* Back-ground of modular p-adic deformation theory and a brief outline

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*Abstract: The deformation theory of modular forms is increasingly attracting many researchers in arithmetic geometry as it has been an important step in the proof of Fermat's last theorem by Wiles (and Taylor) and supplied an effective tool for the study of the p-adic Birch and Swinnerton Dyer conjecture in the proof by Skinner-Urban of divisibility of the characteristic power series of the Selmer group of a rational elliptic curve by its p-adic L-function under appropriate assumptions. I try to give my back-ground motivation of creating the theory and describe an outline of the theory.

§**0. A fundamental question in 1980.** Consider a huge space $X := \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{R}^\times$ for $\mathbb{A} = \mathbb{Q} + \widehat{\mathbb{Z}}\mathbb{R} \subset \mathbb{R} \times \prod_l \mathbb{Q}_l$ ($\widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$) with a Haar measure $d\mu$ and an L^2 Hilbert space

$$L^2_{cusp} := \{f : X \to \mathbb{C} | \int_X |f|^2 d\mu < \infty, \int_{\mathbb{Q}\backslash\mathbb{A}} f(\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) h) du = 0 \ \forall h\}.$$

By Gelfand and Harish-Chandra, the representation $\Pi : \mathrm{GL}_2(\mathbb{A}) \to \mathrm{Aut}_{\mathbb{C}}(L^2_{cusp})$ given by $(\Pi(g)f)(h) = f(hg)$ is a discrete direct sum of irreducible representations. In 1980, I felt that having a discrete spectrum is unfortunate, as one cannot come close to a difficult constituent $\pi \subset \Pi$ from another somehow simpler π_0.

Question. *Is there any other topology, keeping the information of π and π_0 to good extent, to have a continuous spectrum?*

My answer: via p-adic topology, indeed

- Wiles moved from a simpler weight 1 form to the weight 2 cusp form to be associated to an elliptic curve (FLT; 1995);
- Skinner–Urban moved from an Eisenstein series to cusp forms on $U(2,2)$ to compute the order of zero of the L-function of an elliptic curve (BSD; 2014).

§**1.** p**-adic replacement of** L^2_{cusp} can be sections of line bundle over a huge scheme $S(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}) = \varprojlim_\Delta \Delta\backslash\mathfrak{H}$ which is defined over $\mathbb{Z}$. Consider a holomorphic cusp form $f : \mathfrak{H} \to \mathbb{C}$ of weight k on Δ. The factor $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto j(g,\tau)^k = (c\tau + d)^k$ satisfies $j(gh,\tau) = j(g,h(\tau))j(g,\tau)$. So $\Delta\backslash(\mathfrak{H} \times \mathbb{C})$ by the action $(\tau,u) \mapsto (\delta(\tau), j(\delta,\tau)^k u)$ gives a line bundle ω^k on $\Delta\backslash\mathfrak{H}$. Then $\omega = \varphi_*\Omega^1_{\mathbb{E}/S}$ for the universal elliptic curve $\mathbb{E} \xrightarrow{\varphi} S_{/\mathbb{Z}}$. A section f of ω^k satisfies $f|\delta(\tau) = f(\delta(\tau))j(\delta,\tau)^{-k} = f(\tau)$ for $\delta \in \Delta \subset \mathrm{SL}_2(\mathbb{Z})$. The section f is lifted to $f : X \to \mathbb{C}$ right invariant under $\widehat{\Delta}$ (closure of Δ in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$).

Writing $L(n;A) = AX^n + AX^{n-1}Y + \cdots + AY^n$ and letting $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathfrak{H} \times L(n;A)$ by $(\tau, P(X,Y)) \mapsto (\delta(\tau), P((X,Y)^t\delta^{-1}))$, we have the sheaf $\mathcal{L}(n;A)$ of locally constant sections from $\Delta\backslash\mathfrak{H}$ to $\Delta\backslash(\mathfrak{H} \times L(n;A))$. We replace L^2 by the p-adic completion of

$$\begin{cases} V_k := \varinjlim_\alpha H^0(\widehat{Y_1(p^\alpha)}_{/\mathbb{Z}_p}, \omega^k_{cusp}) & \text{coherent cohomology,} \\ TJ_n := \varprojlim_\alpha H^1_{et,!}(Y_1(p^\alpha), \mathcal{L}(n;\mathbb{Z}_p)) & \text{pro-étale cohomology.} \end{cases}$$

Here $H^1_! = \mathrm{Im}(H^1_c \to H^1)$ and $Y_1(p^\alpha) = \Gamma_1(p^\alpha)\backslash\mathfrak{H}$.

