Sample Midterm Problem Solutions (continued)

3. Associated polynomial is $\lambda^2 + a\lambda + b = 0$
(remember: given $a > 0, b > 0$). Roots are $\lambda = -\dfrac{a \pm \sqrt{a^2 - 4b}}{}$

$= -\dfrac{a}{2} \pm \dfrac{\sqrt{a^2 - 4b}}{2}$. Now if $a^2 - 4b < 0$, then

the general solution is $C_1 e^{-ax/2} \cos\left(\tfrac{1}{2}\sqrt{-a^2 + 4b}\; x\right)$
$+ C_2 e^{-ax/2} \sin\left(\tfrac{1}{2}\sqrt{-a^2 + 4b}\; x\right)$. This comes from

$\lambda_1, \lambda_2 = -\dfrac{a}{2} \pm \dfrac{i}{2}\sqrt{-a^2 + 4b}$ so complex solutions (independent)

are $e^{\left(-\frac{a}{2}x + \frac{ix}{2}\sqrt{-a^2 + 4b}\right)}$ and $e^{\left(-\frac{a}{2}x - \frac{ix}{2}\sqrt{-a^2 + 4b}\right)}$.

Since $a > 0$, these solutions are clearly transient.

If $a^2 - 4b \geq 0$ then $\sqrt{a^2 - 4b}$ is real and
$0 \leq \sqrt{a^2 - 4b} < a$ (actually $< a$, since $b > 0$).

So the two independent solutions are
$e^{x(-a + \sqrt{a^2 - 4b})}$ and $e^{x(-a - \sqrt{a^2 - 4b})}$ and

in both the x-coefficients in the exponents are
negative. So these are transient and hence

any constant-coefficient linear combination of
them (the general solution) is also transient.

[Think about what happens when $a \to 0$: the
two roots become purely imaginary and the
general solution becomes the nontransient
$C_1 \cos(\sqrt{b}\; x) + C_2 \sin(\sqrt{b}\; x)$: this is just putting
$a = 0$ in the case of $a^2 - 4b < 0$.]

4. Rough statement (which is all we have at the moment!):
An nth order differential equation $\frac{d^n y}{dx^n} = F\left(\frac{d^{n-1} y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y, x\right)$
has exactly one solution y with $y(x_0), \cdots, y^{(n-1)}(x_0)$
having specified values.

For future reference, here is a more precise statement:
Given x_0 and $a_0, \cdots, a_{n-1}$, suppose F is continuously
differentiable in some neighborhood of the point
$(a_0, \cdots, a_{n-1})$ in $\mathbb{R}^n$. Then there is an $\varepsilon > 0$ and
a function y on $(x_0 - \varepsilon, x_0 + \varepsilon)$ (interval in $\mathbb{R}$)
such that y is n-times differentiable on $(x_0 - \varepsilon, x_0 + \varepsilon)$,
y satisfies the differential equation, and
$y(x_0) = a_0, \quad y'(x_0) = a_1, \quad \cdots \quad y^{(n-1)}(x_0) = a_{n-1}$
(this is "existence").

Uniqueness part: If y_2 is a function on $(x_0 - \varepsilon, x_0 + \varepsilon)$
which satisfies the differential equation and which
has $y_2(x_0) = a_0, \quad y_2'(x_0) = a_1, \cdots, \quad y^{(n-1)}(x_0) = a_{n-1}$, then
$y_2 \equiv y$ on $(x_0 - \varepsilon, x_0 + \varepsilon)$.

5. With the c_j's chosen as in the Suggestion, the function $F = \sum c_j Y_j$ has value 0 at x_0, first derivative $= 0$ at $x_0, \ldots, (n-1)$st derivative $= 0$ at x_0. Since the differential equation is homogeneous linear, F is a solution of the equation. So is the $\equiv 0$ function. Since F and the 0 function have the same value and $(n-1)$ derivatives at x_0, $F =$ the 0 function by uniqueness. Thus
$$\sum c_j Y_j \equiv 0.$$
This implies that at any other point x_1,
$$\left(\sum c_j Y_j\right)(x_1) = \sum c_j Y_j(x_1) = 0$$
$$\left(\sum c_j Y_j\right)'(x_1) = \sum c_j Y_j'(x_1), \ldots \left(\sum c_j Y_j\right)^{(n-1)}(x_1)$$
$$= \sum c_j Y_j^{(n-1)}(x_1) = 0$$
so that the column vectors
$$\begin{pmatrix} Y_j(x_1) \\ Y_j'(x_1) \\ \vdots \\ Y_j^{(n-1)}(x_1) \end{pmatrix} \qquad j = 1, \ldots, n$$

are dependent and hence
$$\det \begin{pmatrix} Y_1(x_1) & \cdots & Y_n(x_1) \\ \vdots & & \vdots \\ Y_1^{(n-1)}(x_1) & & Y_n^{n-1}(x_1) \end{pmatrix} = 0 \qquad \square$$

