{I}'_P)$ cannot exists in $\mathrm{Spec}(\mathbb{T}_P)$. This finishes the proof. □

6. Proof of the theorem

First we list one more fundamental property of $\rho_{\mathbb{I}}$:

(Det) $$\det(\rho_{\mathbb{I}})(\sigma) = (1+T)^{\log_p(\mathcal{N}(\sigma))/\log_p(\gamma)}\chi_1(\sigma) \text{ for all } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

where χ is the Neben character as in the introduction and $\mathcal{N} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character (see the second edition of [GME] Theorem 4.3.1 for this fact). Pick a prime divisor P of $\mathbb{I}$ over (p) in $\mathbb{Z}_p[[T]]$. Define a character $\boldsymbol{\kappa} : \mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \to \mathbb{F}_p[[T]]^\times$ by $\boldsymbol{\kappa}(\sigma) = ((1+T)^{\log_p(\mathcal{N}(\sigma))/2\log_p(\gamma)} \mod (p)$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We put $\varrho_P = \rho_P \otimes \boldsymbol{\kappa}^{-1}$. Since $\boldsymbol{\kappa}$ is unramified outside p, we have the identity of prime-to-p conductors: $C(\rho_P) = C(\varrho_P)$. By (Det), $\det(\varrho_P) = \det(\rho_P)/\boldsymbol{\kappa}^2$ is a finite order character. Recall G_P° in the introduction, which is the connected component of the Zariski closure (over $\overline{\kappa}(P)$) of $S_P := \mathrm{Im}(\rho_P) \cap SL_2(\kappa(P))$.

By (Gal), we have $\tau \in \mathrm{Im}(\rho_P)$ such that $\det(\tau) = (1+T) \in \Lambda$. Then we have the subgroup group $\mathcal{T} = \{\tau^s | s \in \mathbb{Z}_p\}$ in $\mathrm{Im}(\rho_P)$. Since $\det(\rho_P)/\boldsymbol{\kappa}^2$ is a finite order character, $\mathcal{T} \ltimes S_P$ is an open subgroup of $\mathrm{Im}(\rho_P)$ (hence of finite index). Regarding ϱ_P as a representation of $\mathcal{T} \ltimes S_P$, we have $\varrho_P(\mathcal{T} \ltimes S_P) = \mathcal{T}' \cdot S_P$ for $\mathcal{T}' = \{\tau^z z^{-1/2} | z \in 1 + p\mathbb{Z}_p\}$. Since the Zariski closure T (over $\overline{\kappa}(P)$) of $\mathcal{T}'$ is isomorphic to $\mathbb{G}_m$ there are two possibilities:

$$\text{Case I: } T \subset G_P^\circ, \text{ Case II: } G_P^\circ \cap T \text{ is finite.}$$

Since T normalizes G_P° and is outside of G_P° in Case II, G_P° has to be either a unipotent group or trivial.

We now prove

Proposition 6.1. *Suppose* (H) *and that*

(1) *N is cube-free,*
(2) *if $N > 1$, p is prime to $l - 1$ for all prime factors l of N.*

Then if $\mathbb{I}$ is a non CM component of the Hecke algebra $\mathbf{h}$*, for each prime divisor $P \in \mathrm{Spec}(\mathbb{I})$ over $(p) = p\mathbb{Z}_p[[T]]$, G_P° is equal to $SL(2)_{/\overline{\kappa}(P)}$.*

Proof. Since $\rho_{\mathbb{I}} \cong \rho_{\mathbb{I}'}$ as long as $\mathrm{Tr}(\rho_{\mathbb{I}}(Frob_l)) = \mathrm{Tr}(\rho_{\mathbb{I}'}(Frob_l))$ for almost all primes l outside Np, we may assume that N is the minimal prime-to-p level of $\mathbb{I}$; i.e., we may assume that members of the associated p-adic analytic family are N-new forms. If a prime $l \neq p$ is unramified for ρ_P but ramified for $\rho_{\mathbb{I}}$, then $\rho_{\mathbb{I}} \equiv 1 \mod P$ over the inertia subgroup $I_l \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of l. Thus $H := \rho_{\mathbb{I}}(I_l)$ is p-profinite (and hence p-nilpotent); so, its maximal abelian quotient H^{ab} is nontrivial. By local class field theory, H^{ab} is covered by $\mathbb{Z}_l^\times$, we find $p | l - 1$, which is impossible by our assumption (2). Thus ρ_P ramifies at all prime factors of N. Recall $C := C(\rho_P) = C(\varrho_P)$ for the prime-to-p conductors. Thus we have $l | N \Leftrightarrow l | C$ (i.e., they have the same set of prime factors).

