Twistor Spaces and Balanced Metrics on Complex Manifolds

Complex manifolds of complex dimension 1 (Riemann surfaces) are of course always Kähler, that is admit Kähler metrics, on account of the obvious dimension situation: $d\omega=0$ simply because it is a 3-form! This dimensional necessity naturally does not apply in complex dimension 2 or higher, but as it happens, most compact complex surfaces in fact are Kähler. Moreover, the non-Kähler examples occurring in the Kodaira classification are well-understood with the exception of some of the Class VII_0 surfaces, and it seems likely that the classification of these latter will be completed soon.

But in complex dimension 3, things are quite different. There are in fact vast collections of compact complex manifolds of complex dimension 3 that are guaranteed to admit no Kähler metric. One set of examples arises from the "twistor space" construction. This construction associates to certain (real) 4-manifolds a complex manifold of complex dimension 3 that is Kähler only in very specific and special cases.

The twistor space construction begins with a compact 4-manifold M^4 that admits what is called an anti-self-dual metric. Recall here that the Weyl tensor is the tensor the vanishing of which is equivalent to local conformal flatness. This is obtained from the Riemannian curvature tensor R by taking its traceless part. Explicitly it is given by

$$W = R - \frac{1}{n-2}\left(Ric - \frac{s}{n}g\right) \circ g - \frac{s}{2n(n-1)} g \circ g$$

where the circle product is defined by

$$\begin{aligned}(h \circ k)(v_1, v_2, v_3, v_4) = & h(v_1, v_3)k(v_2, v_4) + h(v_2, v_4)k(v_1, v_3) \\ & - h(v_1, v_4)k(v_2, v_3) - h(v_2, v_3)k(v_1, v_4)\end{aligned}$$

The curvature tensor R can be considered as an endomorphism from 2-forms to 2-forms(the curvature operator). Then R on self-dual 2-forms, that is, R restricted to the +1 eigenspace of * on 2-forms, can be considered as a pair of maps, into the self-dual and anti-self dual spaces respectively. The first of these , the map of self dual into the self dual component of the R image, is given by $(s/12) + W^+$, s being the scalar curvature, this notation defining W^+ as a map. Similarly W^- is anti-self-dual part of R on anti-self-dual forms - multiplication by the scalar curvature s /12. A metric is called anti-self-dual if one of W^+ or W^- is 0.

An abundance of such examples is provided by the theorem of Taubes that if N is any compact 4-manifold, then the connected sum of N with sufficiently many copies of S^2 x S^2 has an anti-self-dual metric in the sense indicated.

Now consider an compact oriented 4-manifold M with a Riemannian metric g. Then attached to M is a bundle consisting of , for each x in M, the almost complex structures J on the tangent space T_xM which are compatible with the orientation and which satisfy g(JX, JY)= g(X,Y) for all tangent vectors X and Y in T_xM. In other words. J acts as an orthogonal transformation on T_xM with respect to the metric g. The set of all such almost complex structures on T_xM for a fixed x in M is identifiable with S^2 as follows: Choose some unit vector X in T_xM. Then JX must be unit and perpendicular to X, but subject to that restriction JX can be chosen arbitrarily. So ,since the orthogonal complement of X is a three-dimensional subspace, JX can be chosen arbitrarily within a copy of S^2 . Once JX is chosen, J is determined. For the orthogonal complement of the J-invariant subspace span(X,JX) is necessarily J invariant and on this two-dimensional complement, J acts isometrically with J composed with J = - Id. But which of the two possible such isometries is involved is determined by the requirement that J preserves orientation on all of T_xM.

The total space of this bundle B of g-compatible orientation-compatible almost complex structures on the tangent spaces of M has itself an almost complex structure. This is defined almost tautologically. First note that at any point of B, namely a point of the form (x, J) where J is an almost complex structure of the required sort on T_xM, there is a natural subspace of the tangent space of B at that point, with the subspace being isomorphic to T_xM. This "horizontal subspace" is obtained in the usual way from the natural connection on almost complex structures. Namely, given any vector X in T_xM, and curve C(t) tangent to X in M, we can parallel translate J along C so that we have a parallel lift of C into B which projects back into M as C. (Here the parallel translation is relative to the natural notion of covariant derivative of endomorphisms of the tangent space arising from the Riemannian connection determined by g). The set of B- tangent vectors at (x,J) of such lifts is the "horizontal subspace" we are looking for. Now J itself acts on this horizontal subspace since J acts on T_xM by definition , and we have an obvious identification of T_xM and the horizontal subspace. We extend this J on the horizontal subspace to an almost complex structure on the whole tangent space of B at (x,J) by making this extension act on the tangent space of the fibre via the standard almost complex structure of S^2. In other words , we define $J_{\mathbf{B}}$ on the tangent space of B at (x,J) as the direct sum of J on the horizontal subspace via identification and J on S^2. It is easy to check that the latter is independent of how the identification of the fibre with S^2 is chosen in the discussion above.

