ON THE RANK OF MORDELL–WEIL GROUPS OF ABELIAN VARIETIES OF $GL(2)$-TYPE

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Abstract. Pick an elliptic curve E of conductor N defined over $\mathbb{Q}$ with good ordinary reduction at a prime p. We suppose that E is not anomalous at p up to quadratic unramified twists. Suppose that $E(k)$ is finite for a number field k and p is outside a finite explicit set of primes (independent of k). We will prove that almost all $\mathbb{Q}$-simple abelian varieties A of GL(2)-type (with prime-to-p conductor N) has finite $A(k)$, as long as the p-divisible group $A[p^\infty]$ contains a Galois module isomorphic to $E[p](\overline{\mathbb{Q}})$. We also give a positive rank version of this result.

1. Introduction

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$ and $\overline{\mathbb{F}}_p$ be a fixed algebraic closure of the finite field $\mathbb{F}_p$ of p elements. Throughout this paper, $k_{/\mathbb{Q}}$ denotes a (fixed) field extension inside $\overline{\mathbb{Q}}$ of finite degree. Such a field is called a number field.

An F-simple abelian variety (with a polarization) defined over a number field F is called, in this paper, "*of* GL(2)-*type*" if we have a subfield $K_A \subset \mathrm{End}^0(A_{/F}) = \mathrm{End}(A_{/F}) \otimes_{\mathbb{Z}} \mathbb{Q}$ of degree $\dim A$ (stable under Rosati-involution). Then, for the two-dimensional compatible system ρ_A of Galois representation of A with coefficients in K_A, K_A is generated by traces $\mathrm{Tr}(\rho_A(\mathrm{Frob}_{\mathfrak{l}}))$ of Frobenius elements $\mathrm{Frob}_{\mathfrak{l}}$ for F-primes $\mathfrak{l}$ of good reduction (i.e., the field K_A is uniquely determined by A). We always regard F as a subfield of the algebraic closure $\overline{\mathbb{Q}}$. Thus $O'_A := \mathrm{End}(A_{/F}) \cap K_A$ is an order of K_A. Write O_A for the integer ring of K_A. Replacing A by the abelian variety representing the group functor $R \mapsto A(R) \otimes_{O'_A} O_A$, we may choose A so that $O'_A = O_A$ in the F-isogeny class of A. Since the Mordell–Weil rank $\dim_{\mathbb{Q}} A(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a field extension k/F is determined by the F-isogeny class of A, we hereafter assume that $\mathrm{End}(A_{/F}) \cap K_A = O_A$ for any abelian variety of GL(2)-type over F. For two abelian varieties A and B of GL(2)-type over F, we say that A is *congruent to B modulo a prime p over F* if we have a prime factor $\mathfrak{p}_A$ (resp. $\mathfrak{p}_B$) of p in O_A (resp. O_B) and field embeddings $\sigma_A : O_A/\mathfrak{p}_A \hookrightarrow \overline{\mathbb{F}}_p$ and $\sigma_B : O_B/\mathfrak{p}_B \hookrightarrow \overline{\mathbb{F}}_p$ such that $(A[\mathfrak{p}_A] \otimes_{O_A/\mathfrak{p}_A, \sigma_A} \overline{\mathbb{F}}_p)^{ss} \cong (B[\mathfrak{p}_B] \otimes_{O_B/\mathfrak{p}_B, \sigma_B} \overline{\mathbb{F}}_p)^{ss}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-modules, where the superscript "ss" indicates the semi-simplification. Hereafter in this article, we

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always assume the field of definition F is equal to $\mathbb{Q}$ but that the evaluation field k is any number field (unless otherwise specified).

Let $E_{/\mathbb{Q}}$ be an elliptic curve. Writing the Hasse–Weil L-function $L(s,E)$ as a Dirichlet series $\sum_{n=1} a_n n^{-s}$ with $a_n \in \mathbb{Z}$ (i.e., $1+p-a_p = |E(\mathbb{F}_p)|$ for each prime p of good reduction for E), we call p *admissible* for E if E has good reduction at p and $a_p \mod p$ is not in $\Omega_E := \{\pm 1, 0\}$. Therefore, the maximal étale quotient of $E[p]$ over $\mathbb{Z}_p$ is **not** isomorphic to $\mathbb{Z}/p\mathbb{Z}$ up to unramified quadratic twists. By the Hasse bound $|a_p] \leq 2\sqrt{p}$, $p \geq 7$ is not admissible if and only if $a_p \in \Omega_E$ (so, 2 and 3 are not admissible). Thus if E does not have complex multiplication, the Dirichlet density of non-admissible primes is zero by a theorem of Serre as $L(s,E) = L(s,f)$ for a rational Hecke eigenform f (see [Se81, Théorème 15] and Section 8 in the text). A proto-typical theorem we prove is:

Theorem A. *Let $E_{/\mathbb{Q}}$ be an elliptic curve with $|E(k)| < \infty$. Let N be the conductor of E, and pick an admissible prime p for E. Consider the set $\mathcal{A}_{E,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of* GL(2)*-type with prime-to-p conductor N congruent to E modulo p over $\mathbb{Q}$. Then there exists an explicit finite set S_E of primes depending on N such that if $p \notin S_E$, almost all $A \in \mathcal{A}_{E,p}$ has Mordell–Weil k-rank* 0 *(i.e., we have $|A(k)| < \infty$).*

Here "almost all" means except for finitely many. The set $S = S_E$ will be specified for each elliptic curve $E_{/\mathbb{Q}}$ in Definition 5.1, and the definition of S_E is nothing to do with the $\mathbb{Q}$-rank of E. For a given $\mathbb{Q}$-rational elliptic curve E, there are density one set of primes at which E has ordinary good reduction. According to the minimalist conjecture, the "probability" of rational elliptic curves E with finite $E(\mathbb{Q})$ is expected to be $\frac{1}{2}$. As proven by Bhargava–Shankar, under reasonable ordering of elliptic curves, at least a positive proportion of $\mathbb{Q}$-rational elliptic curves has rank 0 (see [BS14a] and [BS14b]). For each pair (E,p) with $|E(\mathbb{Q})| < \infty$ and an admissible prime p for E, since E can be lifted to an infinite p-adic analytic family of $\mathbb{Q}$-simple abelian varieties of GL(2)-type of prime-to-p conductor equal to N (as we will see later), the set $\mathcal{A}_{E,p}$ is an infinite set. Thus the above theorem produces a lot of examples of $\mathbb{Q}$-simple abelian varieties with trivial Mordell–Weil $\mathbb{Q}$-rank (perhaps, a positive proportion of $\mathbb{Q}$-simple abelian varieties of GL(2)-type once we order them by their conductor and dimension).

Taking $k = \mathbb{Q}$ and applying the above theorem to the modular elliptic curves $X_0(N)$ for small N, we get the following corollary:

Corollary B. *Let N be one of* $11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$ *(all the cases when $X_0(N)$ is an elliptic curve with finite $X_0(N)(\mathbb{Q})$). As long as p is admissible for $X_0(N)$, we have $|A(\mathbb{Q})| < \infty$ for almost all A in $\mathcal{A}_{X_0(N),p}$.*

In these special cases, the set $S_{X_0(N)}$ is contained in the set of non-admissible primes (can be checked by the table by Stein and Cremona, or also theoretically except for $N = 11$ for which $a_5 = 1$, we know that for any square-free prime factor p of N is non-admissble as $a_p = \pm 1$, and the rest is just $2, 3$ which are plainly not admissible), and therefore the corollary follows from Theorem A. It is interesting to know if any exceptional $A \in \mathcal{A}_{X_0(N),p}$ with $|A(\mathbb{Q})| = \infty$ appears for small admissible prime p for

$X_0(N)$ in the above examples. We will prove similar results concerning the vanishing of the k-rank for a number field k, replacing the starting rational elliptic curve of $\mathbb{Q}$-rank 0 by a $\mathbb{Q}$-simple abelian variety B of GL(2)-type with k-rank 0, covering $\mathbb{Q}$-simple abelian varieties of GL(2)-type congruent to B.

In the rank one case, we prove, in Section 6, the following fact:

Theorem C. *Let the notation be as in* Thorem A. *Suppose* $\mathrm{rank}_{\mathbb{Z}} E(k) = 1$ *and that* p *is admissible for* E *outside* S_E. *Then, almost all* $A \in \mathcal{A}_{E,p}$ *has* k*-rank* 0 *if and only if there exists another member* $A' \in \mathcal{A}_{E,p}$ *such that* $|A'(k)| < \infty$. *Otherwise, almost all* $A \in \mathcal{A}_{E,p}$ *has* $\dim_{K_A} A(k) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$.

There is a rational rank 1 elliptic curve E^- which has conductor 37 and has root number -1 (there is another one E^+ with root number $+1$, and $J_0(37)$ is isogenous to $E^+ \times E^-$). The curve E^- has the smallest conductor among rational elliptic curves of positive rank (according to the table of Cremona), and the next one has conductor 43. A positive proportion of rational elliptic curves has both Mordell–Weil and analytic rank 1 by [BS14b], and the "proportion" of having rank 1 is conjectured to be $\frac{1}{2}$; so, assuming to have E with $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ in the theorem is natural. However we deal with the general case where $\mathrm{rank}_{\mathbb{Z}} E(k) = m \geq 1$ in Theorem 6.1. In Section 3, we prove that for almost all $A \in \mathcal{A}_{E,p}$ (for p admissible for E outside S_E), $A(k)$ has constant rank r over O_A, and in Section 6, we prove $r \leq m$ under the assumption that $\mathrm{rank}_{\mathbb{Z}} E(k) = m$. In Section 7, under some different set of assumptions, we extends the result of Theorems A and C to some of the primes in the exceptional set S_E.

We should be able to obtain a better result than Theorem C for rank 1 cases. If we knew an analogue of the parity conjecture for the Mordell–Weil rank for partially ordinary abelian varieties of GL(2)-type, assuming that the root number of $L(s, E_{/k})$ is equal to -1, plainly $|A'(\mathbb{Q})|$ would not be finite (as the root number is constant on $\mathcal{A}_{E,p}$ for admissible p outside S_E); so, we could conclude that almost all $A \in \mathcal{A}_{E,p}$ has $\dim_{K_A} A(k) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$ by Theorem C. Some results on the parity conjecture for the Selmer rank of abelian varieties of GL(2)-type can be found in [N06, Theorem 12.2.8]. Anyway, we hope to deal with the parity question for the Mordell–Weil rank in a future article.

As indicated in [HM97, §1], any given two Hecke eigen cusp form f, g (of weight 2) should have a congruence $f \equiv g \mod \mathfrak{P}$ for some prime $\mathfrak{P}$ of the field $\mathbb{Q}(f, g)$ generated by Hecke eigenvalues of f and g. Thus the attached abelian varieties A_f and A_g are congruent each other. If we could remove the ordinarity assumption (and the level restriction) for $\mathfrak{P}$ in the above results, we would be able to show that most of $\mathbb{Q}$-simple abelian varieties of GL(2)-type have Mordell–Weil rank 0 or 1 over their endomorphism field, since we are fairly close to see that most of rational elliptic curves have Mordell–Weil rank 0 or 1. This paper is written this hope in mind, though our proof really relies on our hypothesis of (partial) ordinarity.