Since $\mathrm{Im}(\varrho_P)$ is compact, it has values in the discrete valuation subring of $\kappa(P)$, which has the form $\mathbb{F}[[X]] \subset \kappa(P)$ for the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ in $\kappa(P)$. By construction, $\det(\varrho_P)$ has finite image; so, in Case I, $G_P^\circ \subset SL(2)$ is the connected component of the Zariski closure of $\mathrm{Im}(\varrho_P)$ in $GL(2)_{/\overline{\kappa}(P)}$ and cannot be inside in a unipotent subgroup. In Case II, G_P° is either unipotent or trivial, as we already remarked.

Write $\mathfrak{s}_P \subset \mathfrak{sl}(2)$ for the Lie algebra of G_P°. The semi-simplification of $\rho_{\mathbb{I}}|_{I_p}$ has values in a torus in $GL(2)$. The torus contains an element with two distinct eigenvalues (which are 1 and $(1+T)^s$ for some $s \neq 0$). Thus the semi-simplification of $\varrho_P|_{I_p}$ has values in a split torus of $SL(2)$, and hence, in Case I, $\mathfrak{s}_P$ contains a split Cartan subalgebra $\mathfrak{H}$; so, $\dim \mathfrak{s}_P \geq 1$. We need to prove $\mathfrak{s}_P = \mathfrak{sl}(2)$. There are four possibilities of proper Lie subalgebras of $\mathfrak{sl}(2)_{/\overline{\kappa}(P)}$: 0, a (nontrivial) nilpotent subalgebra $\mathfrak{U}$, the Borel subalgebra $\mathfrak{B}$ and the Cartan subalgebra $\mathfrak{H}$. In Case II, the algebra $\mathfrak{s}_P$ could be 0 and $\mathfrak{U}$, and in Case I, we could have $\mathfrak{B}$, $\mathfrak{H}$ and $\mathfrak{sl}(2)$ for $\mathfrak{s}_P$. When $\mathfrak{s}_P = 0$ in case II, ρ_P is reducible semi-simple over an open normal subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and if $\mathfrak{s}_P = \mathfrak{U}$, ρ_P is reducible non-semi-simple over an open normal subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

If $\mathfrak{s}_P \subset \mathfrak{H}$ (this include the case $\mathfrak{s}_P = 0$ in Case II), we can find an open normal subgroup $H \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $\varrho_P|_H$ is isomorphic to the direct sum of two abelian characters. Write $\varrho_H = \varrho_P|_H$, then ϱ_H is completely reducible. Write $\varrho_H = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right)$, extending the scalars $\mathbb{F}[[X]]$ if necessary. Since ϱ_P extends ϱ_H, $g \mapsto \varrho_H^h(g) := \varrho_H(hgh^{-1}) = \varrho_P(h)\varrho_H(g)\varrho_P(h)^{-1}$ is equivalent to ϱ_H for all $h \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus $G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\{\delta, \epsilon\}$ by inner conjugation. Indeed,

$$\begin{pmatrix} \epsilon^h & 0 \\ 0 & \delta^h \end{pmatrix} = \varrho_P(h) \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \delta \end{smallmatrix}\right) \varrho_P(h)^{-1}. \tag{6.1}$$

Let $\Delta \subset G$ be the stabilizer of δ. Then $M = \overline{\mathbb{Q}}^{\Delta}$ is at most a quadratic extension of $\mathbb{Q}$. If $M = \mathbb{Q}$, the two characters extend to $\delta, \epsilon : G \to \mathbb{I}^\times$ (e.g. [GME] §5.1.1), and $\varrho_P^{ss} = \delta \oplus \epsilon$. Since $\det(\varrho_P(c)) = \det(\rho_P(c)) = -1$ for complex conjugation c, we conclude $\delta \neq \epsilon$. Thus this case can be classified in the case where $\mathfrak{s}_P \subset \mathfrak{B}$, which will be dealt with later. So, we suppose that ρ_P is absolutely irreducible over the quotient field $\mathbb{F}((X))$ of $\mathbb{F}[[X]]$. Then $[G : \Delta] = 2$ and by

