Lecture I: First order linear equations

$$\frac{dy}{dx} + p(x)\, y(x) = q(x) \qquad (*)$$

How to solve: Suppose $P(x)$ has $\frac{dP}{dx} = p$

(e.g. $P(x) = \int_{x_0}^{x} p(t)\,dt$). Then

$$\frac{d}{dx}\left(e^{P(x)}\, y(x)\right) = p(x) e^{P(x)} y + e^{P(x)} \frac{dy}{dx}$$

for any function y. So: $y(x)$ solves $(*)$

$$\iff \quad e^{P} \frac{dy}{dx} + e^{P} p(x)\, y(x) = e^{P} q(x) \iff$$

$$\frac{d}{dx}\left(e^{P} y(x)\right) = e^{P} q(x) \iff$$

$$e^{P(x)} y(x) = C + \int_{x_0}^{x} e^{P(t)} q(t)\,dt \iff$$

$$y(x) = e^{-P(x)} C + e^{-P(x)} \left(\int_{x_0}^{x} e^{P(t)} q(t)\,dt \right)$$

For the initial value problem $y(x_0) = A$, it must be that

$$C = A e^{P(x_0)}$$

When $P(x) = \int_{x_0}^{x} p(t)\,dt$ as above, $C = A$.

So (for that P)

$$y(x) = y(x_0)\, e^{-\int p} + e^{-\int p} \left(\int_{x_0}^{x} e^{\int p} q \right)$$

in shorthand notation.

Important points: Think of $\frac{dy}{dx} + p(x)y$

as taking the input function $q(x)$ and making it into an output function $y(x)$. Then the solution, the output, has two pieces:

One piece, $y(x_0)e^{-\int p}$ does not "see" what the input function is at all. It is completely determined by $y(x_0)$ and p.

The other piece sees the input $q(x)$. But if $x > x_0$, this piece at time x "sees" q only from time x_0 to time x. What happened before x_0 is not of any significance: And what will happen to q in the future, after time x, has not any influence on $y(x)$. This is the (a) basic reason that differential equations model physical processes.

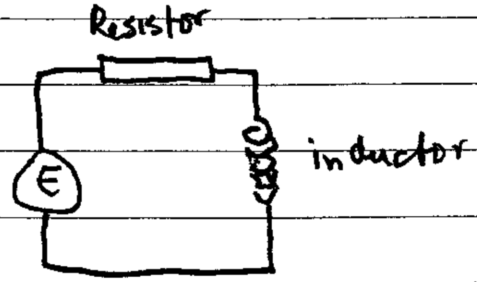
So (1) the solution has "no memory" — nothing before x_0 matters

and

(2) the solution has no knowledge of the future: q beyond time x does not influence $y(x)$.

Of course the system does have memory of q after x_0, with $y(x)$ influenced by $q(t)$, $x_0 \le t \le x$. But the world begins at time x_0.

An example from electricity and magnetism



$$R\,I(t) + L\frac{dI}{dt} = E(t)$$

resistance (R), inductance (L)

R, L constants R, L > 0.

$$\frac{dI}{dt} + \frac{R}{L}I(t) = \frac{1}{L}E(t)$$

Multiply by $e^{\frac{R}{L}t}$ to get

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}I(t)\right) = \frac{1}{L}e^{\frac{R}{L}t}E(t)$$

So

$\boxed{\text{Sol. Formula:}}$ $$I(t) = C e^{-\frac{R}{L}t} + \frac{1}{L}e^{-\frac{R}{L}t}\int_{t_0}^{t} e^{\frac{R}{L}s}E(s)\,ds$$

For convenience, let $t_0 = 0$. Then C = given initial value of current I_0.

The first term goes to 0 as $t \to +\infty$.

The initial current is "transient".

Note that large values of inductance (for fixed resistance) make the transient current decay more slowly: this makes sense because inductance tends to prevent change. (If inductance were 0, I would become 0 instantly if E were to become 0).

There are two interesting cases (actually more than two, but two to begin with) for "the input", the generated voltage $E(t)$.

The first one is $E(t) = E_0 \sin \omega t$, a "sine wave" input (like the AC out of the wall).

To solve this, we need to be able to compute $\int_0^t e^{\frac{R}{L}s} \sin \omega s \, ds$. In calculus, it is shown how to do this, by integrating by parts (twice). But for ease and for future use, let's do this integral by complex numbers. Let $k = R/L$ for notational convenience. Then

$$e^{ks} \sin \omega s = \operatorname{Im}\left(e^{(k+i\omega)s}\right)$$

where we have used the standard formula (a definition from one viewpoint)

$$e^{a+ib} = e^a (\cos b + i \sin b), \quad a, b \text{ real}.$$

Now

$$\int_0^t \operatorname{Im}\left(e^{(k+i\omega)s}\right) ds = \operatorname{Im} \int_0^t e^{(k+i\omega)s} \, ds$$

and one would expect that

$$\int e^{(k+i\omega)s} \, ds = \frac{1}{k+i\omega} e^{(k+i\omega)s}$$

So

$$\operatorname{Im} \int e^{(k+i\omega)s} \, ds = \operatorname{Im} \frac{1}{k+i\omega} e^{(k+i\omega)t} + C$$

$$= \operatorname{Im}\left(\frac{k-i\omega}{k^2+\omega^2} e^{kt} (\cos \omega t + i \sin \omega t)\right)$$

$$= e^{kt} \frac{1}{k^2+\omega^2} (k \sin \omega t - \omega \cos \omega t) + C$$

You can check by differentiating that this worked.

So the second term of the solution formula $\boxed{\text{Sol Formula}}$ on page 3 becomes

$$\frac{1}{L}\frac{E_0}{k^2+\omega^2}(k\sin\omega t-\omega\cos\omega t)\ \left(+\,Ce^{-\frac{R}{L}t}\right)$$

So the steady-state nontransient part is

$$\frac{1}{L}\frac{E_0}{k^2+\omega^2}(k\sin\omega t-\omega\cos\omega t).$$

$$=\frac{1}{L}\frac{E_0}{\sqrt{k^2+\omega^2}}\left(\frac{k}{\sqrt{k^2+\omega^2}}\sin\omega t-\frac{\omega}{\sqrt{k^2+\omega^2}}\cos\omega t\right)$$

$$=\frac{1}{L}\frac{E_0}{\sqrt{k^2+\omega^2}}\left(\sin(\omega t-\phi)\right)=\frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t-\phi)$$

where ϕ has $\cos\phi=\frac{k}{\sqrt{k^2+\omega^2}}$ $\sin\phi=\frac{\omega}{\sqrt{k^2+\omega^2}}$ and $k=\frac{R}{L}$ as before

Thus the "steady state" current solution is another sine wave with the same frequency as the $E_0\sin\omega t$ "input", but "phase shifted"

and with an amplitude changed by $\frac{1}{\sqrt{R^2+\omega^2L^2}}$

Thus $\frac{1}{\sqrt{R^2+\omega^2L^2}}$ is like $\frac{1}{R}$ in ordinary Ohm's Law for resistance along ($\frac{\text{voltage}}{\text{resistance}}$ = current). But the "generalized resistance" or impedance

depends on ω: higher frequencies make the inductive part contribute more. You can also see why E&M people like to think

of R and $L\omega$ as perpendicular vectors: then $\sqrt{R^2+L^2\omega^2}$ corresponds to the Pythagorean

Theorem.

Clearly, a lot of E&M (electricity and magnetism) is packed into that one seemingly simple differential equation

$$L\frac{dI(t)}{dt} + R\,I(t) = E(t).$$