§2. **Why taking the limit for $Y_1(p^n)$.** Limiting myself to a p-power p^α is an analogue from Kummer and Iwasawa, as $\mathbb{Z}_p^\times = \varprojlim_\alpha \Gamma_0(p^n)/\Gamma_1(p^n)$, the modules V and TJ_n are Iwasawa modules over $\mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ for $\Gamma = \mathbb{Z}_p^\times/\{\text{torsion}\} = \gamma^{\mathbb{Z}_p}$. I wanted to make a GL(2)-version of Iwasawa theory. Let

$$\widehat{\Gamma}_0(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Z}})\middle| c \in N\widehat{\mathbb{Z}}\right\},\ \Gamma_?(N) = \widehat{\Gamma}_?(N) \cap \mathrm{SL}_2(\mathbb{Z}),$$
$$\widehat{\Gamma}_1(N) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widehat{\Gamma}_0(N)\middle| d \equiv 1 \mod N\widehat{\mathbb{Z}}\right\}.$$

Then $Y_?(N) = S/\widehat{\Gamma}_?(N) \cong \Gamma_?(N)\backslash\mathfrak{H}$. For each $\pi \subset \Pi$, there is a minimal N with $H^0(\widehat{\Gamma}_1(N), \pi) = \mathbb{C}f_\pi$ (so, sufficient to take limit with respect to $\Gamma_1(p^\alpha)$). Assume that $f = f_\pi$ is holomorphic. On $\mathfrak{H}$, $f = \sum_{n=1}^\infty a(n,f)q^n$ for $q := \exp(2\pi i\tau)$. Decomposing

$$\{\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) |\, \det(\alpha) = n, d \equiv 1 \bmod N, N|c\} = \bigsqcup_i \alpha_i\Gamma_1(N),$$

define the Hecke operator $T(n)$ by $T(n)(f)(h) = \sum_i f(h\alpha_i)$, we get $f|T(n) = a(n,f)f$ if $a(1,f) = 1$ as $H^0(\widehat{\Gamma}_1(N), \pi) = \mathbb{C}f$, and $T(n)$ determines f and hence π. We often write $U(p)$ for $T(p)$.

§**3. Weight reduction to the constant sheaf.** Consider a morphism $I : L(n; \mathbb{Z}/p^\alpha\mathbb{Z}) \to \mathbb{Z}/p^\alpha\mathbb{Z}$ sending $P(X,Y) \mapsto P(1,0)$. Note $\Gamma_1(p^\alpha) \ni \delta \equiv \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ mod p^α. Thus

$$I(\delta P(X,Y)) = P((1,0)\left(\begin{smallmatrix} 1 & 0 \\ -u & 1 \end{smallmatrix}\right)) = I(P(X,Y)),$$

and hence $I : \mathcal{L}(n; \mathbb{Z}/p^\alpha\mathbb{Z}) \to \mathbb{Z}/p^\alpha\mathbb{Z}$ is an étale sheaf morphism over $Y_0(p^\alpha)$. Passing to the limit, we have a Galois and Hecke equivariant map: $I : TJ_n \to TJ_0$. Define, for $k = n+2$ and $\Lambda := \mathbb{Z}_p[[\Gamma]] = \mathbb{Z}_p[[T]]$ by $\langle\gamma\rangle \mapsto 1+T$ $(\Gamma = \gamma^{\mathbb{Z}_p})$,

$$\mathbf{h}_k(p^\infty; \mathbb{Z}_p) = \Lambda[[T(n) : 0 < n \in \mathbb{Z}]] \subset \mathrm{End}_\Lambda(TJ_n).$$