Frobenius reciprocity, $\varrho_P \cong \mathrm{Ind}_M^{\mathbb{Q}} \delta \cong \mathrm{Ind}_M^{\mathbb{Q}} \epsilon$ for the quadratic extension $M = \overline{\mathbb{Q}}^{\Delta}$ of $\mathbb{Q}$; so, $\rho_P = \varrho_P \otimes \boldsymbol{\kappa} \cong \mathrm{Ind}_M^{\mathbb{Q}} \theta$ for a Galois character $\theta : \Delta = \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{F}[[X]]^\times$. We have therefore $\rho_P|_\Delta = \theta \oplus \theta^c$ with $\theta^c(\sigma) = \theta(c\sigma c^{-1})$ $(\sigma \in G)$ for $c \in G$ inducing a generator of $G/\Delta = \mathrm{Gal}(M/\mathbb{Q})$. By (Gal), we may assume that θ is ramified at a prime $\mathfrak{p}$ of M above p and that the other θ^c is unramified at the prime $\mathfrak{p}$. Then, the two images in $\mathrm{Im}(\rho_P)$ of the decomposition groups D_p and $c \cdot D_p c^{-1}$ must be distinct; so, p splits in M into $\mathfrak{p}\overline{\mathfrak{p}}$ with $D_p = D_{\mathfrak{p}}$. If M is a real quadratic field, by the existence of infinite order units in the integer ring of M, θ restricted to the inertia subgroup of $D_{\mathfrak{p}}$ cannot be of infinite order, contradicting (Gal). Thus M must be an imaginary quadratic field.

Let $\mathfrak{C}$ be the prime-to-p conductor of θ. The prime-to-p Artin conductor $N(\mathfrak{C})D$ of $\rho_P = \mathrm{Ind}_M^{\mathbb{Q}} \theta$ is a factor of the prime-to-p conductor N of $\rho_{\mathbb{I}}$ (e.g., [GME] Theorem 5.1.8). Here D is the discriminant of $M/\mathbb{Q}$. As already remarked, $l|N \Leftrightarrow l|C(\rho_P) = N(\mathfrak{C})D$.

As before, let $Z = \varprojlim_n Cl_M(\mathfrak{C}\mathfrak{p}^n)$ and Z_p be the maximal p-profinite quotient of Z. We claim to be able to lift θ to a character $\Theta : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to W[[Z_p]]$ with $\mathrm{Im}(\Theta) \supset Z_p$ so that

$$\theta = \Theta \mod P' \quad \text{for a prime } P' \in \mathrm{Spec}(W[[Z_p]])$$

without changing its ramification outside p. Let us prove this claim via Galois deformation theory. We may assume that W and $\mathbb{I}$ have the same residue field $\mathbb{F}$. Let $(R, \widetilde{\theta} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to R^\times)$ be the universal couple with the universal character unramified outside $\mathfrak{p}\mathfrak{C}$ deforming $(\theta \mod \mathfrak{m})$ over W. This couple $(R, \widetilde{\theta})$ is characterized by the following universal property: For any local artinian W-algebra A with residue field $\mathbb{F}$ and any character $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to A^\times$ unramified outside $\mathfrak{p}\mathfrak{C}$ with $\varphi \mod \mathfrak{m}_A = \theta \mod \mathfrak{m}$ (for the maximal ideal $\mathfrak{m}_A$ of A), there exists a unique W-algebra homomorphism $\iota : R \to A$ such that $\varphi = \iota \circ \widetilde{\theta}$. Such a pair (A, φ) is called a deformation of θ (see [M1] for general theory of Galois deformation).

