**Lecture no. 4**, Professor S.T. Yau , April 12, 2007

Last time we explained how, for a compact complex surface, $b_1$ even implies that the surface is Kähler. The argument we used made use of the classification of surfaces of Kodaira. But in fact it is possible to establish this result without using the classification conclusions. This was done by Buchdahl(Ann. Ins. Fourier 49(1999) ) and independently by Lamari (J. Math.Pures Appl. 78(1999)). This uses a result of Gauduchon that is useful in other contexts: On any compact complex manifold there is a form ω which is the Kähler form of an Hermitian metric and which satisfies $\partial\bar{\partial}\omega^{n-1} = 0$.
This is in fact doable by a conformal change from a given metric form. The equation involved is a scalar equation which is basically elliptic and one can do some estimate to solve for the conformal factor. (The original proof used the Hahn-Banach Theorem).

Now in the complex dimension = 2 case, one has just that $\partial\bar{\partial}\omega = 0$, and from this one gets a closed (positive) "Kähler current". Recall from the earlier discussion of the Harvey-Lawson result that the concept of a current is just the generalization to forms of the idea of a distribution and that a closed current is one that applied to d(form) , the form of the appropriate degree, is 0. A "Kähler current" is a (1,1) form $\tilde{\omega}$ that is closed and that satisfies, for some ω that is the Kähler form of a smooth Hermitian metric, $\tilde{\omega} \geq \omega$ .
(More on this later).
From such a current, one gets a Kähler metric by regularization. The singularities possible for the current are controlled by Siu's structure theorem on closed positive currents and then one uses regularization results of Demailly(1996). Note that the latter depends on Monge Ampere equation results even though there is no explicit use here of e.g., Ricci flat metrics. This method avoids classification results entirely and is this likely to be generalizable. We shall return to this topic later.

Inoue constructed explicit example of class VII surfaces: these will be presented in detail in an appendix below

Bogomolov's original argument for the fact that, in the case indicated((class VII, algebraic dimension=0, $b_2$=0, no curves) , Inoue's surfaces are the only examples involved complicated arguments about the possible representations of the fundamental group into GL(2,C) arising from the affine structure(for how this affine structure arises, look a few paragraphs down). But a much more direct and shorter proof of the result that only Inoue examples appear here was given by Li, Yau, and Zheng, using Hermitian Yang Mills theory. This is based on the result of Inoue that a compact complex manifold of the sort indicated is necessarily one of the Inoue examples if it satisfies the following additional condition:

Inoue Condition: For some line bundle L over M , the space of holomorphic sections of $T_M \otimes L$ has positive dimension.

Bogomolov also used this and tried to show that this was always so: it is not possible that for every L , the space of holomorphic sections of the tensor product bundle is the zero section alone. Li,Yau, Zheng show this by showing that if no line bundle exists with the tensor product having nontrivial sections then in fact that manifold has a Kähler metric. In the present case, this is a contradiction, since of course the surface here cannot have a Kähler metric(since $b_1$=1 is not even).

The essential idea is to use the absence of holomorphic objects of one kind to imply the existence of interesting objects of another kind by using the obstruction to existence to the holomorphic objects. In general terms, one gets something out of the existence of nothing, since the definite nothing implies that the obstruction to something is nonzero.

Specifically, one works with the idea of holomorphically affine connections. A type (1,0) connection on the holomorphic tangent bundle is called holomorphically affine if, with respect to a holomorphic frame, its connection forms are type (1,0). The obstruction to the existence of such a connection lies in $H^1( M, \Omega_M \otimes \Omega_M \otimes T_M)$ and the difference of two such connection lies in $H^0$ of the same bundle. Now it was proved from this by Inoue,Kobayashi, Ochiai(J. Fac. Sci Univ. Tokyo 27(1980)) that: for M of class VII and satisfying the Inoue line bundle condition, there is a unique holomorphic affine connection on the tangent bundle of M. (This is also the way the Bogomolov argument commences: at this point, one notes that this affine structure gives a holonomy representation of the fundamental group into GL(2,C) and analyzes the possibilities for that).

Li,Yau, Zheng then use Yang-Mills ideas to find a Hermitian Yang-Mills connection relative to a specific metric on M and then use uniqueness of holomorphic affine connections to deduce that the metric in question is Kähler.

This construction depends on the stability of the tangent bundle in the present case. Recall the relevant stability definitions as follows:

First, choose a Gauduchon metric, that is an Hermitian metric with Kähler form ω satisfying $\partial\bar{\partial}\omega = 0$. (In higher dimensions, this would be $\omega^{n-1}$). If V is a sub-bundle of the tangent bundle(or more generally a coherent sheaf) set

degree(V)= $\int c_1(V) \wedge \omega$ (in higher dimensions, $\omega^{n-1}$)

This makes sense because the form integrated is top-dimensional.
Now the $\partial\bar{\partial}\omega = 0$ d condition gives that this is independent of the choice of the metric on V (in the vector bundle case of V). Namely, $c_1(V) = \left(\frac{i}{2\pi}\right)\partial\bar{\partial}\log D$ as usual. For any other metric , this changes by a global term of the form $\partial\bar{\partial} f$ for some function f. Hence the integral giving the degree changes by a term of the form
$\left(\frac{i}{2\pi}\right)\int \partial\bar{\partial} f \wedge \omega^{n-1}$ which is necessarily 0 if $\partial\bar{\partial}\omega^{n-1} = 0$.