Here is how to achieve our goal. Fix a prime $p \geq 5$. By the solution of Serre's mod p modularity conjecture by Khare–Wintenberger–Kisin, any $\mathbb{Q}$-simple abelian variety A of GL(2)-type has Hasse–Weil zeta function $L(s, f)$ for a cusp form f of weight 2 with $K_A = \mathbb{Q}(f)$. If f is p-ordinary, for a given integer N prime to p, congruence classes of A with prime-to-p conductor N, is given by a connected component of the big ordinary Hecke algebra $\mathbf{h} = \mathbf{h}(N)$ of prime-to-p level N. Indeed, Shimura's abelian subvarieties of the Jacobians of $X_1(Np^r)$ (for $r > 0$) parameterized by arithmetic points of $\mathrm{Spec}(\mathbb{T})$ form a congruence class. We therefore study irreducible components $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbb{T})$.

The normalization $\mathrm{Spec}(\widetilde{\mathbb{I}})$ of $\mathrm{Spec}(\mathbb{I})$ is finite flat over the Iwasawa algebra $\mathbb{Z}_p[[T]]$, and whose points P of codimension one and not in the special fiber correspond to ordinary p-adic modular eigenforms f_P. Among those points, many corresponds to modular classical eigenforms of weight 2 and level Np^r (for variable r), and such points are Zariski dense in $\mathrm{Spec}(\mathbb{I})$. A classical, well-known, and fundamental construction of Eichler–Shimura attaches to any modular cuspidal eigenform f of weight 2 an abelian variety A_f defined over $\mathbb{Q}$, of dimension the degree of the field $\mathbb{Q}(f)$ generated by the coefficients of f over $\mathbb{Q}$. We call $\mathbb{Q}(f)$ the *Hecke field* of f. For these abelian varieties A_f, one can consider the Mordell–Weil group $A_f(\mathbb{Q})$ and more generally, $A_f(k)$ for a number field k. The group $A_f(k)$ is a finitely generated abelian group. Let us set $\widehat{A}_f(k) = A_f(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We consider the following natural question: how does the Mordell–Weil group $\widehat{A}_f(k)$ varies as f varies among those cuspidal eigenforms of weight 2 in the family? For the Selmer/analytic λ-invariant, the variation was studied by Emerton–Pollack–Weston [EPW06] for $k = \mathbb{Q}$ (and they proved constancy over the irreducible family). Here their Selmer group is relative to the cyclotomic $\mathbb{Z}_p$-extension, and the Mordel–Weil group could be a small part of the Selmer group (concentrated to the zero at $s = 1$). Our partial answer to this question is that $\dim_{\mathbb{Q}(f)} A_f(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the fixed number field k is constant over the family except for finitely many f in the family (Theorems 3.1 and 3.3), and the dimension is often shown to be 0 if the parity of the functional equation of $L(s, A_{f/k})$ is even in the family. We recall the control theorem (Theorem 2.3) for these Mordell–Weil groups proved in [H14b] and apply the theorem to our present problem discussed as above.

Here is the notation used throughout the paper. Fix a prime p. Let $X_r = X_1(Np^r)_{/\mathbb{Q}}$ be the compactified moduli of the classification problem of pairs (E, ϕ) of elliptic curves E and an embedding $\phi : \mu_{Np^r} \hookrightarrow E[Np^r]$ as finite flat group schemes. Since $\mathrm{Aut}(\mu_{p^r}) = (\mathbb{Z}/p^r\mathbb{Z})^\times$, $z \in \mathbb{Z}_p^\times$ acts on X_r via $\phi \mapsto \phi \circ \overline{z}$ for the image $\overline{z} \in (\mathbb{Z}/p^r\mathbb{Z})^\times$. We write X_s^r $(s > r)$ for the quotient curve $X_s/(1+p^r\mathbb{Z}_p)$. The complex points $X_s^r(\mathbb{C})$ contains $\Gamma_s^r\backslash\mathfrak{H}$ as an open Riemann surface for $\Gamma_s^r = \Gamma_0(p^s) \cap \Gamma_1(Np^r)$. Write $J_{r/\mathbb{Q}}$ (resp. $J^r_{s/\mathbb{Q}}$) for the Jacobian of X_r (resp. X_s^r) whose origin is given by the infinity cusp ∞ of the modular curves. We regard J_r as the degree 0 component of the Picard scheme of X_r. For a number field k, we consider the group of k-rational points $J_r(k)$. The Hecke operator $U(p)$ acts on $J_r(k)$ and the p-adic limit $e = \lim_{n\to\infty} U(p)^{n!}$ is well defined on the Barsotti–Tate group $J_r[p^\infty]$ and the completed Mordell–Weil group $\widehat{J}_r(k) = J_r(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For a general abelian variety over a number field k, we put $\widehat{X}(k) = X(k) \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Let Γ be the maximal torsion-free quotient of $\mathbb{Z}_p^\times$ and identify it with $1 + p\mathbb{Z}_p$ if $p > 2$ and $1 + 4\mathbb{Z}_2$ if $p = 2$. Writing $\gamma = 1 + p \in \Gamma$ if p is odd and $\gamma = 5 \in \Gamma$ if $p = 2$, γ is a topological generator of the multiplicative group $\Gamma = \gamma^{\mathbb{Z}_p}$.

Let

$$h_r(\mathbb{Z}) = \mathbb{Z}[T(n), U(l) : l|Np, (n, Np) = 1] \subset \mathrm{End}(J_r),$$

and put $h_r(R) = h_r(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any commutative ring R. Then we define $\mathbf{h}_r = \mathbf{h}_r(\mathbb{Z}_p) = e(h_r(\mathbb{Z}_p))$. The restriction morphism $h_s(\mathbb{Z}) \ni h \mapsto h|_{J_r} \in h_r(\mathbb{Z})$ for $s > r$ induces a projective system $\{\mathbf{h}_r\}_r$ whose limit gives rise to the big ordinary Hecke algebra

$$\mathbf{h} = \mathbf{h}(N) := \varprojlim_r \mathbf{h}_r.$$

Writing $\langle l \rangle$ (the diamond operator) for the action of $l \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ identified with $\mathrm{Gal}(X_r/X_0(Np^r))$, we have an identity $l\langle l \rangle = T(l)^2 - T(l^2) \in h_r(\mathbb{Z}_p)$ for all primes $l \nmid Np$. Thus we have a canonical Λ-algebra structure $\Lambda = \mathbb{Z}_p[[\Gamma]] \hookrightarrow \mathbf{h}$. It is now well known that $\mathbf{h}$ is a free of finite rank over Λ and $\mathbf{h}_r = \mathbf{h} \otimes_\Lambda \Lambda/(\gamma^{p^r} - 1)$ (cf. [H86a], [GK13] or [GME, §3.2.6]). A prime P in $\bigcup_{r>0} \mathrm{Spec}(\mathbf{h}_r)(\overline{\mathbb{Q}}_p) \subset \mathrm{Spec}(\mathbf{h})(\overline{\mathbb{Q}}_p)$ is called an *arithmetic prime* of weight 2. Though the construction of the big Hecke algebra is intrinsic, to relate an algebra homomorphism $\lambda : \mathbf{h} \to \overline{\mathbb{Q}}_p$ killing $\gamma^{p^r} - 1$ for $r \geq 0$ to a classical Hecke eigenform, we need to fix (once and for all) an embedding $\overline{\mathbb{Q}} \xrightarrow{i_p} \overline{\mathbb{Q}}_p$ of the algebraic closure $\overline{\mathbb{Q}}$ in $\mathbb{C}$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We write i_∞ for the inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Picard functoriality gives injective limits $J_\infty(k) = \varinjlim_r \widehat{J}_r(k)$ and $J_\infty[p^\infty](k) = \varinjlim_r J_r[p^\infty](k)$, on which e again acts. Write $\mathcal{G} := e(J_\infty[p^\infty])$, which is called the Λ-adic Barsotti–Tate group in [H14a] and whose integral property was scrutinized there. We define the p-adic completion of $J_\infty(k)$:

$$\check{J}_\infty(k) = \varprojlim_n J_\infty(k)/p^n J_\infty(k).$$

These groups we call ind (limit) MW-groups. Since projective limit and injective limit are left-exact, the functor $R \mapsto J_\infty(R)$ is a sheaf with values in abelian groups on the fppf site over $\mathbb{Q}$ (we call such a sheaf an fppf abelian sheaf). Adding superscript or subscript "ord", we indicate the image of e. We studied in [H14b] the control theorems of

$$\check{J}_\infty(k)^{\mathrm{ord}} \text{ and its dual } \check{J}_\infty(k)^*_{\mathrm{ord}} := \mathrm{Hom}_{\mathbb{Z}_p}(\check{J}_\infty(k)^{\mathrm{ord}}, \mathbb{Z}_p), \tag{1.1}$$

which we recall in the following section.

The compact cyclic group Γ acts on these modules by the diamond operators. In other words, we identify canonically $\mathrm{Gal}(X_r/X_0(Np^r))$ for modular curves X_r and $X_0(Np^r)$ with $(\mathbb{Z}/Np^r\mathbb{Z})^\times$, and the group Γ acts on J_r through its image in $\mathrm{Gal}(X_r/X_0(Np^r))$. Thus $\check{J}_\infty(k)^{\mathrm{ord}}$ is a module over $\Lambda := \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ by $\gamma \leftrightarrow t = 1 + T$. The big ordinary Hecke algebra $\mathbf{h}$ acts on $\check{J}^{\mathrm{ord}}_\infty$ and J^{ord}_∞ as endomorphisms of functors.