We now show $R \cong W[[Z_p]]$ by class field theory. To see this, we pick a deformation φ : $\mathrm{Gal}(\overline{\mathbb{Q}}/M) \to A^\times$ of θ unramified outside $\mathfrak{p}\mathfrak{C}$; thus, A is a local artinian W-algebra sharing the residue field $\mathbb{F}$ with W and $\varphi \mod \mathfrak{m}_A = \theta \mod \mathfrak{m}$ for the maximal ideal $\mathfrak{m}_A$ of A. Let θ_0 be the Teichmüller lift of $\theta \mod \mathfrak{m}$; so, $\varphi' = \varphi\theta_0^{-1}$ has p-power order. For a prime $\mathfrak{l}|\mathfrak{C}$, by class field theory, the image $I_{\mathfrak{l}}^{ab}$ of the inertia group $I_{\mathfrak{l}} \subset \mathrm{Gal}(\overline{\mathbb{Q}}/M)$ in the Galois group of the maximal abelian extension of M over M is isomorphic to the multiplicative group $\mathfrak{O}_{\mathfrak{l}}^\times$ of the $\mathfrak{l}$-adic integer ring of $M_{\mathfrak{l}}$. Since φ' has p-power order and $p \neq l$, φ' must be trivial on $1 + \mathfrak{l}\mathfrak{O}_{\mathfrak{l}} \subset \mathfrak{O}_{\mathfrak{l}}^\times$. Thus $\mathfrak{l}$-conductor of φ' is at most $\mathfrak{l}$, and hence $\varphi = \varphi'\theta_0$ factors through Z. Thus φ' factors through the maximal p-profinite quotient Z_p and extends to a unique W-algebra homomorphism $\iota = \iota_\varphi : W[[Z_p]] \to A$ such that $\iota|_{Z_p} = \varphi'$. Since Z_p is the maximal p-profinite quotient of Z, by class field theory, we have the corresponding subfield $\widetilde{M}$ of the ray class field modulo $\mathfrak{p}^\infty\mathfrak{C}$ such that $\mathrm{Gal}(\widetilde{M}/M) \cong Z_p$ by Artin symbol. Writing the inclusion $Z_p \subset W[[Z_p]]$ as $\gamma \mapsto [\gamma]$ and identifying $\mathrm{Gal}(\widetilde{M}/M) = Z_p$, define a character $\Theta : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to W[[Z_p]]$ by $\Theta(\sigma) = \theta_0(\sigma)[\sigma|_{\widetilde{M}}]$. Then by our construction $\iota \circ \Theta = \varphi$; so, $(W[[Z_p]], \Theta)$ satisfies the universal property of $(R, \widetilde{\theta})$ for deformations φ of θ. Then the claim follows from the universality. In other words, taking $\iota : W[[Z_p]] \to \mathbb{F}[[X]]$ such that $\iota \circ \Theta = \theta$, we have $P' = \mathrm{Ker}(\iota)$.

Since $l|N \Leftrightarrow l|N(\mathfrak{C})D$, the Hecke operators at $l|N$ of level N and of level $N(\mathfrak{C})D$ are both $U(l)$ (not $T(l)$), therefore, the character Θ gives rise to an algebra homomorphism $\mathbf{h} \to W[[Z_p]]$ as in (3.1), which we still write Θ. Since each irreducible component $\mathbb{I}''$ of $\mathrm{Spec}(W[[Z_p]])$ gives a character $\Psi_{\mathbb{I}''} : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}''^\times$ and $\mathrm{Ind}_M^{\mathbb{Q}} \Psi_{\mathbb{I}''}$ is modular giving rise to an irreducible component of $\mathrm{Spec}(\mathbf{h})$ and hence of $\mathrm{Spec}(\mathbb{T})$, the morphism Θ is dominant. Indeed, by [H86c] Theorem 4.3, the map $\Theta : \mathbb{T} \to W[[Z_p]]$ always has finite cokernel and actually surjective under the condition (s). This shows the existence of an irreducible component $\mathrm{Spec}(\mathbb{I}'')$ of $\mathrm{Spec}(W[[Z_p]])$ giving rise to an irreducible component $\mathrm{Spec}(\mathbb{I}')$ of $\mathrm{Spec}(\mathbf{h})$ carrying the point P' through the projection Θ. We write ρ' for $\rho_{\mathbb{I}'}$, $\rho'_{P'}$ for $\rho' \mod P'$ and $a'(l)$ for the image of $U(l)$ in $\mathbb{I}'$. Then we have the identity of Galois representations $\rho_P \cong \rho'_{P'} = \mathrm{Ind}_M^{\mathbb{Q}} \Theta_{P'}$. This implies

$$\mathrm{Tr}(\rho_P(Frob_l)) = \mathrm{Tr}(\rho'_{P'}(Frob_l)) \quad \text{for all } l \text{ prime to } Np. \tag{6.2}$$