So B , the so-called "twistor space", is thus an almost complex manifold, for any oriented 4-manifold M. It is natural to ask when is the almost complex structure of B integrable. This turns out necessarily to happen if the metric of M is anti-self-dual in the sense defined earlier. (result of Atiyah-Hitchin-Singer, 1986).

We introduce the notation for B , "Tw(M)" ,for "twistor space of M".

Note that by the result of Taubes, given any oriented 4-manifold M, there is a manifold of the form Tw(M # S^2 x S^2 #...... #S^2 x S^2) which is a complex manifold(of complex dimension 3) .

These twistor spaces have natural Hermitian metrics. Namely, one gets a metric on the twistor space by the combination of the metric of M and of the natural metric on metric-compatible almost complex structures(which have a natural metric induced from the M metric, since they are tensors). This corresponds to the splitting of the tangent space of Tw(M) at each point already discussed: we put the M-metric on the horizontal subspace and the metric on the fibre tangent space that arises from the fact that the fibres are M-tensors. This is easily checked to be Hermitian relative to the almost complex structure that we have discussed already.

Now in general these twistor spaces cannot be expected to be Kähler relative to any Hermitian metric at all. For one thing, van Kampen's Theorem shows that M # S^2 x S^2 #...... #S^2 x S^2 has the same fundamental group as M itself and hence so does the twistor space associated(this follows from the homotopy exact sequence of the fibration) . In particular, it could surely be that the first Betti number of Tw(M # S^2 x S^2 #...... #S^2 x S^2) could be odd, and thus the twistor space could not be Kähler in this case. But actually the manifold Tw(M # S^2 x S^2 #...... #S^2 x S^2) is even more rarely Kähler than this type of example suggests. Indeed , Hitchin has shown that a twistor space obtained in this way can be Kähler only when the underlying 4-manifold is CP^2 or S^4 . In the first case, the twistor space Tw(CP^2) is a flat manifold and in the second case Tw(S^4) is CP^3 .

Speculatively, one might try to prove the smooth 4-dimensional Poincare Conjecture (that a smooth manifold homeomorphic to S^4 is diffeomophic to the standard S^4) using these general ideas. One could try to deform the metric to be anti-self-dual(interesting question in itself) and then prove that the twistor space is Kähler and of the homotopy type of CP^3 . From the construction, it seems likely that it would in fact be diffeomorphic to (standard) CP^3 , associated to the idea that exotic differentiable structures, e.g., on the sphere, tend not to happen in dimension 6. Then one could try to reason back to the original S^4 situation.

One might also be able to approach the famous question of whether S^6 has a complex structure via similar ideas. Namely ,supposing that such a structure existed, one could "blow up" one point to get a topological (indeed diffeomorphic) CP^3 and then hope to study the question of complex structures on CP^3 to see whether this was possible in the first place.

We have already introduced the idea of a balanced metric on a complex manifold, namely a metric with the property that the n-1 power of its Kähler form is closed, where n is the complex dimension: i.e., $d\omega^{n-1} = 0$. Now for twistor spaces that arise from the anti-self-dual situation and are hence complex, we have observed that, except for a couple of special instance, $d\omega$ fails to be zero on the whole complex 3-dimensional twistor space, no matter what Hermitian metric is chosen on the space. But the natural metric on the whole space does have $d\omega^2 = 0$. Complex twistor spaces have balanced metrics!!

There are other reasons why balanced metrics are interesting:

First, the existence of a balanced metric does detect explicitly a nontrivial property of the complex manifold. In a general sense, there is an "obstruction" to the existence of a balanced metric. Namely, on a compact complex manifold with a balanced metric no compact complex submanifold of codimension 1(or more generally no compact codimension 1 subvariety) can be homologous to 0. (Of course on a Kähler manifold the statement holds with the restriction "of codimension 1" deleted). This follows the usual patter of the similar result for submanifolds of Kähler manifolds, regardless of codimension:
Specifically, let D be a compact subvariety of codimension 1 in M. Then $\int_D \omega^{n-1} > 0$ since ω^{n-1} is a positive form. On the other hand, if D were homologous to 0, then by Stokes Theorem, $d\,\omega^{n-1} = 0$ would imply that the integral was 0.