Contents

2. Control theorems

For a $Z[U]$-modules X and Y, we call a $\mathbb{Z}[U]$-linear map $f : X \to Y$ a U-injection (resp. a U-surjection) if $\mathrm{Ker}(f)$ is killed by a power of U (resp. $\mathrm{Coker}(f)$ is killed by a power of U). If f is an U-injection and U-surjection, we call f is a U-isomorphism. In other words, f is a U-injection (resp. a U-surjection, a U-isomorphism) if after tensoring $\mathbb{Z}[U, U^{-1}]$, it becomes an injection (resp. a surjection, an isomorphism). We apply this notion of U-isomorphisms to the operator $U(p)$

As before, let k be a finite extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ or a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$ for a prime l. Let A_r be a abelian subvariety of J_r defined over k. Write A_s $(s > r)$ for the image of A_r in J_s under the morphism $\pi^* = \pi^*_{s,r} : J_r \to J_s$ induced by Picard functoriality from the projection $\pi = \pi_{s,r} : X_s \to X_r$. If A_r is Shimura's abelian subvariety [IAT, Theorem 7.14] attached to a Hecke eigenform f, we write $A_{f,s}$ for A_s to indicate this fact. We assume the following condition to have a good control of the Mordell–Weil group of $\widehat{A}_s(k)$ when s varies:

(A) We have a coherent sequence $\alpha_s \in \mathrm{End}(J_{s/\mathbb{Q}})$ (for all $s \geq r$) having the limit $\alpha = \varprojlim_s \alpha_s \in \mathrm{End}(J_{\infty/\mathbb{Q}})$ such that
  (a) A_s is the connected component of $J_s[\alpha_s] := \mathrm{Ker}(\alpha_s)$ with $J_s = A_s + \alpha_s(J_s)$ so that the inclusion: $A_s[p^\infty] \cong J_s[\alpha][p^\infty]$ is a $U(p)$-isomorphism,
  (b) the restriction $\alpha_s|_{\alpha(J_s)} \in \mathrm{End}(\alpha(J_s))$ is a self-isogeny.

Here for $s' > s$, coherency of α_s means the following commutative diagram:

$$\begin{array}{ccc} J_s & \xrightarrow{\pi^*} & J_{s'} \\ {\scriptstyle \alpha_s}\downarrow & & \downarrow{\scriptstyle \alpha_{s'}} \\ J_s & \xrightarrow[\pi^*]{} & J_{s'}. \end{array}$$

The Rosati involution $h \mapsto h^*$ and $T(n) \mapsto T^*(n)$ (with respect to the canonical divisor on J_r) brings $h_r(\mathbb{Z})$ to $h^*_r(\mathbb{Z}) \subset \mathrm{End}(J_{r/\mathbb{Q}})$. Define A^*_s to be the identity connected component of $J_s[\alpha^*]$. The condition (A) is equivalent to

(B) The abelian quotient map $J_s \twoheadrightarrow B_s = \mathrm{Coker}(\alpha)$ dual to $A^*_s \subset J_s$ induces an $U(p)$-isomorphism of p-adic Tate modules: $T_p(J_s/\alpha_s(J_s)) \to T_p B_s$ and α_s induces an automorphism of the $\mathbb{Q}_p$-vector space $T_p\alpha_s(J_s) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Again if A_r is Shimura's abelian subvariety of J_r associated to a Hecke eigenform f, we sometimes write $B_{f,s}$ for B_s as above. This abelian variety $B_{f,s}$ is the abelian variety quotient studied in [Sh73].

Take a connected (resp. an irreducible) component $\mathrm{Spec}(\mathbb{T})$ (resp. $\mathrm{Spec}(\mathbb{I})$) of $\mathrm{Spec}(\mathbf{h})$ and assume that $\mathbb{I}$ is primitive in the sense of [H86a, Section 3] or [H88, page 317]. For each arithmetic $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$, the corresponding cusp form f_P is a p-stabilized Hecke eigenform of weight 2 new at each prime $l|N$ if and only if $\mathbb{I}$ is primitive. We quote the following fact from [H14b, Proposition 5.1] giving sufficient conditions for the validity of (A) for $A_{f,s}$ when $f = f_P$ is in a p-adic anlytic family indexed by $P \in \mathrm{Spec}(\mathbb{I})$.

Proposition 2.1. *Let* $\mathrm{Spec}(\mathbb{T})$ *be a connected component of* $\mathrm{Spec}(\mathbf{h})$ *and* $\mathrm{Spec}(\mathbb{I})$ *be a primitive irreducible component of* $\mathrm{Spec}(\mathbb{T})$*. Then the condition* (A) *holds for the following choices of* (α, A_s, B_s)*:*

(1) *Suppose that an eigen cusp form $f = f_P$ new at each prime $l|N$ belongs to* $\mathrm{Spec}(\mathbb{T})$ *and that $\mathbb{T} = \mathbb{I}$ is regular (or more generally a unique factorization domain). Then writing the level of f_P as Np^r, the algebra homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ given by $f|T(l) = \lambda(T(l))f$ gives rise to the prime ideal $P = \mathrm{Ker}(\lambda)$. Since P is of height 1, it is principal generated by $\varpi \in \mathbb{T}$. This ϖ has its image $\varpi_s \in \mathbb{T}_s = \mathbb{T} \otimes_\Lambda \Lambda_s$ for $\Lambda_s = \Lambda/(\gamma^{p^{s-1}} - 1)$. Since $\mathbf{h}_s = \mathbf{h} \otimes_\Lambda \Lambda_s = \mathbb{T}_s \oplus X_s$ as an algebra direct sum, $\mathrm{End}(J_{s/\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \supset h_s(\mathbb{Z}_p) = \mathbb{T}_s \oplus Y_s$ with Y_s projecting down onto X_s. Then, we can approximate $a_s = \varpi_s \oplus 1_s \in h_s(\mathbb{Z}_p)$ for the identity 1_s of Y_s by $\alpha_s \in h_s(\mathbb{Z})$ so that $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$ (hereafter we call α_s "sufficiently close" to a_s if $\alpha_s h_s(\mathbb{Z}_p) = a_s h_s(\mathbb{Z}_p)$). For this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*
(2) *More generally than* (1)*, we pick a general connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h})$*. Pick a (classical) Hecke eigenform $f = f_P$ (of weight 2) for $P \in \mathrm{Spec}(\mathbb{T})$. Assume that $\mathbf{h}_s$ (for every $s \geq r$) is reduced and $P = (\varpi)$ for $\varpi \in \mathbb{T}$, and write ϖ_s for the image of ϖ in $h_s(\mathbb{Z}_p)$. Take the complementary direct summand Y_s of $\mathbb{T}_s$ in $h_s(\mathbb{Z}_p)$ and approximate $a_s := \varpi_s \oplus 1_s$ in $h_s(\mathbb{Z}_p)$ to get α_s sufficiently close to a_s. Then for this choice of α_s, $A_s := A_{f,s}$ and $B_s := B_{f,s}$.*
(3) *Fix $r > 0$. Then $\alpha_s = \alpha$ for a factor $\alpha|(\gamma^{p^{r-1}} - 1)$ in Λ, let $A_s = J_s[\alpha]^\circ$ (the identity connected component) and $B_s = \mathrm{Pic}^0_{A_s/\mathbb{Q}}$ for all $s \geq r$.*

Consider the Hecke algebra $h_2(\Gamma_1(N)) = \mathbb{Z}[T(n)|n = 1, 2, \ldots] \subset \mathrm{End}(J_1(N))$. Then by the diamond operators, $h_2(\Gamma_1(N))$ is naturally an algebra over the group algebra $\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times]$. For each character χ of $(\mathbb{Z}/N\mathbb{Z})^\times$, writing $\mathbb{Z}[\chi]$ for the subalgebra of $\overline{\mathbb{Q}}$ generated by the values of χ, we put

$$h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi]) = h_2(\Gamma_1(N); \mathbb{Z}) \otimes_{\mathbb{Z}[(\mathbb{Z}/N\mathbb{Z})^\times],\chi} \mathbb{Z}[\chi].$$

Let D_χ be the discriminant of the reduced quotient of $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$ over $\mathbb{Z}[\chi]$. Here is an easy criterion from [F02, Theorem 3.1] for the condition (1) in the above proposition to be met:

Theorem 2.2. *Let f be a Hecke eigenform of conductor N, of weight 2 and with Neben character χ, and define $a_p \in \overline{\mathbb{Q}}$ by $f|T(p) = a_p f$. Let p be a prime outside*

$2D_\chi N\varphi(N)$ (for $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$). Suppose that for the prime ideal $\mathfrak{p}$ of $\mathbb{Z}[a_p]$ induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, ($a_p$mod $\mathfrak{p}$) is different from 0 and $\pm\sqrt{\chi(p)}$. Then for the connected component Spec($\mathbb{T}$) *acting non-trivially on the p-stabilized Hecke eigenform corresponding to f in $S_2(\Gamma_0(Np), \chi)$, $\mathbb{T}$ is a regular integral domain isomorphic to $W \otimes_{\mathbb{Z}_p} \Lambda = W[[T]]$ for a complete discrete valuation ring W unramified at p.*

Here is a short proof of this fact since the statement of [F02, Theorem 3.1] is slightly different from the above theorem.

Proof. Since $p \nmid 2D_\chi N\varphi(N)$, we have $p > 2$ and $p \nmid \varphi(Np)$. By the diamond operators $\langle z\rangle$ for $z \in (\mathbb{Z}/Np\mathbb{Z})^\times$, $\mathbf{h}$ is an algebra over $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times]$. Thus we can decompose $\mathbf{h} = \oplus_\psi \mathbf{h}(\psi)$ so that the diamond operator $\langle z\rangle$ for $z \in (\mathbb{Z}/Np\mathbb{Z})^\times$ acts by $\psi(z)$ on $\mathbf{h}(\psi)$, where ψ runs over all even characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$. From the exact control $\mathbf{h}/T\mathbf{h} \cong \mathbf{h}_1$ ($T = \gamma - 1 \in \Lambda$), we thus get

$$\mathbf{h}(\chi)/T\mathbf{h}(\chi) \cong h_2^{\mathrm{ord}}(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) =: h$$

for the character χ of $(\mathbb{Z}/Np\mathbb{Z})^\times$, where

$$h_2(\Gamma_0(Np), \chi; \mathbb{Z}_p[\chi]) = h_2(\Gamma_1(Np), \chi; \mathbb{Z}) \otimes_{\mathbb{Z}[(\mathbb{Z}/Np\mathbb{Z})^\times],\chi} \mathbb{Z}_p[\chi]$$

and $\mathbb{Z}_p[\chi]$ is the $\mathbb{Z}_p$-subalgebra of $\overline{\mathbb{Q}}_p$ generated by the values of χ. Here the tensor product is with respect to the algebra homomorphism $\mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \to \mathbb{Z}_p[\chi]$ induced by χ. Writing $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(\mathbf{h}(\chi), \overline{\mathbb{F}}_p)$, for each $\lambda \in \Sigma$, $\overline{\Sigma} := \{\mathfrak{m}_\lambda = \mathrm{Ker}(\lambda) | \lambda \in \Sigma\}$ is the set of all maximal ideals of $\mathbf{h}(\chi)$. Thus we have compatible decompositions $\mathbf{h}(\chi) = \bigoplus_{\mathfrak{m}\in\overline{\Sigma}} \mathbf{h}(\chi)_\mathfrak{m}$ and $h = \bigoplus_{\mathfrak{m}\in\overline{\Sigma}} h_\mathfrak{m}$ (see [BCM, III.4.6]). Here the subscript "$\mathfrak{m}$" indicates the localizations at the maximal ideal $\mathfrak{m}$.