Let $\mathbf{h}' \subset \mathbf{h}$ be the Λ-subalgebra generated by $T(l)$ for all l prime to Np. The identity (6.2) implies $P' \cap \mathbf{h}' = P \cap \mathbf{h}'$. Since the possibility of $a(l)$ for $l|N$ in $\mathbb{I}/P$ is uniquely determined by $\rho_{\mathbb{I}}$ as $\mathbb{I}$ is associated to a family of N-new Hecke eigenforms. Indeed, by the solution of the local Langlands conjecture (see [GME] §5.1.3 for some explanation on this and references for the local Langlands conjecture), we have, for $l|N$,

$$a(l) = \begin{cases} 0 & \text{if } H_0(I_l, \rho_{\mathbb{I}}) = 0, \\ \alpha & \text{for the eigenvalue } \alpha \text{ of } Frob_l \text{ on } H_0(I_l, \rho_{\mathbb{I}}) \neq 0, \text{ otherwise.} \end{cases}$$

If $\dim_{\kappa(P)} H_0(I_l, \rho_P) = \dim H_0(I_l, \rho_{\mathbb{I}})$, by this, we have $a(l) \mod P = a'(l) \mod P'$. If we have an inequality $\dim_{\kappa(P)} H_0(I_l, \rho_P) > \dim H_0(I_l, \rho_{\mathbb{I}})$, we have $\dim H_0(I_l, \rho_{\mathbb{I}}) = 0$ as $l|N \Leftrightarrow l|N(\mathfrak{C})D$ (so, $\dim_{\kappa(P)} H_0(I_l, \rho_P)$ is at most 1). Then $\mathrm{ord}_l(N) = 1 + \mathrm{ord}_l(N(\mathfrak{C})D) = 2$, and we have a l-old component $\mathbb{I}''$ with $\rho_{\mathbb{I}''} = \rho_{\mathbb{I}'}$ in which $U(l) = 0$. Indeed, writing F for the $\mathbb{I}'$-adic form associated to the component $\mathbb{I}'$, $\mathbb{I}''$ is associated to $F' = F - a'(l)F|[l](q)$ which still has level a factor of N (see the proof of Corollary 1.2). In this way, we can find an irreducible component $\mathrm{Spec}(\mathbb{I}'') \subset \mathrm{Spec}(\mathbb{I}^M_{cm})$ with which $\mathrm{Spec}(\mathbb{I})$ intersects at P. However, as we have proven in Theorem 5.1, this is impossible, and $\mathfrak{s}_P \subset \mathfrak{H}$ with absolutely irreducible ρ_P does not happen.

Now we may assume that $\mathfrak{s}_P \subset \mathfrak{B}$ (including $\mathfrak{s}_P = 0, \mathfrak{U}$ in Case II) and either (i) that $\mathfrak{s}_P$ has nontrivial nilpotent radical or (ii) that ϱ_P is completely reducible over the entire $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (as these cases are the only cases we have not dealt with). Since conjugation by $\varrho_P(\sigma)$ has to preserve $\mathfrak{s}_P$ and its nilpotent radical, in Case (i), ϱ_P has to be reducible. In Case (ii), it is reducible by assumption. Again write

$$\rho_P^{ss} = \begin{pmatrix} \theta_P & 0 \\ 0 & \psi_P \end{pmatrix}.$$

The prime-to-p conductor of ρ_P^{ss} is the product $C(\theta_P)C(\psi_P)$ of the prime-to-p conductors $C(\theta_P)$ and $C(\psi_P)$. Thus we have $C(\theta_P)C(\psi_P)|N$. By (Gal), we may assume that ψ_P is unramified at p. Thus ψ_P is only ramified at prime factors of N prime to p. By class field theory, the image of the inertia group I_l at l in the abelianization of the decomposition group D_l at l is isomorphic to the almost l-profinite group $\mathbb{Z}_l^\times$. Thus $\psi_P|_{I_l}$ with values in almost p-profinite group $\mathbb{F}[[X]]^\times$ has to be of finite order. Then by global class field theory, ψ_P is of finite order. Thus ψ_P has values in $\mathbb{F}^\times$, and we have a unique Teichmüller lift $\psi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W^\times$ of ψ_P. Now we consider the ray class group $Cl_{\mathbb{Q}}(Np^n)$ and $Y = \varprojlim_n Cl_{\mathbb{Q}}(Np^n) \cong (\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times)/\{\pm 1\}$. By class field theory, for the maximal ray class field $\mathbb{Q}[\mu_{Np^\infty}]/\mathbb{Q}$ of $\mathbb{Q}$ modulo Np^∞, we have a canonical isomorphism $\mathrm{Gal}(\mathbb{Q}[\mu_{Np^\infty}]/\mathbb{Q}) \cong Y$. We identify these two groups. Write Y_p for the Sylow p-profinite subgroup of Y; so, $Y = Y^{(p)} \times Y_p$ canonically for the finite group $Y^{(p)}$ of order prime to p. We consider the group algebra $W[[Y_p]]$ and for $u \in Y$, we write $[u_p]$ for the group element in $Y_p \subset W[[Y_p]]^\times$ represented by the projection of u in Y_p. By the same deformation argument as above, for the Teichmüller lift $\theta_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W^\times$ of $\theta_P \mod \mathfrak{m}_{\mathbb{I}}$, $(W[[Y_p]], \Theta)$ for $\Theta([u, \mathbb{Q}_p]) = \theta_0(u)[u_p] \in W[[Y_p]]$ for $u \in Y$ is the universal couple among all deformations $(A, \epsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to A^\times)$ of $(\mathbb{F}, \theta_0 \mod \mathfrak{m}_W)$.