6. This follows from the argument used for problem 5 since if all the $y_j(x_0)$ are 0, then the vectors

$$\begin{pmatrix} y_j(x_0) \\ y_j'(x_0) \\ \vdots \\ y_j^{(n-1)}(x_0) \end{pmatrix} = \begin{pmatrix} 0 \\ y_j'(x_0) \\ \vdots \\ y_j^{(n-1)}(x_0) \end{pmatrix} \qquad j = 1, 2, \ldots, n$$

are necessarily dependent (reason: associated determinant clearly $= 0$ $\underline{or}$ you can observe that they are n vectors in the $(n-1)$ dimensional subspace consisting of all vectors with first component $= 0$).

7. $k = 1$:
$$(D - \alpha)(x^1 e^{\alpha x}) = e^{\alpha x} + x \alpha e^{\alpha x} - \alpha(x \cdot e^{\alpha x})$$
$$= e^{\alpha x} = k! \, e^{\alpha x} \text{ for } k = 1.$$

Assuming works for $k \geq 1$, then
$$(D - \alpha)^{k+1}(x^{k+1} e^{\alpha x}) = (D - \alpha)^k \left[(D - \alpha)(x^{k+1} e^{\alpha x}) \right]$$
$$= (D - \alpha)^k \left[(k+1) x^k e^{\alpha x} + x^{k+1} \cdot \alpha e^{\alpha x} - \alpha x^{k+1} e^{\alpha x} \right]$$
$$= (D - \alpha)^k \left[(k+1) x^k e^{\alpha x} \right] = (k+1)(D - \alpha)^k (x^k e^{\alpha x})$$
$$= (k+1) \, k! \, e^{\alpha x} = (k+1)! \, e^{\alpha x} \qquad \square$$

↑ induction hypothesis

8. If $P(\lambda) = (\lambda - \alpha)^k \, Q(\lambda) = Q(\lambda)(\lambda - \alpha)^k$

then $P(D) = Q(D)(D - \alpha)^k$.

Then $P(D)\left(\dfrac{1}{k! \, Q(\alpha)} \, x^k e^{\alpha x}\right)$

$$= \dfrac{1}{k! \, Q(\alpha)} \, Q(D)\left[(D-\alpha)^k (x^k e^{\alpha x})\right]$$

$$= \dfrac{1}{k! \, Q(\alpha)} \, Q(D)\left[k! \, e^{\alpha x}\right] = \dfrac{1}{k! \, Q(\alpha)} \, k! \, Q(\alpha) e^{\alpha x} = e^{\alpha x}$$

Prob 7

where $Q(D) \, e^{\alpha x} = Q(\alpha) e^{\alpha x}$ follows immediately

from $D^\ell e^{\alpha x} = \alpha^\ell e^{\alpha x}$, every $\ell \geq 0$.

9. Actually, one of the ways problem 2 was solved there was a direct application of the P, Q method of problem 8. So this is already done!

10. General solution of of $\dfrac{d^2 y}{dx^2} - 2 \dfrac{dy}{dx} + y = 0$:

$P(D) = D^2 - D + 1 = (D-1)^2$. So independent solutions are e^x and $x e^x$. So $y_1 = e^x$, $y_2 = x e^x$

$y_1' = e^x$, $y_2' = e^x + x e^x$. Solve $\quad e^x v_1' + x e^x v_2' = 0$

$$(e^x) v_1' + (e^x + x e^x) v_2' = e^x$$

Taking difference $\quad v_2' \, (\underset{-x e^x}{e^x + x e^x}) = e^x$ or $v_2' = 1$

So $v_1' = -x$. Thus $v_1 = -\dfrac{x^2}{2} + C_1$, $v_2 = x + C_2$.