Identify Σ with $\mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{F}}_p)$. Write Σ° for the subset of $\Sigma = \mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{F}}_p)$ made of λ's such that there exists

$$\lambda^\circ \in \mathrm{Hom}_{\mathrm{alg}}(h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]), \overline{\mathbb{F}}_p)$$

with $\lambda(T(l)) = \lambda^\circ(T(l))$ for all primes $l \nmid pN$. Here we put

$$h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{F}_p[\chi]) := h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi]) \otimes_{\mathbb{Z}} \mathbb{F}_p.$$

Accordingly let $\overline{\Sigma}^\circ$ denote the set of maximal ideals corresponding to $\lambda \in \Sigma^\circ$. Since p-new forms in $S_2(\Gamma_0(Np), \chi)$ have $U(p)$-eigenvalues $\pm\sqrt{\chi(p)}$ (see [MFM, Theorem 4.6.17]), by $a_p \not\equiv \pm\sqrt{\chi(p)} \mod \mathfrak{p}$, we have further decomposition $h = h_N \oplus h'$ so that h_N is the direct sum of $h_\mathfrak{m}$ for $\mathfrak{m}$ running over $\overline{\Sigma}^\circ$. Since $\mathbf{h}(\chi)/T\mathbf{h}(\chi) \cong h$, by Hensel's lemma (e.g., [BCM, III.4.6]), we have a unique algebra decomposition $\mathbf{h}(\chi) = \mathbf{h}_N \oplus \mathbf{h}'$ so that $\mathbf{h}_N/T\mathbf{h}_N = h_N$ and $\mathbf{h}'/T\mathbf{h}' = h'$.

Since $T(p) \equiv U(p) \mod (p)$ in h_N, we have $h_N \cong h_2^{\mathrm{ord}}(\Gamma_0(N), \chi; \mathbb{Z}_p[\chi])$. Since $p \nmid D_\chi$, the reduction map modulo p: $\mathrm{Hom}_{\mathrm{alg}}(h, \overline{\mathbb{Q}}_p) \to \Sigma$ is a bijection. In particular, we have $h = h^{new} \oplus h^{old}$ where h^{new} is the direct sum of $h_{\mathfrak{m}_\lambda}$ for λ coming from the eigenvalues of N-primitive forms. Again by Hensel's lemma, we have the algebra decomposition $\mathbf{h}_N = \mathbf{h}^{new} \oplus \mathbf{h}^{old}$ with $\mathbf{h}^?/T\mathbf{h}^? = h^?$ for $? = new, old$. Since h^{new} is reduced by the theory of new forms ([H86a, §3] and [MFM, §4.6]) and unramified over $\mathbb{Z}_p$ by $p \nmid D_\chi\varphi(N)$, we conclude $h^{new} \cong \bigoplus_W W$ for discrete valuation rings W finite unramified over $\mathbb{Z}_p$ (one of the direct summand W acts on f non-trivially; i.e.,

W given by $\mathbb{Z}_p[f] = \mathbb{Z}_p[a_n|n = 1, 2, \dots] \subset \overline{\mathbb{Q}}_p$ for $T(n)$-eigenvalues a_n of f). Thus again by Hensel's lemma, we have a unique algebra direct factor $\mathbb{T}$ of $\mathbf{h}^{new}$ such that $\mathbb{T}/T\mathbb{T} = \mathbb{Z}_p[f] = W$. Since W is unramified over $\mathbb{Z}_p$, by the theory of Witt vectors [BCM, IX.1], we have a unique section $W \hookrightarrow \mathbb{T}$ of $\mathbb{T} \twoheadrightarrow \mathbb{Z}_p[f] = W$. Then $W[[T]] \subset \mathbb{T}$ which induces a surjection after reducing modulo T. Then by Nakayama's lemma, we have $\mathbb{T} = W[[T]] = W \otimes_{\mathbb{Z}_p} \Lambda$ as desired. □

Recall the module $\check{J}_\infty(k)^*_{\mathrm{ord}}$ defined in (1.1). We define, for an $\mathbf{h}$-algebra A,

$$\check{J}_\infty(k)^*_{\mathrm{ord},\,A} := \check{J}_\infty(k)^*_{\mathrm{ord}} \otimes_{\mathbf{h}} A \quad\text{and}\quad \mathcal{G}_A(k) = \mathcal{G}(k) \otimes_{\mathbf{h}} A. \tag{2.1}$$

For the use in the following section, we quote here the control theorem in [H14b, Theorem 6.5] as follows:

Theorem 2.3. *Assume that* (α_s, A_s, B_s) *satisfies the condition* (A), *and let* k *be a number field. Write* $\mathrm{Spec}(\mathbb{T})$ *for a connected component of* $\mathrm{Spec}(\mathbf{h})$ *such that* α *projected to the complement of* $\mathbb{T}$ *in* $\mathbf{h}$ *is a unit. Then, the following sequence*

$$\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\alpha} \check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \widehat{A}_r(k)^*_{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0$$

is an exact sequence of p*-adic* $\mathbb{Q}_p$*-Banach* $\mathbf{h}$*-modules (with respect to the Banach norm having the image of* $\check{J}_\infty(k)^*_{\mathrm{ord}}$ *in* $\check{J}_\infty(k)^*_{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *as its closed unit ball), and the module* $\check{J}_\infty(k)^*_{\mathrm{ord},\mathbb{T}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is a* $\Lambda[\frac{1}{p}]$*-module of finite type.*

3. Constancy of Mordell–Weil rank and naïve questions

We continue to use the notation introduced at the end of the previous section. Take a connected (resp. a primitive irreducible) component $\mathrm{Spec}(\mathbb{T})$ (resp. $\mathrm{Spec}(\mathbb{I})$) of $\mathrm{Spec}(\mathbf{h})$ (resp. of $\mathrm{Spec}(\mathbb{T})$) and assume that $\mathbb{I}$ is primitive in the sense of [H86a, Section 3]. Let $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ be an arithmetic point of weight 2. We write $\kappa(P)$ for the residue field of P (i.e., the quotient field of the image $P(\mathbb{I})$ in $\overline{\mathbb{Q}}_p$). Then we have a unique classical Hecke eigenform $f_P \in S_2(\Gamma_0(Np^{r(P)}), \chi\epsilon_P)$ for a character χ of $(\mathbb{Z}/Np\mathbb{Z})^\times$ and a character $\epsilon_P : \Gamma \to \mu_{p^\infty}(\overline{\mathbb{Q}}_p)$ such that $f_P|T(l) = P(T(l))f_P$ for all primes l. The p-power root of unity $\epsilon_P(\gamma)$ has order $\le p^{r(P)}$. We suppose that $Np^{r(P)}$ is the minimal possible level of f_P (indeed, if $\chi\epsilon_P|_{\mathbb{Z}_p^\times} \neq 1$, f_P is primitive of conductor $Np^{r(P)}$, and otherwise, f_P is the p-stabilized form associated to a primitive form of conductor C with $N|C|Np$). Thus $\mathbb{I}$ gives rise to a family of p-adic Hecke eigen cusp forms f_P new at each prime $l|N$ which in turn has the associated abelian subvariety $A_P := A_{f_P}$ (of J_r for $r = r(P)$) and the (isogenous) abelian variety quotient $B_P := B_{f_P}$ associated to f_P. The abelian varieties A_P and B_P are all $\mathbb{Q}$-simple (e.g., [R75], [R80] and [R81]). Write $\mathbb{Q}(f_P)$ for the Hecke field of f_P generated by all Hecke eigenvalues of f_P. We also define $\mathbb{Q}_p(f_P)$ for the p-adic closure of $i_p(\mathbb{Q}(f_P))$ in $\overline{\mathbb{Q}}_p$. Then $\mathbb{Q}_p(f_P) = \kappa(P)$.

Let $\widetilde{\mathbb{I}}$ be the normalization of $\mathbb{I}$ (in its quotient field). Then $\widetilde{\mathbb{I}}[\frac{1}{p}]$ is a Dedekind domain (cf. [CRT, Theorem 11.6]). Since $\check{J}_\infty(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}[\frac{1}{p}]}$ is an $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module of finite type by Theorem 2.3, its localization at any prime divisor P is isomorphic to $\widetilde{\mathbb{I}}[\frac{1}{p}]_P^{r_k(\mathbb{I})} \times X_{\mathbb{I},P}$

for a torsion $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module $X_{\mathbb{I}}$ of finite type and an integer $r_k(\mathbb{I}) \geq 0$ independent of P (see [BCM, VII]). Indeed, $X_{\mathbb{I}}$ is the maximal $\widetilde{\mathbb{I}}[\frac{1}{p}]$-torsion submodule of $\check{J}_\infty(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}[\frac{1}{p}]}$. The support $\mathrm{Supp}(X_{\mathbb{I}})$ (if non-empty) is a union of finitely many prime divisors of $\mathrm{Spec}(\widetilde{\mathbb{I}})$ (and the maximal point). We call $r_k(\mathbb{I})$ the $\widetilde{\mathbb{I}}[\frac{1}{p}]$-*free rank* of $\check{J}_\infty(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}[\frac{1}{p}]}$.

We start with $A_r = A_P$ and $B_r = B_P$ ($r = r(P)$). If (A) is satisfied for $\{A_s\}_s$, the abelian variety A_s in (A) is given by $\pi^*_{s,r}(A_P)$, and let B_s denote the dual quotient $J_s \twoheadrightarrow B_s$ of $A^*_s = w_s(A_s) \subset J_s$ for Weil involution w_s (cf. [H14b, §5]). We sometimes write $A_{P,s} = A_{f_P,s}$ for A_s and $B_{P,s} = B_{f_P,s}$ for B_s. In the first of the following two theorems, we assume the condition (A) in Section 2 for $(A_s, B_s, J_s)_s$. In Theorem 3.3, we do not assume (A) for A_P and B_P.

Theorem 3.1. *Let the notation be as above. Suppose infinity of the set $\mathcal{A}$ of arithmetic points P of weight 2 of* $\mathrm{Spec}(\mathbb{I})$ *for which the condition* (A) *with $A_s = A_{P,s} = A_{f_P,s}$ ($s \geq r = r(P)$) is satisfied. Then, for a number field k, except for finitely many arithmetic points $P \in \mathcal{A}$ inside* $\mathrm{Supp}(X_{\mathbb{I}})$*, the Mordell–Weil rank* $\dim_{\mathbb{Q}(f_P)} A_P(k)\otimes_{\mathbb{Z}}\mathbb{Q}$ *of A_P (and hence of B_P) over $\mathbb{Q}(f_P)$ is constant equal to the number $r_k(\mathbb{I})$.*

As in Proposition 2.1 (1), if $\mathbb{I} = \mathbb{T}$ and $\mathbb{I}$ is regular, $\mathcal{A}$ is the entire set of arithmetic points of $\mathrm{Spec}(\mathbb{I})$ of weight 2. By Theorem 2.2, most of the cases, this condition is satisfied.