Let Y_t be the maximal torsion subgroup of Y; so, $Y_t \supset Y^{(p)}$. We may assume that any character: $Y_t \to \overline{\mathbb{Q}}_p^\times$ actually has values in $W^\times$ by extending scalars if necessary. The maximal torsion free quotient of Y_p is canonically isomorphic to Γ, and we have a non-canonical decomposition $Y = Y_t \times \Gamma$. We identify $W[[\Gamma]]$ with $W[[T]]$ by $\gamma \mapsto (1+T)$. Since $W[[Y_p]] = W[Y_t][[\Gamma]]$, geometrically irreducible components of $\mathrm{Spec}(W[[Y_p]])$ are indexed by characters $\theta : Y_t \to \overline{\mathbb{Q}}_p^\times$ so that the component is given by the projection $\theta_* : W[[Y_p]] \twoheadrightarrow W[[T]]$ sending $y \in Y_t$ to $\theta(y)$ and γ to $(1+T)$. We call this component the θ-component. Take θ such that $\theta_* \circ \Theta \mod P' = \theta_P$ for $P' = P \cap \Lambda$ in $\mathbb{I}$. By (Gal), we have $\theta_* \circ \Theta(Frob_l) = \theta(Frob_l)\langle l \rangle$ for all primes l outside Np. Since θ_P gives rise to a point P' of an irreducible component $\mathrm{Spec}(W[[T]])$ of the universal deformation space $\mathrm{Spec}(W[[Y_p]])$ so that $\theta_P \equiv \theta_* \circ \Theta \mod P'$ (with $P' = P \cap \Lambda$). Consider the Λ-adic Eisenstein series $E(\theta, \psi)$. By our construction, ρ_P is isomorphic to $\psi \oplus (\theta_* \circ \Theta) \mod P'$. Then in a way similar to the CM case, we can find a possibly "old" Eisenstein component $\mathbb{I}'$ with Galois representation $\psi \oplus (\theta_* \circ \Theta)$ which intersects with $\mathbb{I}$ at P. Indeed, again by $l|C := C(\psi_P)C(\theta_P) \Leftrightarrow l|N$, mismatch of $\dim H_0(I_l, \rho_P)$ and $\dim H_0(I_l, \rho_{\mathbb{I}})$ could occur only when $l|(N/C)$ and $l|C(\xi)$ but $l \nmid C(\eta)$ for $\{\xi, \eta\} = \{\psi_P, \theta_P\}$. Then writing $\Xi(\eta)$ for the set of primes $l|(N/C)$ with the above divisibility/non-divisibility property, we consider the imprimitive characters ψ' (resp. θ') induced by ψ (resp. θ)

modulo $M_1 := C(\psi)\prod_{l\in\Xi(\psi)} l$ (resp. $M_2 := p\cdot C(\theta)\prod_{l\in\Xi(\theta)} l$). Then the Eisenstein series $E(\theta',\psi')$ has congruence modulo P with the $\mathbb{I}$-adic form.

By Corollary 2.2, under the assumption (2), the Eisenstein prime gets trivial after localization at P over (p). Thus ρ_P cannot be diagonal or upper triangular. The only remaining possibility is $\mathfrak{s}_P = \mathfrak{sl}(2)$, and hence $G_P^\circ = SL(2)$. □

Recall the involution $*$ of $\mathbb{F}_p((T))$ sending $\phi(T)$ to $\phi((1+T)^{-1}-1)$ and the subfield $\mathbb{F}_p((T))^* \subset \mathbb{F}_p((T))$ fixed by $*$. We now state a slightly stronger version of the theorem and give a proof.

