Constant-Coefficient Linear Differential Equations: Lecture II

Equations of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = F(x) \qquad (*)$$

where $p_1, \ldots, p_n$ are constants (numbers).

Later we shall have a procedure for solving for y, given n "independent" solutions $y_1, \ldots, y_n$ of the equation, the associated "homogeneous" equation

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0 . \qquad (**)$$

(We shall define "independent" in a moment. The procedure from going from the general solution of the homogeneous equation to the solution of the equation when $F \not\equiv 0$ actually will work when the $p_1, \ldots, p_n$ are functions, not necessarily constants. This will be covered later).

Notation: Write $D = \frac{d}{dx}$ and the homogeneous equation $(**)$ as

$$D^n y + p_1 D^{n-1} y + \cdots + p_{n-1} D y + p_n y = 0 .$$

We associate to this a polynomial $P(\lambda)$ defined by

$$P(\lambda) = \lambda^n + p_1 \lambda^{n-1} + \cdots + p_{n-1} \lambda + p_n$$

The polynomial $P(\lambda)$ can be factored

$$P(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$$

where the λ_i's are numbers, but they may need to be complex numbers, e.g.

$$D^2 + 1 , \quad \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

Equation $\quad \dfrac{d^2 y}{dx^2} + y = 0$

The factorization of $P(\lambda)$ corresponds to a factorization of the left hand side of the equation:

$$\frac{d^n y}{dx^n} y + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdot \quad + p_n y$$

$$= \left(\frac{d}{dx} - \lambda_1\right) \left[\left(\frac{d}{dx} - \lambda_2\right) \left\{ \quad \cdots \quad \right] \right]. \; y$$

$$= (D - \lambda_1)\,((D - \lambda_2) \cdots \quad (D - \lambda_n) y))\cdots$$

written as

$$(D - \lambda_1)(D - \lambda_2) \cdots (D - \lambda_n)\, y$$

Example $\quad \dfrac{d^2 y}{dx^2} - 5 \dfrac{dy}{dx} + y$

$$= \left(\frac{d}{dx} - 3\right) \left[\left(\frac{d}{dx} - 2\right) y \right]$$

$$\doteq (D - 3)(D - 2) y .$$

This makes sense because, since all coefficients are constant,

$$(D-3)\left[(D-2)y\right] = \left(\frac{d}{dx}-3\right)\left[\frac{dy}{dx}-2y\right]$$

$$= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3\left(\frac{dy}{dx}-2\right) = \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y.$$

The $(D-\lambda_i)$ factors commute as "operators" just the way the $\lambda-\lambda_i$ factors commute in writing the polynomial $P(\lambda)$ as a product.

Why is this useful? The reason is that we already know how to solve $(D-\lambda_i)\,G = 0$ (or $= F(x)$) given a number λ_i complex or not. So we can solve (**) (or for that matter (*)) by solving first order linear equations successively.

Example: Find the general solution of

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

Answer: $(D^2 - 5D + 6)y = (D-3)(D-2)y$.
Let $G(x) = (D-2)y$. Then $(D-3)G = 0$
So (as in Lecture I)

$$G(x) = Ae^{3x} \quad \text{for some constant } A$$

Then $(D-2)y = G = Ae^{3x}$. Solving (using "integrating factor" e^{-2x}) gives

$$D(e^{-2x}y) = Ae^{x} \quad \text{or} \quad e^{-2x}y = Ae^{x} + B$$

for B another constant or $\quad y = Ae^{3x} + Be^{2x}$.

If we did this in the opposite order $(D-2)[(D-3)y]=0$:

$G = (D-3)y$ so $G(x) = Ae^{2x}$

and

$$D(e^{-3x}G) = Ae^{2x}e^{-3x} = Ae^{-x}$$

so

$$e^{-3x}y = B + \int Ae^{-x} = B - Ae^{-x}$$

and

$$y = Be^{3x} - Ae^{2x}$$

The answer looks different. For one thing, there is a minus sign. But since A and B are arbitrary constants, the sets of all solutions you get are really the same in both cases.

Clearly, this process works for the general 2nd order case and the $n>2$ cases, too. One just peels off the layers of $(D-\lambda_i)$'s by solving successive first-order linear equations.

What you end up with in general looks as though it might be complicated. But in fact, for the homogeneous case (**) at least, it is relatively simple to describe:

(1) If the roots $\lambda_1, \ldots, \lambda_n$ of $P(\lambda)=0$ (i.e. $P(\lambda)=(\lambda-\lambda_1)\ldots(\lambda-\lambda_n)$) are all real and different from each other, then the general solution of $P(D)y=0$ is

$$C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} + \cdots + C_ne^{\lambda_n x}$$

the C_i's numbers. We say the general solution is a linear combination of $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$.

(2) If some of the λ_i's in the factorization $P(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n)$ are the same, then each group of, say, k λ_i's all equal (but different from the rest of the λ_i's) gives "particular solutions"

$$e^{\lambda_i x}, \; x e^{\lambda_i x}, \; \ldots, \; x^{k-1} e^{\lambda_i x}$$

And the general solution is the set of all linear combinations of these.