Proof. In the proof, we write $X := X_{\mathbb{I}}$ and $R := r_k(\mathbb{I})$ for simplicity. Thus, for $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ with $X_P = X \otimes_{\widetilde{\mathbb{I}}} \widetilde{\mathbb{I}}_P = 0$ (for the localization $\widetilde{\mathbb{I}}_P$ of $\widetilde{\mathbb{I}}$), we have

$$\dim_{\kappa(P)} \widehat{A}_P(k)^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \dim_{\kappa(P)} J_\infty(k)^*_{\mathrm{ord},\,\kappa(P)} = \dim_{\kappa(P)} \kappa(P)^R = R$$

for $\kappa(P) = \widetilde{\mathbb{I}}_P/P\widetilde{\mathbb{I}}_P$.

Because of infinity of the set $\mathcal{A}$, we have infinity of $P \in \mathcal{A}$ with $X \otimes_{\widetilde{\mathbb{I}}} \widetilde{\mathbb{I}}_P = 0$. Suppose this for $P \in \mathcal{A}$. Since $\mathbb{I}$ is primitive, it is étale over at each arithmetic point of Λ (see [HMI, Proposition 3.78]). Thus we have $\widetilde{\mathbb{I}}_P = \mathbb{I}_P$. Therefore $\kappa(P) = \mathbb{I}_P/P\mathbb{I}_P = \mathbb{Q}_p(f_P)$ and hence

$$\dim_{\mathbb{Q}_p(f_P)} \widehat{A}_P(k)^{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \dim_{\kappa(P)} \widehat{A}_P(k)^{\text{co-ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = R.$$

Recall the Hecke field $\mathbb{Q}(f_P)$ of f_P generated over $\mathbb{Q}$ by all $T(n)$-eigenvalues of f ($n = 1, 2, \dots$). Then $A_P(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}(f_P)$-vector space. Write d for its dimension, and fix an isomorphism $A_P(k) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}(f_P)^d$. The field $\kappa(P)$ is a completion of $\mathbb{Q}(f_P)$ at the prime $\mathfrak{p}$ over p (induced by i_p), and writing $a(p)$ for the eigenvalue of $U(p)$ for f_P, we have $a(p) \notin \mathfrak{p}$. Thus $(\widehat{A}_P(k)^*_{\mathrm{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cong \mathbb{Q}(f_P)^d_{\mathfrak{p}} = \mathbb{Q}_p(f_P)^d = \kappa(P)^R$. This shows the result. □

Definition 3.2. *Suppose $p \nmid N$. We call a connected component* $\mathrm{Spec}(\mathbb{T})$ *new if f_P is new at all primes $l|N$ for all arithmetic points $P \in$* $\mathrm{Spec}(\mathbb{T})(\overline{\mathbb{Q}}_p)$*.*

Instead of assuming infinity of the set $\mathcal{A}$ and primitivity of $\mathbb{I}$, to obtain a simialr constancy of the Mordell–Weil rank, we could assume that $\mathbb{T}_s = \mathbb{T}/(\gamma^{p^{s-1}} - 1)\mathbb{T}$ is reduced for all $s > 0$ (and hence $\mathbb{T} = \varprojlim_s \mathbb{T}_s$ is reduced). This condition of reducedness of $\mathbb{T}_s$ is known either if N is cube-free (see [H13a, Corollary 1.2]) or if $\mathbb{T}$ is new (see the theory of primitive components in [H86a, Section 3]). There could be a rare exception of the reduced-ness of $\mathbb{T}_s$ if ρ_P (with $s \geq r(P)$) is unramified at some primes $l|N$ (so, f_P and $\mathbb{I}$ are l-old) such that the semi-simplification of $\rho(\mathrm{Frob}_l)$ is a scalar for some arithmetic point P of weight 2 (such a case is not expected to occur in the elliptic modular case and is proven to be never the case if N is cube-free [CE98]). By Proposition 2.1 (3), the condition (A) is satisfied for any factor of $\gamma^{p^{r-1}} - 1$ ($r \geq 0$) in Λ as long as $\mathbb{T}_s$ is reduced for all $s \geq 0$. Note that $\Lambda[\frac{1}{p}]$ is a principal ideal domain. Since $J := \check{J}_\infty(k)^*_{\mathrm{ord},\ \mathbb{T}[\frac{1}{p}]}$ is a $\Lambda[\frac{1}{p}]$-module of finite type unconditionally by Theorem 2.3, we have an isomorphism $J \cong \Lambda[\frac{1}{p}]^R \times X$ for a torsion $\Lambda[\frac{1}{p}]$-module X.

Theorem 3.3. *Let the notation be as above. Pick a local ring $\mathbb{T}$ of $\mathbf{h}$, and assume either that $\mathbb{T}_s = \mathbb{T}/(\gamma^{p^{s-1}} - 1)\mathbb{T}$ is reduced for all $s > 0$ or N is cube-free or $\mathbb{T}$ is new. Let k be a number field. Then for each irreducible component $\mathrm{Spec}(\mathbb{I})$ of $\mathrm{Spec}(\mathbb{T})$, if an arithmetic points P of $\mathrm{Spec}(\mathbb{I})$ of weight 2 is outside the support of X in $\mathrm{Spec}(\Lambda[\frac{1}{p}])$, we have $\dim_{\mathbb{Q}(f_P)} A_P(k) \otimes_{\mathbb{Z}} \mathbb{Q} = r_k(\mathbb{I})$.*

As already remarked, if N is cube-free or $\mathbb{T}$ is new, $\mathbb{T}_s$ is reduced for all $s > 0$. Thus we prove the theorem assuming the reduced-ness of $\mathbb{T}_s$ all $s > 0$.

Proof. Write $M_{\mathbb{T}}$ for $M \otimes_{\mathbf{h}} \mathbb{T}$ for an $\mathbf{h}$-module M. Pick a prime factor $(\varpi)|(\gamma^{p^r} - 1)$ outside the support of X in $\mathrm{Spec}(\Lambda)$. Since $\gamma^{p^r} - 1 \in \mathrm{End}(J_s)$ can be factored into a product of primes inside $\mathrm{End}(J_s)$, we may assume $\varpi \in \mathrm{End}(J_s)$, and write this choice of ϖ as ϖ_s for each $s \geq r$. By Proposition 2.1 (3), $A_s = J_s[\varpi_s]^\circ$ satisfies the condition (A); so, we can apply Theorem 2.3 to $\{A_s\}_{s \geq r}$. Writing the localization of a $\mathbb{T}$-module M at (ϖ) as $M_{(\varpi)}$, we get $J_{(\varpi)} \cong \Lambda^r_{(\varpi)}$. Write $\mathrm{Spec}(\mathbb{T}) = \bigcup_{\mathbb{I}} \mathrm{Spec}(\mathbb{I})$ for its irreducible components $\mathbb{I}$. Then we have $\mathbb{T}_{(\varpi)} = \bigoplus_{\mathbb{I}} \mathbb{I}_{(\varpi)}$ and $\mathbb{I}$'s are domains (from reduced-ness of $\mathbb{T}$). Since $\mathrm{Spec}(\mathbb{T}_{(\varpi)})$ is étale over $\mathrm{Spec}(\Lambda_{(\varpi)})$ by [HMI, Proposition 3.78], $\mathbb{I}_{(\varpi)}$ is a Dedekind domain with finitely many maximal ideals. Then as $\mathbb{T}_{(\varpi)}$-modules, we have $J_{(\varpi)} = \bigoplus_{\mathbb{I}} \mathbb{I}_{(\varpi)}^{r_{(\varpi)}(\mathbb{I})}$ for $0 \leq r_{(\varpi)}(\mathbb{I}) \in \mathbb{Z}$ with $R = \sum_{\mathbb{I}} r_{(\varpi)}(\mathbb{I})[Q(\mathbb{I}) : Q]$ for the quotient field $Q(\mathbb{I})$ (resp. Q) of $\mathbb{I}$ (resp. Λ). Writing $\widetilde{\mathbb{I}}$ for the normalization of $\mathbb{I}$ in $Q(\mathbb{I})$, we have $\widetilde{\mathbb{I}}_{(\varpi)} = \mathbb{I}_{(\varpi)}$ as $\mathrm{Spec}(\mathbb{T}_{(\varpi)})$ is étale over Λ. Thus $r_{(\varpi)}(\mathbb{I})$ is equal to $r_k(\mathbb{I}) = \dim_{Q(\mathbb{I})} J \otimes_{\mathbb{T}} Q(\mathbb{I})$, and it is independent of arithmetic primes (ϖ) outside the support of X. Thus, sometimes, we simply write $r(\mathbb{I}) = r_k(\mathbb{I}) = r_{(\varpi)}(\mathbb{I})$.

For $s \geq r$, we consider the identity connected component $A_s \subset J_s[\varpi]$ and its dual quotient $J_s \twoheadrightarrow B_s$. Since $\mathbb{T}$ is a direct factor of $\mathbf{h}$, tensoring the exact sequence in Theorem 2.3 with $\mathbb{T}$ over $\mathbf{h}$, we get the following exact sequence

$$0 \to J \xrightarrow{\varpi} J \xrightarrow{\pi_\infty} \widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}[\frac{1}{p}]} \to 0.$$

Localizing at P, we get another exact sequence

$$0 \to J_{(\varpi)} \xrightarrow{\varpi} J_{(\varpi)} \xrightarrow{\pi_\infty} \widehat{A}_r(k)^*_{\mathrm{ord},\ \mathbb{T}_{(\varpi)}} \to 0.$$

Since $\mathbb{T}_{(\varpi)} = \bigoplus_{\mathbb{I}} \mathbb{I}_{(\varpi)}$, we can further take the $\mathbb{I}_{(\varpi)}$-component producing one more exact sequence

$$0 \to J \otimes_{\mathbb{T}} \mathbb{I}_{(\varpi)} \xrightarrow{\varpi} J \otimes_{\mathbb{T}} \mathbb{I}_{(\varpi)} \xrightarrow{\pi_\infty} \widehat{A}_r(k)^*_{\text{ord}, \mathbb{I}_{(\varpi)}} \to 0.$$

This shows

$$J \otimes_{\mathbb{T}} \mathbb{I}_{(\varpi)} \otimes_{\Lambda} \kappa((\varpi)) = \widehat{A}_r(k)^*_{\text{ord}, \mathbb{I}_{(\varpi)}},$$

where $\kappa((\varpi))$ is the quotient field of $\Lambda/(\varpi)$. Decompose $(\varpi) = \prod_P P^{e(P)}$ in $\widetilde{\mathbb{I}}$ as a product of (arithmetic) primes P in $\widetilde{\mathbb{I}}$. Again by étaleness of $\mathrm{Spec}(\mathbb{T})$ over arithmetic primes, we have $e(P) = 1$. Thus

$$\widehat{A}_r(k)^*_{\text{ord}, \mathbb{I}_{(\varpi)}} = \prod_P \widehat{A}_r(k)^*_{\text{ord}} \otimes_{\mathbb{T}} \mathbb{I}_P = \prod_P \widehat{A}_r(k)^*_{\text{ord}} \otimes_{\mathbb{T}} \kappa(P).$$

Since $J_{(\varpi)} = \bigoplus_{\mathbb{I}} \mathbb{I}^{r(\mathbb{I})}_{(\varpi)}$, we get

(3.1)

$$\dim_{\kappa(P)} \widehat{A}_r(k)^*_{\text{ord}} \otimes_{\mathbb{T}} \kappa(P) = \dim_{\kappa(P)} J \otimes_{\mathbb{T}} \kappa(P) = \dim_{\kappa(P)} J \otimes_{\mathbb{T}} \mathbb{I}_{(\varpi)} \otimes_{\mathbb{I}_{(\varpi)}} \kappa(P) = r_k(\mathbb{I}).$$

Let H be the subalgebra of $\mathrm{End}(A_r)$ generated over $\mathbb{Z}$ by all Hecke operators $T(n)$. We put $H(R) = H \otimes_{\mathbb{Z}} R$ and $H^{\text{ord}}(R) = e(H(R))$ if e is well defined over R. By the control theorem of the big Hecke algebra (e.g., [H86b, Theorem 1.2]), we have $\mathbb{T}/(\varpi)\mathbb{T} \cong H(\mathbb{Z}_p)^{\text{ord}}$ by an isomorphism sending $T(l)$ to $T(l)$ for all primes l.