Note that a compact complex manifold definitely can have codimension 1 compact complex submanifolds that are homologous to 0. For example, consider the complex structure on $S^{2p+1} \times S^1$, $p>0$, obtained by regarding $S^{2p+1} \times S^1$ as C^{p+1} with the origin removed divided out by the action of scalar multiplication by 2, say. Then the image in $S^{2p+1} \times S^1$ of a complex dimension p complex linear subspace of C^{p+1} with the origin removed is a compact complex submanifold of codimension 1. And of course it is homologous to 0 since $S^{2p+1} \times S^1$ has homology =0 in real dimension 2p.

A similar construction can be used to find codimension 1 compact submanifolds of the Calabi Eckmann complex manifold structures on $S^{2p+1} \times S^{2q+1}$, $p,q>0$. Namely, there is in this case a fibration with torus fibres
$F: S^{2p+1} \times S^{2q+1} \to CP^p \times CP^q$, and if N is any codimension 1 submanifold of $CP^p \times CP^q$ (of which there are of course many), then $F^{-1}(N)$ is a codimension 1 complex submanifold of $CP^p \times CP^q$. And this is of course homologous to 0 since again the homology of $S^{2p+1} \times S^{2q+1}$ is 0 in real dimension $2(p+q)-2$.

Thus the Calabi –Eckmann manifolds have no balanced metrics, extending the well-known fact that they have no Kähler metrics.

A second reason for the importance of the balanced metric condition is that it is a birational invariant: if a compact complex manifold M admits a balanced metric then so does any compact complex manifold birationally equivalent to M. This is not obvious, however!

In the case of complex dimension 2, this reduces to the result shown by Kodaira, that being Kähler is a birational invariant for compact , complex surfaces.

This is shown as follows: If a complex surface is Kähler, then the blow up of a point(obtained by replacing a point of the surface by a CP^1) is also Kähler: Kodaira proved this as part of his proof of the projective embedding result now known as the Kodaira Embedding Theorem. Actually , he proved this particular thing for all dimensions, and indeed showed that the same is true for blowing up along the normal bundle of a complex sub-manifold of codimension at least two. Now every compact complex surface is obtained by blowing up points of one of the minimal models that occur in the Kodaira classification of surfaces result. And blowing up is a birational transformation: the manifold obtained by blowing up is birationally equivalent to the original manifold. So to prove that being Kähler is a birational invariant one need only check that the blowing up of points on the non-Kähler surfaces of the classification does not lead to Kähler manifolds, which can be done explicitly.

But in higher dimensions, a difficulty arises. While it is still true that the blowing up preserves being Kählerian(for some metric), Hironaka (Ann. Math 1962) gave an example where the blow-up was Kählerian but the original manifold was not. The example was birational to CP^3 in fact. The way that the example was shown to be non-Kähler was to exhibit a holomorphic curve that was homologous to 0. The construction actually involves a family M_t , t in a neighborhood of 0, with all M_t except M_0 , which is nonKähler.

Note that the existence of the curve described is consistent with the birational invariance of balanced metrics which implies that the example has a balanced metric. The example is birational to CP^3 . So the curve homologous to 0 has codimension two while it is only for a submanifold of codimension 1 that the existence a balanced metric implies non-zeroness of the homology class of the submanifold.

It is worthwhile looking at the proof of the birational invariance of the property of admitting a balanced metric. The argument involves the characterization by M.L. Michelson of when a balanced metric exists in terms of currents. (This is related to the Harvey-Lawson Kähler criterion discussed earlier, that a complex manifold admits a Kähler metric if and only if it has no closed positive (1,1) current which is the (1,1) part of a boundary). The result for balanced metrics is this:

Define a compact complex manifold M to be homologically balanced if every d-closed 2n-2 current with its (n-1,n-1) component nonzero and positive represents a nonzero class in the 2n-2 homology of M.

Then:

(Michelson) A compact complex manifold M has a balanced metric if and only if it is homologically balanced.