Independence Theorem: [EMI, Theorem 4.2.17]. We have a canonical isomorphism $i : \mathbf{h}_k(p^\infty; \mathbb{Z}_p) \cong \mathbf{h}_2(p^\infty; \mathbb{Z}_p)$ such that $i(T(n)) = T(n)$, $i(U(p)) = U(p)$ and $i(\langle z\rangle) = z^n\langle z\rangle$ for $z \in \mathbb{Z}_p^\times$, where $\langle z\rangle \in \varprojlim \Gamma_0(p^\alpha)/\Gamma_1(p^\alpha) \cong \mathbb{Z}_p^\times$ and $z^n \in \mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p$.

Hereafter we just write $\mathbf{h} := \mathbf{h}_2(p^\infty; \mathbb{Z}_p)$.

§**4. Finite level Hecke algebras.** Define

$$\mathbf{h}_k(p^\alpha;\mathbb{Z}_p) = \mathbb{Z}_p[T(n)|0 < n \in \mathbb{Z}] \subset \mathrm{End}_{\mathbb{Q}_p}(H^1_!(Y_1(p^\alpha),\mathcal{L}(n;\mathbb{Q}_p))).$$

Then by definition

$$\mathbf{h} = \varprojlim_{\alpha} \mathbf{h}_k(p^\alpha;\mathbb{Z}_p) \quad \text{for all } 2 \leq k \in \mathbb{Z}.$$

By Hodge decomposition (Eichler–Shimura):

$$H^1_!(Y_1(p^\alpha),\mathcal{L}(n;\mathbb{C})) \cong H^0(Y_1(p^\alpha)_{/\mathbb{C}},\omega^k_{cusp}) \oplus \overline{H^0(Y_1(p^\alpha)_{/\mathbb{C}},\omega^k_{cusp})},$$

alternatively, we can define

$$\mathbf{h}_k(p^\alpha;\mathbb{Z}_p) := \mathbb{Z}_p[T(n)|0 < n \in \mathbb{Z}] \subset \mathrm{End}_{\mathbb{Z}_p}(H^0(Y_1(p^\alpha)_{/\mathbb{Z}_p},\omega^k_{cusp})).$$

Thus $\mathbf{h}$ acts on V_k.

§**5.** q-**Expansion.** By the J-invariant, $\mathbf{P}^1(J)_{/\mathbb{Z}} = Y_1(1) \sqcup \{\infty\}$. Write hereafter $\mathbf{P}$ for $\mathbf{P}^1(J)$ and $X_1(N)(\mathbb{C}) = Y_1(N) \times_{\mathbf{P}} \mathbf{P}$ (compactification). We have a parameter $q = \exp(2\pi i\tau)$ at $\infty \in \mathbf{P}_{/\mathbb{C}}$. By John Tate, $\widehat{\mathcal{O}}_{X_1(N),\infty/\mathbb{Z}} = \mathbb{Z}[[q]]$. So $f \in H^0(X_1(p^\alpha)_{/\mathbb{Z}_p}, \omega^k_{cusp})$ has q-expansion $\sum_{n=1}^\infty a(n,f)q^n \in \mathbb{Z}_p[[q]]$, and $V_k \hookrightarrow \mathbb{Z}_p[[q]]$ as $\mathbb{Z}_p[[q]]$ is p-adically complete.

- Serre defined his space of p-adic modular forms as a p-adic completion of $\sum_k H^0(\mathbf{P}_{/\mathbb{Z}_p}, \omega^k_{cusp}) \subset \mathbb{Z}_p[[q]]$ in 1973.
- In 1975, Katz generalized this to the p-adic completion $V(p^\alpha)$ of $\sum_k H^0(Y_1(p^\alpha)_{/\mathbb{Q}_p}, \omega^k_{cusp}) \cap \mathbb{Z}_p[[q]]$ inside $\mathbb{Z}_p[[q]]$, and via his notion of geometric modular forms, he remarked

$$V(p^\alpha) = V(1) \quad \text{inside } \mathbb{Z}_p[[q]].$$

Duality Theorem: [EMI, Remark 4.2.18]. We have a perfect Banach duality $\langle\cdot,\cdot\rangle : \mathbf{h}_k(p^\infty; \mathbb{Z}_p) \times V_k \to \mathbb{Z}_p$ given by $\langle h, f\rangle = a(1, f|h)$.