Theorem 6.2. *Suppose* (H) *and that*

(1) *N is cube-free,*
(2) *if $N > 1$, p is prime to $l-1$ for all prime factor l of N.*

Let $\mathrm{Spec}(\mathbb{I})$ *be a non CM component and P be any prime divisor $P \in \mathrm{Spec}(\mathbb{I})$ over (p). Write E for the field generated over $\mathbb{F}_p((T))^*$ by $\mathrm{Tr}(Ad(\rho_P)(\sigma))$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inside an algebraic closure of $\kappa(P)$. If either $E \supset \mathbb{F}_p((T))$ or one of the conditions* (s) *or* (v) *in the introduction is satisfied, then the image of ρ_P contains an open subgroup of $SL_2(\mathbb{F}_p[[T]])$. If neither $E \supset \mathbb{F}_p((T))$ nor any of* (s) *and* (v) *is satisfied, there exists a quaternion algebra D over E splitting over $E[T]$ such that $\mathrm{Im}(\rho_P)$ contains an open subgroup of $D^\times \cap SL_2(E[T])$, where D is embedded in $M_2(E[T])$ by identifying $D \otimes_E E[T]$ with $M_2(E[T])$.*

Proof. Write $Ad : PGL(2) \to \mathrm{End}(\mathfrak{sl}(2))$ for the adjoint representation. Let $\mathcal{G}$ be a compact subgroup Zariski dense in $PGL_2(\mathbb{F}((X)))$, and define $E \subset \mathbb{F}((X))$ be the closed subfield generated by $\mathrm{Tr}(Ad(g))$ for all $g \in \mathcal{G}$. Then by Proposition 0.6 and Theorem 0.7 in [P], if the Zariski closure of $\mathcal{G}$ is $PGL(2)$, there exists an algebraic group $H_{/E}$ such that $H \times_E \mathbb{F}((X)) = PGL(2)$ and that $\mathcal{G}$ contains an open subgroup of $H(E)$.

We now apply Pink's results to $\mathcal{G}$ given by $\mathrm{Im}(\rho_P)$ modulo center. By Proposition 6.1, the Zariski closure of the image of ρ_P contains $SL(2)$. Since $\det(\rho_P)$ has infinite order, the Zariski closure of $\mathrm{Im}(\det(\rho_P))$ in $\mathbb{G}_m$ is $\mathbb{G}_m$; so, the Zariski closure of $\mathcal{G}$ is $PGL(2)$. Since $\kappa(P)$ is a local function field of characteristic p, the integral closure of $\mathbb{F}_p[[T]]$ in $\kappa(P)$ is isomorphic to $\mathbb{F}[[X]]$ for a variable $X \in \kappa(P)$ with a finite field extension $\mathbb{F}/\mathbb{F}_p$; so, $\kappa(P) = \mathbb{F}((X))$. Thus we may assume that the image $\mathcal{G}$ is contained in $GL_2(\mathbb{F}[[X]])$. Let $Ad(\varrho_P) = Ad(\rho_P) = Ad \circ \rho_P$ be the adjoint representation of ρ_P on $\mathfrak{sl}(2)$. By (Gal), we have $Tr(Ad(\rho_P)([\gamma^s,\mathbb{Q}_p])) = 1+(1+T)^s+(1+T)^{-s}$. Thus $\mathbb{F}_p((T))^*$ is contained in the closed subfield E in $\mathbb{F}((X))$ generated by $\mathrm{Tr}(Ad(\rho_P)|_{I_p})$ over $\mathbb{F}_p$ in $\mathbb{I}/P$. Since any closed subfield of $\mathbb{F}((X))$ is of the form $\mathbb{F}_q((x))$ for $x \in X\cdot\mathbb{F}[[X]]$, we find $E = \mathbb{F}_q((x)) \supset \mathbb{F}_p((T))^*$ with $0 \neq x \in X\cdot\mathbb{F}[[X]]$. Again by (Gal), the semi-simple part of $\varrho_P([\gamma^s,\mathbb{Q}_p])$ is conjugate to $\begin{pmatrix} (1+T)^{s/2} & 0 \\ 0 & (1+T)^{-s/2}\end{pmatrix}$. Therefore the Zariski closure of the semi-simplification of $\varrho_P|_{I_p}$ is a split torus $\mathcal{T}$ of $SL(2)_{/E[T]}$. Thus its intersection $\mathcal{T}_H = \mathcal{T} \cap H$ in $H_{/E}$ falls in the following two cases:

(1) $\mathcal{T}_H$ is split over E, and then the group H is split; so, $H_{/E} \cong PGL(2)_{/E}$;
(2) $\mathcal{T}_H$ is non-split in $H_{/E}$ and $\mathcal{T} \times_E E[T]$ becomes split.