Since A_r is isogenous to a sum $\bigoplus_{[f_P]} A_P$ for the Galois conjugacy classes $[f_P]$ of Hecke eigenforms f_P associated to an arithmetic point $P|(\varpi)$ of $\mathrm{Spec}(\mathbb{T})$, we have $H(\mathbb{Q}_p) = \bigoplus_{[f_P]} \mathbb{Q}_p(f_P)$ as an algebra direct sum. Note that $\kappa(P) = \mathbb{Q}_p(f_P)$. Since A_P is the factor of A_r corresponding to $\mathbb{Q}(f_P)$, we have

$$\widehat{A}_r(k)^*_{\text{ord}} \otimes_{\mathbb{T}} \kappa(P) \cong \widehat{A}_r(k)^*_{\text{ord}} \otimes_{H(\mathbb{Z}_p)} \mathbb{Q}_p(f_P) \cong \widehat{A}_P(k)^*_{\text{ord}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Then by the same argument at the end of the proof of Theorem 3.1, we conclude from (3.1) the desired identity $\mathrm{rank}\, A_P(k) = [\mathbb{Q}(f_P) : \mathbb{Q}] r(\mathbb{I})$ ($\Leftrightarrow \dim_{\mathbb{Q}(f_P)} A_P(k) \otimes_{\mathbb{Z}} \mathbb{Q} = r_k(\mathbb{I})$). □

Let ε_P be the root number of the functional equation of $L(s, f_P^\circ)$ for the unique primitive Hecke eigenform f_P° associated to f_P. If $\varepsilon_P = \pm 1$, this number is independent of P. Let $Q = Q(\mathbb{I})$ be the quotient field of $\mathbb{I}$. Here is a conjectural description of $r_{\mathbb{Q}}(\mathbb{I})$.

Conjecture 3.4. *Pick a rational elliptic curve E of ordinary good reduction at p with conductor N. Write $\mathbb{I}_E$ for the unique irreducible component of* $\mathrm{Spec}(\mathbf{h}(N))$ *such that we have an arithmetic point $P \in \mathrm{Spec}(\mathbb{I}_E)(\overline{\mathbb{Q}}_p)$ of weight 2 with A_P isogenous to E; so, $\varepsilon_P = \pm 1$. Then for a set of rational elliptic curves with ordinary reduction at p having 100% "proportion" in Bargava's sense in* [BS14a] *and* [BS14b], *we have*

$$r_{\mathbb{Q}}(\mathbb{I}_E) \le 1 \text{ and } r_{\mathbb{Q}}(\mathbb{I}_E) \equiv \frac{1 - \varepsilon_P}{2} \mod 2.$$

We prove this conjecture under some mild conditions in Sections 5 and 7 if $|E(k)| < \infty$ (see Theorem 5.2 and Propositions 7.1 and 7.4). This also proves Theorem A in the introduction. In this special case, of course, we have $r_k(\mathbb{I}) = 0$ (and necessarily $\varepsilon_P = 1$).

There is a related conjecture on the analytic rank made by Greenberg in [Gr94], and one can make a similar conjecture for the Λ-adic Selmer group (of the Galois representation attached to $\mathbb{I}$). Some positive result for the Selmer group is obtained in [Hw07a, Corollary 3.4.4] and [Hw07b] towards the conjecture. In the cases where Howard proved his conjecture, it implies that the Tate–Shafarevich part of the Λ-adic Selmer group is Λ-torsion. This conjecture is a version for Mordell–Weil groups. It would be interesting to study the limit Tate–Shafarevich group directly by our method (which we hope to do in future). If we start with a rational elliptic curve E with ordinary good reduction at p with $L(1, E) \neq 0$, under mild assumptions, Kolyvagin (and Rubin in the CM case) proved the finiteness of the p-part of the Tate–Shafarevich group of E. After that, Skinner and Urban proved cases of the p-adic Birch-Swinnerton Dyer conjecture. Taking $\mathbb{I}$ whose family contains the cusp form attached to E of rank 0, we conclude from [SU13, Theorems in §3.6] combined with Theorem A that $\check{J}_\infty(\mathbb{Q})^*_{\mathrm{ord},\,\mathbb{I}}$ is $\mathbb{I}$-torsion and hence the above conjecture holds.

One can ask the following naive questions for a general number field $k \supsetneq \mathbb{Q}$:

- (Q1) What is $r_k(\mathbb{I})$? It could be equal to 0 or 1 most of the time if k is totally real (see [N06]). If k is an anticyclotomic abelian extension of an imaginary quadratic field, [Hw07a] contains some answer that the rank could grow dependent on $[k : \mathbb{Q}]$.
- (Q2) If $r_\mathbb{Q}(\mathbb{I}) = 0$, does the characteristic power series of $\check{J}_\infty(\mathbb{Q})^*_{\mathrm{ord},\,\mathbb{I}}$ give a factor of (the two variable) $\mathbb{I}$-adic standard p-adic L-function (of Mazur–Kitagawa) restricted to the self-dual line? Similarly, if $r_\mathbb{Q}(\mathbb{I}) = 1$, does the characteristic power series of the torsion part of $\check{J}_\infty(\mathbb{Q})^*_{\mathrm{ord},\,\mathbb{I}}$ give a factor of the first derivative (with respect to the cyclotomic variable) of the $\mathbb{I}$-adic standard p-adic L-function of two variables (restricted to the self-dual line)?

Again we can answer (Q2) affirmatively in the cases where the p-adic Birch-Swinnerton Dyer conjecture is proven.

4. Preliminary lemmas

Let $B_{/\mathbb{Q}}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type. We assume that $O_B = \mathrm{End}(B_{/\mathbb{Q}}) \cap K_B$ is the integer ring of its quotient field K_B. Then the compatible system of two dimensional Galois representations $\rho_B = \{\rho_{B,\mathfrak{l}}\}_\mathfrak{l}$ realized on the Tate module of B has its L-function $L(s, B)$ equal to $L(s, f)$ for a primitive form $f \in S_2(\Gamma_1(C))$ for the conductor $C = C_B$ of ρ_B (see [KW09a, Theorem 10.1]). Thus B is isogenous to A_f over $\mathbb{Q}$ (by a theorem of Faltings). Let A be another $\mathbb{Q}$-simple abelian variety of GL(2)-type. Thus A is isogenous to A_g for another primitive form $g \in S_2(\Gamma_1(C_A))$ of conductor C_A. Without losing generality, we may (and do) assume that $O_A = \mathrm{End}(A_{/\mathbb{Q}}) \cap K_A$. Note that $K_B = Q(f)$ and $K_A = \mathbb{Q}(g)$ which are subfield of $\overline{\mathbb{Q}}$.

Suppose A is congruent to B modulo p with $(B[\mathfrak{p}_B]\otimes_{\kappa(\mathfrak{p}_B)}\overline{\mathbb{F}}_p)^{ss}\cong(A[\mathfrak{p}_A]\otimes_{\kappa(\mathfrak{p}_A)}\overline{\mathbb{F}}_p)^{ss}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules. Here, for any ring R and a prime ideal $\mathfrak{p}$ of R, we write $\kappa(\mathfrak{p})$ for the residue field of $\mathfrak{p}$.

Choosing g (resp. f) well in the Galois conjugacy class of g (resp. f), we may assume that $\mathfrak{p}_A$ and $\mathfrak{p}_B$ are both induced by the fixed embedding $i_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. We suppose that B is $\mathfrak{p}_B$-ordinary in the sense that we have $\dim_{\kappa(\mathfrak{p}_B)}H_0(I_p,B[\mathfrak{p}_B])\geq 1$ for the inertia group I_p of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Lemma 4.1. *Suppose that C_A/C_B is in $\mathbb{Z}[\frac{1}{p}]^\times$ and that B is $\mathfrak{p}_B$-ordinary. Write $C_B=Np^r$. Then there exists a connected component* $\mathrm{Spec}(\mathbb{T})$ *of* $\mathrm{Spec}(\mathbf{h}(N))$ *such that for some primes $P,Q\in\mathrm{Spec}(\mathbb{T})$, $f=f_P$ and $g=f_Q$.*

Proof. Let $\overline{\rho}$ be the two dimensional Galois representation into $\mathrm{GL}_2(\mathbb{F})$ realized on $B[\mathfrak{p}_B]$ for $\mathbb{F}=O_B/\mathfrak{p}_B$. Replacing $\overline{\rho}$ by its semi-simplification, we may assume that $\overline{\rho}$ is semi-simple. Since $(B[\mathfrak{p}_B]\otimes_{\kappa(\mathfrak{p}_B)}\overline{\mathbb{F}}_p)^{ss}\cong(A[\mathfrak{p}_A]\otimes_{\kappa(\mathfrak{p}_A)}\overline{\mathbb{F}}_p)^{ss}$, $L(s,A)=L(s,g)$ and $L(s,B)=L(s,f)$ imply $f \mod \mathfrak{p}_B=g \mod \mathfrak{p}_B$. Moreover writing $f=\sum_{n=1}^\infty a(n,f)q^n$ for the q-expansion of f at the infinity cusp, we have $a(p,f)\not\equiv 0 \mod \mathfrak{p}_A$ as B is $\mathfrak{p}_B$-ordinary. Thus g (resp. f) is lifted to a p-adic analytic family parameterized by an irreducible component $\mathrm{Spec}(\mathbb{I})$ (resp. $\mathrm{Spec}(\mathbb{J})$) of $\mathrm{Spec}(\mathbf{h}(N))$. Since $f \mod \mathfrak{p}_B=g \mod \mathfrak{p}_B$, the algebra homomorphisms $\lambda_?:\mathbf{h}(N)\to\overline{\mathbb{Q}}_p$ realized as $f|T(n)=\lambda_f(T(n))f$ and $g|T(n)=\lambda_g(T(n))g$ satisfy $\lambda_f\equiv\lambda_g \mod \mathfrak{m}$ for a maximal ideal $\mathfrak{m}$ of $\mathbf{h}(N)$. Then, $P=\mathrm{Ker}(\lambda_f)$ and $Q=\mathrm{Ker}(\lambda_g)$ belongs to the connected component $\mathrm{Spec}(\mathbb{T})$ given by $\mathbb{T}=\mathbf{h}(N)_\mathfrak{m}$, since the local rings of $\mathbf{h}(N)$ corresponds one-to-one to the maximal congruence classes of Hecke eigenforms of prime-to-p level N modulo p just because the set of maximal ideals $\overline{\Sigma}$ of $\mathbf{h}(N)$ is made of $\mathrm{Ker}(\lambda)$ for $\lambda\in\Sigma=\mathrm{Hom}_{\mathrm{alg}}(\mathbf{h}(N),\overline{\mathbb{F}}_p)$. The maximal ideal $\mathfrak{m}$ is given by $\mathrm{Ker}(\lambda_f \mod \mathfrak{P})=\mathrm{Ker}(\lambda_g \mod \mathfrak{P})$ for $\mathfrak{P}=\{x\in\overline{\mathbb{Q}}_p:|x|_p<1\}$. □

The following result is just the combination of the above Lemma 4.1 and Theorem 2.2.