This implies $V := V_k = V_2 = V(p^\infty) \overset{\text{Katz}}{=} V(1) \subset \mathbb{Z}_p[[q]]$.

§**6. Ordinary part.** The algebra $\mathbf{h}$ is a little too big to have an exact structure theorem. We consider the ordinary projection $e := \lim_{n\to\infty} U(p)^{n!}$ inside $\mathbf{h}$, which cuts down V and $\mathbf{h}$ to a reasonable size.

Write $\mathbf{h}^{ord} = e\mathbf{h}$ and $V^{ord} = e(V)$. For whatever $\alpha > 0$, we have

$$\Gamma_0(p^\alpha)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_0(p^\alpha)/\Gamma_0(p^\alpha)$$
$$= \Gamma_0(p^\alpha)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma_0(p^{\alpha-1})/\Gamma_0(p^{\alpha-1}) = \left\{\left(\begin{smallmatrix} p & u \\ 0 & 1 \end{smallmatrix}\right)\right\}_{u=0}^{p-1}$$

independent of α; so, a power of $U(p)$ reduces the level of $f \in S_k(\Gamma_1(p^\alpha))$ down to $\Gamma_1(p)$ (contraction property of $U(p)$).

Control theorem:

$$\mathbf{h}^{ord}/(\langle\gamma\rangle - \gamma^n)\mathbf{h}^{ord} \cong \mathbf{h}_k^{ord}(p;\mathbb{Z}_p) \quad (k = n + 2 \geq 2).$$

This is, for example, proven as [EMI, Theorem 4.1.29], and hence $\mathbf{h}^{ord}$ is Λ-free of rank $r = \mathrm{rank}_{\mathbb{Z}_p}\mathbf{h}_2(p;\mathbb{Z}_p)$ by ring theory.

§7. Galois representations.

Similarly $TJ_n^{ord} \xrightarrow[I]{\sim} TJ_0^{ord} \cong \Lambda^{2r}$, which is a Galois module (Tate module of the Jacobian of $X_1(Np)$). By Eichler–Shimura, Frob_l satisfies $X^2 - T(l)X + l\langle l\rangle = 0$ on TJ_0 with the image $\langle l\rangle$ of l in $\varprojlim_\alpha \Gamma_0(p^\alpha)/\Gamma_1(p^\alpha) \cong \mathbb{Z}_p^\times$ [GME, Theorem 4.2.2].

We have a canonical exact sequence of $\mathbf{h}$-modules:

$$0 \to \mathbf{h}^{ord} \to TJ_0^{ord} \xrightarrow{\mathrm{red}} \mathrm{Hom}_\Lambda(\mathbf{h}^{ord}, \Lambda) \to 0$$

[H13, Lemma 4.2]. Take a local ring $\mathbb{T}$ of $\mathbf{h}^{ord}$. If $\mathbb{T} \cong \mathrm{Hom}_\Lambda(\mathbb{T}, \Lambda)$ (i.e., $\mathbb{T}$ is Gorenstein), we have $TJ_0^{ord} \otimes_{\mathbf{h}^{ord}} \mathbb{T} \cong \mathbb{T}^2$ and we get a Galois representation $\rho_\mathbb{T} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{T})$ unramified outside p, and if $\mathbb{T}$ is not Gorenstein, localize $\mathbb{T}$ into $\mathbb{T}_P$ at a good prime P, we have $\rho_\mathbb{T} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{T}_P)$ such that

$$\det(1 - \rho_\mathbb{T}(\mathrm{Frob}_l)X) = 1 - T(l)|_\mathbb{T} X + l\langle l\rangle X^2 \quad \text{for all primes } l \neq p.$$

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