Sample Problems for Midterm (continued)

14 Solution (particular) is $\frac{1}{P(i\omega)} e^{i\omega x}$ for RHS = $e^{i\omega x}$ where $P(\lambda) = \lambda^2 + \tau\lambda + k$. (Note that $P(i\omega)$ $\neq 0$ since roots of $P(\lambda)$ are $-\frac{\tau}{2} \pm \sqrt{\tau^2 - 4k}$ which cannot be pure imaginary [Another reason: $P(i\omega)$ $= -\omega^2 + k + i\tau\omega$ which cannot be 0 since Im part $= \tau\omega$ and $P(0) \neq 0$ so $\omega \neq 0$ and hence $\tau\omega \neq 0$ is only possibility].

Now $\frac{1}{k-\omega^2+i\tau\omega}(\cos\omega x + i\sin\omega x) = \frac{1}{P(i\omega)} e^{i\omega x}$

So $\frac{1}{P(i\omega)} e^{i\omega x} = \frac{(k-\omega^2) - i\tau\omega}{(k-\omega^2)^2 + (\tau\omega)^2}(\cos\omega x + i\sin\omega x)$. Taking Imaginary parts:

$$\text{Im}\,\frac{1}{P(i\omega)} e^{i\omega x} \text{ thus } = \frac{1}{(k-\omega^2)^2 + \tau^2\omega^2}\left[(k-\omega^2)\sin\omega x - (\tau\omega)(\cos\omega x)\right]$$

$$= \frac{1}{\sqrt{(k-\omega^2)^2 + \tau^2\omega^2}}\left[\frac{k-\omega^2}{\sqrt{(k-\omega^2)^2+\tau^2\omega^2}}\sin\omega x - \frac{\tau\omega}{\sqrt{(k-\omega^2)^2+\tau^2\omega^2}}\cos\omega x\right]$$

There is an angle ϕ with $\cos\phi =$ and $\sin\phi =$ because sum of squares of these two things $= 1$. Then item in brackets [ ] $= \sin(\omega x - \phi)$ by usual formula for $\sin(A-B)$. So particular solution of diff eq, RHS $= \sin\omega x$ is

$$\frac{1}{\sqrt{(k-\omega^2)^2+\tau^2\omega^2}}\sin(\omega x - \phi).$$

General sol = this + general solution of homogeneous equation which (as was shown earlier) gen. homo. sol is transient.

This finishes part (a). For part (b), note that max amplitude of the $\frac{1}{\sqrt{}}\sin(\omega t-\phi)$ part occurs for that ω value (if any) for which $\sqrt{}$ is minimum, i.e. $(k-\omega^2)^2+\tau^2\omega^2$ attains its minimum. This $= \omega^4+(\tau^2-2k)\omega^2+k^2$

Think of this as a quadratic polynomial in $\omega^2 (\geq 0)$

Its minimum point on $[0,+\infty)$ is either $\omega=0$

or where $\omega^2 = -\frac{\text{coefficient of } \omega^2}{2} = \frac{2k-\tau^2}{2}$

So condition for a "nondegenerate" minimum ($\omega^2_{min}>0$) is $-\tau^2+2k>0$. Assuming this condition is met, minimum is $\left(k-\frac{\tau^2}{2}\right)^2+\frac{1}{2}(\tau^2-2k)(2k-\tau^2)+k^2$

$= k^2-\tau^2k+\frac{\tau^4}{4}+\frac{1}{2}(-\tau^4-4k^2+4k\tau^2)+k^2$

$=-\frac{1}{4}\tau^4-0k^2+k\tau^2 = (1/4)(-\tau^4+4k\tau^2) =$

$= \frac{1}{4}\tau^2(4k-\tau^2)$

Note that $4k-\tau^2>0$

since $2k-\tau^2>0$.

So max amplitude is $\frac{1}{\sqrt{\frac{1}{4}\tau^2(4k-\tau^2)}} = \frac{\sqrt{4}}{\sqrt{\tau^2(4k-\tau^2)}} = \frac{1}{\tau\left(k-\frac{\tau^2}{2}\right)^{1/2}}$

Note that when τ is small (not much damping), this is large. Also when τ is small, max. amplitude occurs near $\omega^2=k$, the undamped resonance.