Example: (a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ $\quad (D^2 - 4D + 4)y = 0$

$(D-2)^2 y = 0$. General sol: $C_1 e^{2x} + C_2 x e^{2x}$

(b) $(D-i)^2 (D+i)^2 y = 0$ or $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

General solution

$$C_1 e^{ix} + C_2 x e^{ix} + C_3 e^{-ix} + C_4 x e^{-ix}$$

In real terms, this becomes for the general real solution

$$A_1 \cos x + A_2 \sin x + A_3 x \cos x + A_4 x \sin x$$

There are four (independent) real constants, corresponding to the equation $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$ being of degree 4. This real form comes from writing $e^{ix} = \cos x + i \sin x$, $e^{-ix} = \cos x - i \sin x$ and collecting terms and then seeing what is needed for the result to be real-valued.

The occurrence of n real constants in getting the general solution of the nth order differential (homogeneous) equation is guaranteed by the following:

The set of all solutions of

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

is a vector space: solutions can be added and multiplied by constants to give other solutions. The general theorem on existence and uniqueness says that this vector space S (for solutions) is mapped one-to-one (uniqueness) and onto (existence) $\mathbb{R}^n$ by the linear transformation

$$y \longrightarrow \left(y(0), \left.\frac{dy}{dx}\right|_0, \ldots, \left.\frac{d^{n-1}y}{dx^{n-1}}\right|_0 \right)$$

$y \in S$. So the vector space S is dimension exactly n.

[Note: This reasoning does not depend on the p_i's being constant: it works for linear homogeneous equations in general].

It is interesting to watch this general idea in action in concrete cases. Consider, for example, the case where the p_i's are constants and all the λ_i's are distinct and real. In this case, the space S is supposed to consist of all functions of the form

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_n e^{\lambda_n x}$$

C's numbers. The linear transformation

$$y \to \left(y(0), \frac{dy}{dx}\Big|_0, \dots, \frac{d^{n-1}y}{dx^{n-1}}\Big|_0 \right)$$

sends $C_1 e^{\lambda_1 x} + \dots + C_n e^{\lambda_n x}$ to

$$\left(C_1 + C_2 + \dots + C_n,\ \lambda_1 C_1 + \dots + \lambda_n C_n,\ \lambda_1^2 C_1 + \dots + \lambda_n^2 C_n,\ \dots,\ \lambda_1^{n-1} C_1 + \dots + \lambda_n^{n-1} C_n \right)$$

This transformation is supposed to be 1-1, onto. So the system of equations, C's regarded as unknowns, $A_0, \dots A_{n-1}$ arbitrarily given

$$\begin{aligned}
C_1 + C_2 + \dots + C_n &= A_0 \\
\lambda_1 C_1 + \lambda_2 C_2 + \dots + \lambda_n C_n &= A_1 \\
&\ \vdots \\
\lambda_1^{n-1} C_1 + \lambda_2^{n-1} C_2 + \dots + \lambda_n^{n-1} C_n &= A_{n-1}
\end{aligned}$$

ought to have one and only one solution for each given set of $A_0, \dots, A_{n-1}$. This will be true (by linear algebra) if the determinant of the coefficients

$$\det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \vdots & & & \vdots \\ \lambda_1^{n-1} & \dots & & \lambda_n^{n-1} \end{pmatrix} \neq 0.$$

This determinant is the famous van der Monde determinant, which is well known to be nonzero exactly when the $\lambda_1, \dots, \lambda_n$ are all distinct.

In fact

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} = \text{the product of all } \lambda_j - \lambda_i, \ j > i.$$

Examples: $\det\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \lambda_2 - \lambda_1$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} = (\lambda_2\lambda_3^2 - \lambda_3\lambda_2^2) - \lambda_1(\lambda_3^2 - \lambda_2^2) + \lambda_1^2(\lambda_3 - \lambda_2)$$

$$= (\lambda_3 - \lambda_2)\left[\lambda_2\lambda_3 - \lambda_1(\lambda_3 + \lambda_2) + \lambda_1^2\right]$$

$$= (\lambda_3 - \lambda_2)\left[-\lambda_1(\lambda_3 - \lambda_1) + \lambda_2(\lambda_3 - \lambda_1)\right]$$

$$= (\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1).$$

The proof in general of the det = product formula comes from observing that, since det $= 0$ if $\lambda_i = \lambda_j$, $j > i$, the det as a polynomial in $\lambda_1, \dots, \lambda_n$ must be divisible by $\lambda_j - \lambda_i$, $j > i$. Hence det must be divisible by prod, and counting degree = constant multiple of prod. Constant is easily checked to be $+1$.

Namely, the term of the form

$$\lambda_n^{n-1} \lambda_{n-1}^{n-2} \cdots \lambda_2$$

occurs only once in the determinant expansion (as the main diagonal) and it occurs clearly with coefficient $+1$.

In the product, this term also appears only once, by choosing λ_n from all $\lambda_j - \lambda_i$ terms with $i<j$, and $j=n$, choosing λ_{n-1} from all $\lambda_j - \lambda_i$ terms with $i<j$ and $j=n-1$, and so on.

There the $\lambda_n^{n-1} \cdots \lambda_2$ term also appears with coefficient $+1$. So

$$\det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} = \underset{\text{all } i,j \text{ with } j>i}{\text{product}} (\lambda_j - \lambda_i)$$

[Noting that both det and product are antisymmetric under interchanges of a pair of λs one can prove the whole formula this way with appealing to the divisibility business at the end of the previous page, if desired].