Corollary 4.2. *Let the notation and the assumptions be as in* Lemma 4.1 *and* Theorem 2.2. *Write χ for the Neben character of f. Suppose that $N=C_B$ is prime to p and write $f|U(p)=a_pf$. If $p\nmid 2D_\chi N\varphi(N)$ and $(a_p \mathrm{mod}\ \mathfrak{p}_B)\notin\Omega_{B,p}:=\{0,\pm\sqrt{\chi(p)}\}$, then $\mathbb{T}$ is a regular integral domain $\mathbb{I}$ and f and g belongs to* $\mathrm{Spec}(\mathbb{I})$.

5. Proof of Theorem A

Let $B_{/\mathbb{Q}}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type such that $O_B=\mathrm{End}(B_{/\mathbb{Q}})\cap K_B$ is the integer ring of its quotient field K_B. Let $\rho_B=\{\rho_{B,\mathfrak{l}}\}$ be the two dimensional compatible system of Galois representations associated to B. Fix an embedding $O_B\hookrightarrow\overline{\mathbb{Q}}$ and write $\mathfrak{p}_B$ for the prime ideal of O_B induced by $i_p:\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$. Write $\det\rho_{B,\mathfrak{p}_B}=\nu\chi$ for the $\mathfrak{p}_B$-adic cyclotomic character ν. Then χ gives the Neben character of the cusp form f with the identity $L(s,B)=L(s,f)$ under $\mathbb{C}\xleftarrow{i_\infty}\overline{\mathbb{Q}}\xrightarrow{i_p}\overline{\mathbb{Q}}_p$. We write $f=\sum_{n=1}^\infty a_nq^n$.

Definition 5.1. *Let $S = S_B$ be the set of prime factors of $2D_\chi N\varphi(N)$ for the conductor N of ρ_B, where D_χ is the discriminant of the reduced part of $h_2(\Gamma_0(N), \chi; \mathbb{Z}[\chi])$.*

The prime p is *admissible* for B if B has good reduction modulo p (so, $p \nmid N$) and $(a_p \bmod \mathfrak{p}_B) \notin \Omega_{B,p} := \{0, \pm\sqrt{\chi(p)}\}$ (so, B has partially $\mathfrak{p}_B$-ordinary reduction modulo p). We prove the following result slightly more general than Theorem A:

Theorem 5.2. *Let p be a prime outside S_B admissible for B and N be the conductor of B. Suppose $|B(k)| < \infty$ for a number field k. Consider the set $\mathcal{A}_{B,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of $\mathrm{GL}(2)$-type congruent to B modulo p over $\mathbb{Q}$ with prime-to-p conductor N. Then almost all $A \in \mathcal{A}_{B,p}$ has Mordell–Weil $\mathbb{Q}$-rank 0 (i.e., we have $|A(\mathbb{Q})| < \infty$).*

Theorem A follows from this theorem taking $B = E$ in Theorem A. As is well known, there are density one (partially) ordinary primes in O_B if B does not have complex multiplication (e.g., [H13b, Section 7])

Proof. Since $B[\mathfrak{p}_B^\infty]$ is an ordinary Barsotti–Tate group by our assumption, $A[\mathfrak{p}_A^\infty]$ is potentially ordinary by the congruence modulo p between A and B. Here we say $A[\mathfrak{p}_A^\infty]$ "potentially ordinary" if $H_0(I_p, A[\mathfrak{p}_A^\infty](\overline{\mathbb{Q}}_p))$ has non-trivial p-divisible rank and $A[\mathfrak{p}_A^\infty]$ over $\mathbb{Q}_p$ extends to a Barsotti–Tate group with non-trivial étale quotient over a finite extension of $\mathbb{Z}_p$. Choosing the embedding $O_A \hookrightarrow \overline{\mathbb{Q}}$ well, we may assume that $\mathfrak{p}_A$ is induced by $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Thus A and B are isogenous to a modular abelian variety $A_{P'}$ and A_P, respectively, for two points $P, P' \in \mathrm{Spec}(\mathbb{T})$ of a connected component $\mathrm{Spec}(\mathbb{T})$ of $\mathrm{Spec}(\mathbf{h}(N))$ for the big p-adic Hecke algebra $\mathbf{h}(N)$. Thus we conclude

$$\mathcal{A}_{B,p} = \{A_Q | Q \in \mathrm{Spec}(\mathbb{T}) \text{ and } Q \text{ is arithmetic of weight } 2\}$$

by the theorem of Khare–Wintenberger [KW09a, Theorem 10.1] (combined with the proof of the Tate conjecture for abelian varieties by Faltings).

Since p is outside S_B, by Corollary 4.2, $\mathbb{T}$ is a regular integral domain $\mathbb{I}$. Thus $A_1 = A_P$ satisfies the condition (A) (in particular, $P = (\alpha)$ for $\alpha \in \mathbb{I}$). In other words, for $\alpha \in \mathbb{I} = \mathbb{T}$, by the control theorem Theorem 2.3, we have $\widehat{A}^*_{P,\mathrm{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \check{J}_\infty(k)^*_{\mathrm{ord},\, \widetilde{\mathbb{I}}[\frac{1}{p}]} \otimes_{\widetilde{\mathbb{I}}[\frac{1}{p}]} \kappa(P)$ as $P = (\alpha)$. Here "$*$" indicates the $\mathbb{Z}_p$-dual module of the module $*$ attached. Since $\widehat{A}^*_{P,\mathrm{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subset A_P(k) \otimes_{\mathbb{Z}} \mathbb{Q}_p = 0$, we conclude $r_k(\mathbb{I}) = 0$ for $r_k(\mathbb{I})$ in Theorem 3.1. Thus $X_{\mathbb{I}} := \check{J}_\infty(k)^*_{\mathrm{ord},\, \widetilde{\mathbb{I}}[\frac{1}{p}]}$ is a torsion $\widetilde{\mathbb{I}}[\frac{1}{p}]$-module of finite type. In particular, $\mathrm{Supp}(X_{\mathbb{I}})$ contains only finitely many maximal ideals of the Dedekind domain $\widetilde{\mathbb{I}}[\frac{1}{p}]$. By the above argument,

$$\mathcal{A}_{B,p} - \{A_P \in \mathrm{Supp}(X_{\mathbb{I}}) | P\text{: arithmetic}\}$$

is the set of A with finite $A(\mathbb{Q})$. Since $\mathrm{Supp}(X_{\mathbb{I}})$ contains only finitely many primes, this concludes the proof. □

As for Corollary B, we remark that $h_2(\Gamma_0(N); \mathbb{Z}) = \mathbb{Z}$ for the values N in the corollary. Then we can check easily from the table of Stein and Cremona that either $p \in S \Rightarrow (a_p \bmod p) \in \Omega_E$ or $X_0(N) \bmod p$ is singular. Thus the condition that $p \notin S$ is not necessary for the corollary.

Though we formulated Theorem 5.2 insisting that A has prime-to-p conductor equal to the conductor of B, enlarging the exceptional set S_B, we can allow the case where B has conductor divisible by the conductor of A under some extra assumptions, in particular, that $r_k(\mathbb{I})$ is independent of the irreducible components of $\mathrm{Spec}(\mathbb{T})$ (see Proposition 7.1). For a level-raising prime p, B could have conductor Cp for the conductor C of A. In such a case, the Selmer rank of B can be higher than that of A [Z14, §5.3] (and also Mordell-Weil rank when $p = 2$ by a work of Chao Li). Thus the assumption of constancy of $r_k(\mathbb{I})$ is necessary.[1]

6. Proof of Theorem C.

We keep the notation in the previous section. Again we prove the following slightly stronger result.

Theorem 6.1. *Let p be a prime outside S_B admissible for B and N be the conductor of B. Suppose $\dim_{K_B} B(k) \otimes_{\mathbb{Z}} \mathbb{Q} = m > 0$ for a number field k. Consider the set $\mathcal{A}_{B,p}$ made up of all $\mathbb{Q}$-isogeny classes of $\mathbb{Q}$-simple abelian varieties $A_{/\mathbb{Q}}$ of $\mathrm{GL}(2)$-type congruent to B modulo p with prime-to-p conductor N. Then we have $r_k(\mathbb{I}) \leq m$. We have $r_k(\mathbb{I}) < m$ if and only if there exists another $A' \in \mathcal{A}_{B,p}$ with $\dim_{K_{A'}} A'(k) \otimes_{\mathbb{Z}} \mathbb{Q} < m$.*

Applying this result to $B = E$, $k = \mathbb{Q}$ and $m = 1$, we obtain Theorem C in the introduction. Under the assumption of the theorem, if $r_k(\mathbb{I}) = r$ for the component $\mathrm{Spec}(\mathbb{I})$ containing B, for almost all $A \in \mathcal{A}_{B,p}$ we have $\dim_{K_A} A(k) \otimes_{\mathbb{Z}} \mathbb{Q} = r$. Thus if we prove $r \leq m$, we get the theorem by an obvious induction on m (with the start step given by Theorem 5.2).