In any case, if $E \supset \mathbb{F}_p((T))$, we are done. In Case (1), this shows the Galois image contains an open subgroup of $SL_2(E)$ for $E \supset \mathbb{F}_p((T))^*$. If $\overline{\epsilon}/\overline{\delta}(\sigma)$ is in $\mathbb{F}_p$ and has order ≥ 3 for $\sigma \in D_p$ (i.e., the condition (s) is satisfied), choosing $\rho_{\mathbb{I}}$ in its isomorphism class so that $\rho_P|_{D_p} = \left(\begin{smallmatrix}\epsilon & * \\ 0 & \delta\end{smallmatrix}\right)$ with δ unramified, we have $g = \rho_P(\sigma) \in \rho_P(D_p)$ such that $g = \left(\begin{smallmatrix}a & * \\ 0 & b\end{smallmatrix}\right)$ having eigenvalues a and b with $a^2 \not\equiv b^2 \mod \mathfrak{m}_{\mathbb{I}}$ (so, $a^{-1}b \not\equiv ab^{-1} \mod \mathfrak{m}_{\mathbb{I}}$). If further $a \mod \mathfrak{m}_{\mathbb{I}}$ and $b \mod \mathfrak{m}_{\mathbb{I}}$ both belong to $\mathbb{F}_p$, replacing g by $\lim_{n\to\infty} g^{p^n}$, we may assume that a, b are in $\mathbb{Z}_p^\times$. Then

$$\mathrm{Tr}(Ad(j\cdot\varrho_P([\gamma,\mathbb{Q}_p])) = 1+(1+T)ab^{-1}+(1+T)^{-1}a^{-1}b.$$

This shows that $T \in E$, and hence, $E \supset \mathbb{F}_p((T))$, and we are done under (s).

If E does not contain T, then $E[T]/E$ is a quadratic extension; so, we may regard $*$ as an automorphism of $E[T]$ over E. Let $\widetilde{H}$ be the pull-back of $H \subset H \times_E E[T] = PGL(2)$ in $GL(2)_{/E[T]}$. Then $\widetilde{H}$ is an inner form of $GL(2)$ over E whose E-points can be embedded into $GL_2(E[T])$. Let D be the E-subalgebra in $M_2(E[T])$ generated by $\widetilde{H}(E)$. Then D is a quaternion algebra. Thus we have

$$D = \left\{ \left(\begin{smallmatrix} a & b \\ \xi b^* & a^* \end{smallmatrix}\right) \middle| a, b \in E[T] \right\}$$

for some $\xi \in E^\times$, and $\widetilde{H}(A) = (D \otimes_E A)^\times$ for any E-algebra A. If $\xi \notin N_{E[T]/E}(E[T]^\times)$, D is a division algebra; so, H does not contain unipotent elements. Under (v), a non-trivial upper unipotent element u exists in $\mathrm{Im}(\rho_P)$. We may assume that $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Conjugating u by $\rho_P([\gamma^s, \mathbb{Q}_p])$, $\mathrm{Im}(\rho_P)$ contains $u = \left(\begin{smallmatrix} 1 & (1+T)^s \\ 0 & 1 \end{smallmatrix}\right)$. Thus under (v), $\mathrm{Im}(\rho_P)$ contains the full unipotent group

$$U = \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \middle| a, b \in \mathbb{F}_p[[T]] \right\}$$

and an open subgroup of $H(E)$, which generates an open subgroup of $PGL_2(E[T])$; so, we are done. If neither (s) nor (v) holds (and $T \notin E$), we can only say that $\mathrm{Im}(\rho_P)$ contains an open subgroup of $D^\times \cap SL_2(E[T])$. □

Remark 6.1. If (H) fails (or N is not cube-free), the μ-invariant of $L(\Psi_{\mathbb{I}}^-)$ could be positive as explained at the end of [H10]. Therefore, we could have a mod p congruence of the CM component of $\Psi_{\mathbb{I}}$ and a non CM component. However, this congruence might again be exhausted by congruences between CM components with different prime-to–p conductors, but we need to further scrutinize this complicated situation.

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