The proof (of the if part, the only if being clear from a previous argument) of this begins with the observation from linear algebra that a (real) positive (n-1,n-1) form is the n-1 power of one and only one (real) positive (1,1) form. Thus one can attempt to find the balanced metric with Kähler form ω by looking for the (n-1, n-1) form Ω that is going to be ω^{n-1} . For this, one uses a Hahn-Banach argument similar to the one used to justify the Harvey-Lawson Kähler criterion (cf. Lecture 3: this type of argument was introduced by D. Sullivan in 1976). Namely, one wishes to find a positive (n-1,n-1) form that is closed. For this, one notes that the set of positive (n-1,n-1) forms constitutes a (real) convex (half) cone. If this failed to intersect the subspace of closed forms then an application of the Hahn-Banach Theorem would produce a codimension 1 subspace of the (n-1,n-1) forms, one part of the complement of which contained the cone and with the hyperplane containing the closed (n-1, n-1) forms. Then +- the defining function of this hyperplane would be positive on the cone. And the defining function would be in the image of (adjoint) of d acting on currents, projected into its (n-1, n-1) part. This would contradict the hypothesis of the theorem. (see M.L. Michelson, Acta. Math. 149(1983) for details).

This characterization makes it possible to deal with the "blowing down" question, that is , to show that if $\overline{M} \to M$ is a blow-up along the normal bundle of a submanifold of a compact complex manifold M and if $\overline{M}$ admits a balanced metric then so does M. (Note that this was precisely what went wrong with birational invariance of being Kähler in Hironaka's example). This is related to the fact that currents push forward so that the Michelson condition can be moved from $\overline{M}$ to M. From this, one obtains the remarkable result that admitting a balanced metric is a birational invariant.

This has, in particular, relevance to Moishezon spaces. Recall that a Moishezon space is by definition a compact complex manifold(or other places, manifold with singularities, but we restrict our attention to manifolds) of complex dimension n for which the transcendence degree of the field of meromorphic functions is n . (C.L. Siegel proved long ago that the transcendence degree was $\leq$ n . A Moishezon space is one where the maximum possible value is attained.) Such a space is bimeromorphic to a projective algebraic manifold. So one deduces that Moishezon spaces admit balanced metrics. (In general, they need not admit Kähler metrics.)

A second interesting aspect of the balanced metric situation involves questions of deformation invariance. Kodaira proved that a small deformation of a Kähler manifold admits a Kähler metric. This is referred to as "local deformation". Explicitly , it means that if M_t is a family defined for t near 0 (and equal to 0), then if M_0 is Kähler, so is M_t for t near 0. (This is actually natural to expect since Kähler metrics are given locally by the Levi form(s) of potential functions on the open sets of some open covering, and the Levi Laplacian of each such a local potential will be a positive definite form for any sufficiently nearby complex structure, on a slightly smaller open set, the collection of all of which will still cover M). But "global deformation" is wrong: the fact that M_t is Kähler for all t near 0 need not imply that M_0 is Kähler. (One sees easily how the idea of using local potentials fails: there may be no limiting local potential and even if there were, it might not have a positive definite Levi form.)

For balanced metrics, one has the following result: In the same notation as in the previous paragraph, if M_0 is balanced(admits a balanced metric) and if M_0 satisfies the d dbar Lemma, then M_t is balanced and satisfies the d dbar Lemma. (Recall that the d dbar Lemma is the assertion that if a type (p,p) form φ is d-exact, that is $\varphi = d\alpha$ for some 2p-1 form α, then there is a form β such that φ is $\partial\bar{\partial}\beta$. This is always true on Kähler manifolds as shown earlier and is important for Kähler geometry.) This shows that balanced metric plus the d dbar Lemma gives one a situation similar to Kähler in a sense.

There are in fact counterexamples to the local deformation stability of balanced in the absence of the d dbar Lemma. Also, the global situation is unclear: is it the case that if M_t is balanced and has the d dbar Lemma for all t near 0 that M_0 is balanced and has the d dbar Lemma satisfied? This is unknown(the Hironaka example referred to earlier does not apply, since there is a curve in the M_0 which is homologous to 0 so that manifold is not balanced).

Michelson also showed that the balanced condition had a useful property relative to fibre spaces: If $f: M \to C$ is a fibre space with irreducible fibres over a curve C ,possibly with some singular fibres, and if the nonsingular fibres are balanced, then M is balanced, provided that the fibration is essential topologically, that is, that the pull-back of the fundamental class in C is not homologically 0 in M.

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