Proof. As explained above, we first prove $r_k(\mathbb{I}) \leq m$ under the assumption of the theorem. Let $P \in \mathrm{Spec}(\mathbb{I})$ be the arithmetic point associated to B. Then after localizing at P, by Nakayama's lemma applied to the valuation ring $\mathbb{I}_P$ and its maximal ideal, the m-dimensionality of

$$K_{A_P}^m = \kappa(P)^m \cong \widehat{A}^*_{P,\mathrm{ord}}(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \check{J}_\infty(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}[\frac{1}{p}]} \otimes_{\widetilde{\mathbb{I}}[\frac{1}{p}]} \kappa(P)$$

tells us that $\check{J}_\infty(k)^*_{\mathrm{ord},\,\widetilde{\mathbb{I}}[\frac{1}{p}]} \otimes_{\widetilde{\mathbb{I}}[\frac{1}{p}]} \mathbb{I}_P$ is generated by m elements over $\mathbb{I}_P$. From which, we conclude $r_k(\mathbb{I}) \leq m$.

If we find $A' \in \mathcal{A}_{B,p}$ as in the theorem, we find $r_k(\mathbb{I}) \leq m - 1$ from the above argument applied to A' in place of A. Since $r_k(\mathbb{I}) \leq m - 1$, almost all $A'' \in \mathcal{A}_{B,p}$ satisfies $\dim_{K_{A''}} A''(k) \otimes_{\mathbb{Z}} \mathbb{Q} = r_k(\mathbb{I}) < m$. Therefore for almost all $A'' \in \mathcal{A}_{B,p}$ has this property. This finishes the proof. □

As discussed in (Q1), we expect to have $0 \leq r_k(\mathbb{I}) \leq 1$ all the time as long as k is totally real. This theorem is plainly short of this goal (because we do not have any effective method to calculate the rank of $A'(k)$ for other members A' of $\mathcal{A}_{B,p}$).

[1] Ashay Burngale has pointed out the author that the Mordell-Weil rank could jump for level raising primes; so, the constancy of $r_k(\mathbb{I})$ over $\mathrm{Spec}(\mathbb{T})$ is necessary. The author is grateful for his timely remark.

7. Good cases for non-regular $\mathbb{T}$

In this section, we suppose that $\mathbb{T}$ is not regular integral domain; so, $p \in S_B$ by Theorem 2.2. Under some different sets of assumptions, we prove the assertions of Theorem 5.2 and Theorem 6.1 for such primes in p. For simplicity, throughout this section, we suppose one of the following two conditions:

- N is cube-free,
- $\mathbb{T}$ is new (see Definition 3.2).

By the above assumption, $\mathbb{T}_s = \mathbb{T}/(\gamma^{p^{s-1}} - 1)\mathbb{T}$ is reduced for all s (see [H13a, §1] and [H86a, §3]).

If $p \in S_B$, then $\mathbb{T}$ is often not regular. We divide our consideration into the following two cases.

(R) $\mathrm{Spec}(\mathbb{T})$ is reducible;
(I) $\mathrm{Spec}(\mathbb{T})$ is an integral domain.

Proposition 7.1. *Suppose that we are in* Case R *for a prime p admissible for B, and let N be the conductor of B. Suppose $|B(k)| < \infty$ for a number field k. If $r_k(\mathbb{T}) := r_k(\mathbb{I})$ is independent of the irreducible components* $\mathrm{Spec}(\mathbb{I})$ *of* $\mathrm{Spec}(\mathbb{T})$*, the assertion of* Theorem 5.2 *is still valid for $\mathcal{A}_{B,p}$; that is for almost all $A \in \mathcal{A}_{B,p}$, we have $|A(k)| < \infty$.*

Proof. The generic rank $r_k(\mathbb{I})$ is well defined by Theorem 3.3. Note that

$$\mathcal{A}_{B,p} = \{A_P | \text{arithmetic points } P \text{ of weight 2 of } \mathrm{Spec}(\mathbb{T})\}.$$

If $|B(k)| < \infty$, we have an arithmetic point $P_B \in \mathrm{Spec}(\mathbb{T})$ of weight 2 for which A_{P_B} is isogenous to B over $\mathbb{Q}$. By the étaleness of $\mathbb{T}$ over $P_B \cap \Lambda$ (see [HMI, Proposition 3.78]), this P_B lies on a unique irreducible component $\mathrm{Spec}(\mathbb{I})$. By the same argument proving Theorem 5.2, we conclude $r_k(\mathbb{I}) = 0$. Since the generic rank is independent of irreducible components of $\mathrm{Spec}(\mathbb{T})$ by our assumption, $r_k(\mathbb{I}') = 0$ for all other irreducible components $\mathrm{Spec}(\mathbb{I}')$ of $\mathrm{Spec}(\mathbb{T})$, and thus again by the proof of Theorem 5.2, we conclude there are only finitely many exceptional abelian varieties $A \in \mathcal{A}_{B,p}$ with positive k-rank, as desired. □

Remark 7.2. Replacing the assumption $|B(k)| < \infty$ by $\mathrm{rank}_{O_B} B(k) = 1$ (and keeping all other assumptions) in the above proposition, we conclude $r_k(\mathbb{T}) \leq 1$ in the same way as the proof of Theorem 6.1.

Remark 7.3. For any pair of irreducible component $\mathrm{Spec}(\mathbb{I})$ and $\mathrm{Spec}(\mathbb{I}')$ of $\mathrm{Spec}(\mathbb{T})$, if each $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p) \cap \mathrm{Spec}(\mathbb{I}')(\overline{\mathbb{Q}}_p)$ is outside of $\mathrm{Supp}(X_{\mathbb{I}}) \cup \mathrm{Supp}(X_{\mathbb{I}'})$, we have the equality of the generic ranks $r_k(\mathbb{I}) = r_k(\mathbb{I}')$, and hence the generic rank for irreducible components are independent of the components.

We assume that we are now in Case I.

Proposition 7.4. *Suppose that we are in* Case I *for a prime p admissible for B, and let N be the conductor of B. Suppose $|B(k)| < \infty$ for a number field k. Then, for almost all $A \in \mathcal{A}_{B,p}$, we have $|A(k)| < \infty$.*

Proof. Again we find $r_k(\mathbb{I})$ defined in Theorem 3.3 is zero. Then the conclusion holds by the argument proving Theorem 5.2. $\square$

In the same way as in Theorem 6.1,we get

Proposition 7.5. *Suppose that we are in* Case I *for a prime p admissible for B, and let N be the conductor of B. Suppose $\dim_{K_B} B(k)\otimes_{\mathbb{Z}}\mathbb{Q} = m > 0$ for a number field k and that N is cube-free. Then we have $r_k(\mathbb{I}) \le m$. We have $r_k(\mathbb{I}) < m$ if and only if there exists another $A' \in \mathcal{A}_{B,p}$ with $\dim_{K_{A'}} A'(k)\otimes_{\mathbb{Z}}\mathbb{Q} < m$.*

The proof is left to the attentive reader.

Remark 7.6. By the above propositions, only cases left open are when Spec($\mathbb{T}$) is not an integral domain which does not satisfy the assumption of Proposition 7.1. Such cases may occur as the λ-invariant of the standard p-adic L-functions indexed by arithmetic points of Spec($\mathbb{T}$) may depends on irreducible components as shown in [EPW06]. For $\mathbb{T}$ is not new but with mixed new and old irreducible components (i.e., when we have level $N = N_0$ with level raising prime $l \nmid N_0$), this happens as verified by Le Hung and Chao Li [HuL14] when $p = 2$ (see also [Z14] for such phenomena for Selmer rank).

8. Zero density of primes p with $(a_p \bmod \mathfrak{p}_B) \in \Omega_B$

Let $B_{/\mathbb{Q}}$ be a $\mathbb{Q}$-simple abelian variety of GL(2)-type of conductor N. Since B is a factor of $J_1(N)$ by [KW09a, Theorem 10.1] (combined with the Tate conjecture proven by Faltings), it has a $\mathbb{Q}$-rational polarization λ induced from the canonical polarization of $J_1(N)$. We assume that $\mathrm{End}(B_{/\mathbb{Q}})\cap K_B$ is the integer ring O_B of K_B. Write $\det\rho_B = \nu\chi$ for a character χ modulo N and the p-adic cyclotomic character ν. Let

$$\Sigma = \{p : \text{rational prime outside } N | (a_p \bmod \mathfrak{p}) \in \Omega_{B,p} \ \text{ for all } O_B\text{-prime ideal } \mathfrak{p}|p\}.$$

We prove the following lemma in this section.

Lemma 8.1. *Assume that $B\times_{\mathbb{Q}}\overline{\mathbb{Q}}$ does not have complex multiplication. The subset Σ has Dirichlet density 0 in the set of all rational primes.*

Proof. Suppose that $p \in \Sigma$ and $\mathfrak{p}|p$ be a prime of O_B with $(a_p \bmod p) \in \Omega_{B,p}$. We may assume that $\mathfrak{p}$ is unramified over $\mathbb{Z}$.

The reduction $B_p = B \mod p$ has the p-power Frobenius endomorphism ϕ whose eigenvalues α satisfies $|\alpha^\varphi| = \sqrt{p}$ for all $\varphi \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Since $a_p = \alpha + \chi(p)\alpha^c$ for the complex conjugation c (i.e., the Rosati involution of O_B with respect to the polarization λ of B; see [GME, Theorem 4.2.1]), we have an estimate $|a^\varphi| \le 2\sqrt{p}$. Taking the norm to $\mathbb{Q}$, we have $|N_{K_B/\mathbb{Q}}(a)| \le 2^d p^{d/2}$ for $d = [K_B : \mathbb{Q}] = \dim B$.

We write ∞ for the set of all field embeddings of K_B into $\mathbb{C}$, and put

$$\Omega_B := \bigcup_{p\nmid N}\Omega_{B,p} = \{0, \pm\sqrt{\chi(p)}|p : \text{ prime outside } N\} \subset \overline{\mathbb{Q}}_p.$$

Note that Ω_B is a finite set. Since α is a Weil p-number of weight 1, If $a_p \equiv \delta_\mathfrak{p} \mod \mathfrak{p}$ for some $\delta_\mathfrak{p} \in \Omega_B$ (which may depend on $\mathfrak{p}$) for all $\mathfrak{p}|p$, we have $p|(a_p - \delta_\mathfrak{p})$

(i.e., $|a_p^\sigma| \geq p-1$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$). Thus $\prod_{\sigma\in\infty} |a_p^\sigma| \geq (p-1)^d$. Therefore we have $(p-1)^d \leq |N_{F/\mathbb{Q}}(a_p)|$; so, $(p-1)^d \leq |N_{F/\mathbb{Q}}(a_p)| \leq 2^d p^{d/2}$ as long as $a_p \notin \Omega_B$. Thus if $a_p \notin \Omega_B$, we have $p \leq 6$, and therefore we may assume that $a_p \in \Omega_B$ if $p \geq 7$. This shows

$$\Sigma \subset \{p \leq 6\} \bigcup_{a\in\Omega_B} \{P|a_p = a\}.$$

For any given constant C, the set of primes $\{p|a_p = C\}$ has Dirichlet density 0 by [Se81, Théorème 15] applied to the cusp form f having A_f isogenous to B. This finishes the proof, since Ω_B is a finite set. □

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