Solution $= -\dfrac{x^2}{2} e^x + x^2 e^x + C_1 e^x + C_2 x e^x = \dfrac{x^2}{2} e^x +$ gen sol of homog eq

Check $y = \frac{x^2}{2} e^x$ $\quad y' = \frac{x}{1} e^x + \frac{x^2}{2} e^x$ $\quad y'' = e^x + x e^x + x e^x$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \frac{x^2}{2} e^x$$

$$y'' - 2y' + y = \begin{array}{l} e^x + 2x e^x + \frac{x^2}{2} e^x \\ \quad - 2x e^x + - x^2 e^x \\ \quad + \frac{x^2}{2} e^x \end{array} = e^x \quad \checkmark$$

11. Since determinant $= 0$ if $\lambda_i = \lambda_j$ $\quad i > j$

it follows that (as polynomial in $\lambda_1, \ldots \lambda_n$)

det is divisible by $\lambda_i - \lambda_j$ all i, j with $i > j$.

So det is divisible by $\prod_{i > j} (\lambda_i - \lambda_j)$.

Now det has total degree $1 + 2 + \ldots n - 1 = n(n-1)/2$;

and so does $\prod_{i > j} (\lambda_i - \lambda_j)$ — factors are linear

and there are $\frac{n(n-1)}{2}$ of them. So

$\prod_{i > j} (\lambda_i - \lambda_j)$ divides det $\Rightarrow$ det $= (\text{cons.}) \prod (\lambda_i - \lambda_j)$

Checking coefficient of $\lambda_n^{n-1} \lambda_{n-1}^{n-2} \ldots \lambda_1$ in both

(coefficient is 1 in both cases) gives cons. $= 1$.

Hence det $= 0 \iff$ some $\lambda_i =$ some λ_j $i > j$.

And if λ's are all distinct, then det $\neq 0$.

12. (Done in notes) If $C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x} = 0$
then (supposing without loss of generality that
$\lambda_1 < \lambda_2 < \dots < \lambda_n$) multiplying by $e^{-\lambda_n x}$ gives
$C_1 e^{(\lambda_1 - \lambda_n)x} + \dots + C_{n-1} e^{(\lambda_1 - \lambda_n)x} + C_n = 0$. Let
$x \to +\infty$ to get $C_n = 0$. Continue to get all
C's $= 0$.

13. By prob. 12, $e^{\lambda_1 x}, \dots e^{\lambda_n x}$ are independent
solutions.

$$\det \begin{pmatrix} y_1(0) & \cdots & y_n(0) \\ y_1'(0) & \cdots & y_n'(0) \\ & \vdots & \\ y_1^{(n-1)}(0) & \cdots & y_n^{(n-1)}(0) \end{pmatrix} = \det \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \lambda_1^2 & \cdots & \lambda_n^2 \\ \lambda_1^{(n-1)} & \cdots & \lambda_n^{(n-1)} \end{pmatrix}$$

with $y_j = e^{\lambda_j x}$. Now problem 11 verifies the nonzeroness
of the determinant, which confirms the independence
of the solutions. Problem 5 then implies that

$$\det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & & y_n^{(n-1)}(x) \end{pmatrix}$$

should be nonzero for every x.

This

$$= \begin{pmatrix} e^{\lambda_1 x} & \cdots & e^{\lambda_n x} \\ \lambda_1 e^{\lambda_1 x} & & \vdots \\ \vdots & & \\ \lambda_1^{n-1} e^{\lambda_1 x} & & \lambda_n^{n-1} e^{\lambda_n x} \end{pmatrix} = \left(e^{\lambda_1 x} \cdots e^{\lambda_n x} \right) \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

so this works. And this latter calculation shows that it really is true that nonzero at one point implies nonzero at every point since the values are all nonzero (exponential) multiples of the value at 0. [In fact if we write $W(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$ for the determinant, then

$$W = (cons.)\, e^{(\lambda_1 + \ldots + \lambda_n)x}$$

So $\qquad W' = \left(\sum_{j=1}^{n} \lambda_j\right) W \qquad$ where $\qquad \sum \lambda_j =$ satisfies

— the λ^{n-1} coefficient of $P(\lambda)$, or W

$W' = -p_1 W$ if the differential equation is

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots p_n y = 0.$$

This turns out to work even when the p's are functions, not constants: i.e. $W' = -p_1 W$ in the general case, even though the form of the solutions y, as exponentials may not hold in this more general situation]

Note: Remarks in the bracket [ ] will <u>not</u> be on the midterm.