[For this in detail: recall that $\int d\left(\alpha \wedge \beta\right) = 0$ implies $\int d\alpha \wedge \beta = \pm\int \alpha \wedge d\beta$. If α is a (p-1,q) form and β a (n-p,n-q)form, this gives $\int \partial\alpha \wedge \beta = \pm\int \alpha \wedge \partial\beta$ since the other terms are 0 by type considerations. Similarly, $\bar{\partial}$ "moves to the other side" as $\pm\bar{\partial}$. So $\partial\bar{\partial}\omega^{n-1} = 0$ implies $\int \partial\bar{\partial} f \wedge \omega^{n-1} = 0$].

So for a Gauduchon metric, the concept of degree is well-defined independently of metric choices.

By definition, a vector bundle V is "stable" if for all coherent sub-sheaves F of V,

degree (F)/rank (F) < deg(V)/ rank V

(This concept was introduced by Mumford in Geometric Invariant Theory and written more precisely by Gieseker for algebraic surfaces)

Now as before, let M be a compact complex surface which definitely fails to satisfy the Inoue condition: that is, for every line bundle L over M , the space of holomorphic sections of $T_M \otimes L$ consists of the 0 section alone. (Here $T_M$ is the holomorphic tangent bundle of M). Note that under this hypothesis, there is no (holomorphic) sub-bundle of $T_M$ with fibre dimension 1. [The reason is that if L were such a sub-bundle, then L would be a holomorphic line bundle with $T_M \otimes L^*$ containing as a sub-bundle $L \otimes L^*$, and this latter being trivial has of course a holomorphic section. This would be a contradiction.]

Thus in this case, the (holomorphic) tangent bundle of M is automatically stable.

Stability corresponds to the obstruction to the existence of Hermitian Yang-Mills connections(which are themselves a generalization of Kähler-Einstein metrics). This was initiated by Donaldson (unpublished, his masters thesis!) for K3 surfaces, where he showed that there is an Hermitian Yang-Mills connection on the tangent bundle. (For the tangent bundle, Kähler Einstein gives Hermitian Yang-Mills automatically. So for Kähler K3 surfaces, the conclusion follows from Yau's solution of the Calabi Conjecture in the 0 case. But at the time , Siu's result that all K3 surfaces are Kähler had not been established.)

Donaldson extended this to stable bundles over algebraic surfaces in general. This makes use of the transgression for $c_2 = i\partial\bar{\partial}\Theta$, where $\Theta$ is locally defined, and then using variational principles and the heat equation to deform the connection on the original bundle to get a Hermitian Yang-Mills connection on another vector bundle(different complex structure) ,which is isomorphic to the original bundle in the stable case. So stability implies the existence of an Hermitian Yang-Mills connection(Donaldson's thesis).

Uhlenbeck and Yau extended this to the general case of any Kähler manifold(algebraic was not needed) by showing that if the process did not converge then there would be a sub-sheaf that would violate stability. So either there is an Hermitian Yang-Mills connection or there is a holomorphic object violating stability.

J.Li and Yau proved this also holds in the non-Kähler case(assuming stability).

The complex dimension 2 situation is easier than higher dimensions because $c_1^2$ and $c_2$ are numbers in that case. For generalization to higher dimensions, one has to work with wedge-producting in a suitable power of the Kahler form of an Hermitian metric . In particular the condition that if V has rank r, then $\int\left(c_2-\frac{2r}{r-1}c_1^2\right)\wedge\omega^{n-2}=0$ is equivalent to the existence of an Hermitian Yang Mills connection with curvature $F_h$ of the form $\alpha\otimes Id_v$ ,where $\alpha$ is an Hermitian (1,1) form.

The number $\int\left(c_2-\frac{2r}{r-1}c_1^2\right)\wedge\omega^{n-2}$ is independent of choice of (fibre) metric as before if $\partial\bar{\partial}\omega^{n-2}=0$ [Here, $c_2$ and $c_1$ are the forms representing the Chern classes, so the integral is that of an (n,n) form and the integral is number-valued.] (This is an automatic condition if n=2). So now one starts with $\partial\bar{\partial}\omega^{n-2}=0$. But one really wants $\partial\bar{\partial}\omega^{n-1}=0$ (different condition) .

So one modifies $\omega$ by a conformal factor $e^{\rho}$. Namely set $\tilde{\omega}=e^{\rho}\omega$ with $\rho$ chosen to get $\partial\bar{\partial}\tilde{\omega}^{n-1}=0$ . One still has the relevant properties since this is only a conformal change. In particular one obtains the result :

**Theorem** (Li, Yau): Suppose that there is an Hermitian form $\omega$ such that $\partial\bar{\partial}\omega^{n-2}=0$ and V is stable , then the bundle V has a metric h such that the curvature form $F_h$ is of the form $\alpha\otimes Id_v$ .

In the class $VII_0$ case, $b_2=0$ and $b_1=1$ so $c_2=0$ by Euler characteristic while $c_1=0$ and so its square is 0 automatically. Thus the Chern class condition is automatically satisfied. And hence there exists a connection of the type described. This gives a complex projectively flat connection. (These will be classified next time).

An appendix covering the details of Inoue surfaces will be added here later.

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