Variation of Arc Length and Geodesics.

In the same spirit that we motivated the Isoperimetric Inequality by variation methods, we are going now to look into the question of connecting two points in a surface by a curve. We start with curves in $\mathbb{R}^3$. Of course, we know the answer in this case from Euclid: shortest connections are straight line segments. But let us look at this from a "variational" viewpoint: Let $\gamma_t(s)$, $s \in [a,b]$, $t \in (-\varepsilon, \varepsilon)$ be a smooth family of curves with

(1) $\gamma_t(a) = \gamma_0(a)$, $\gamma_t(b) = \gamma_0(b)$, all $t \in (-\varepsilon, \varepsilon)$
["fixed endpoints"]

and

(2) $\gamma_0(s)$ is arclength parameter.

Set $L(t)$ = length of γ_t on $[a,b]$.
{Note that γ_t is not assumed to have arclength parameter, so $L(t)$ may not be $\equiv L(0)$}.

We compute $\left.\frac{dL}{dt}\right|_{t=0}$:

$$L(t) = \int_a^b \langle \gamma_t'(s), \gamma_t'(s) \rangle^{\frac{1}{2}} ds . \qquad ' = s \text{ derivative}$$

So
$$\frac{dL}{dt} = \int_a^b 2 \cdot \frac{1}{2} \left\langle \frac{\partial \gamma'}{\partial t}, \gamma' \right\rangle \langle \gamma', \gamma' \rangle^{-\frac{1}{2}} ds$$

Since $\langle \gamma_0', \gamma_0' \rangle \equiv 1$ for all s,

$$\left.\frac{dL(t)}{dt}\right|_{t=0} = \int_a^b \left\langle \frac{\partial \gamma'}{\partial t}, \gamma' \right\rangle ds$$

$$= \int_a^b \left\langle \frac{\partial^2}{\partial t \partial s}\gamma, \frac{\partial \gamma}{\partial s} \right\rangle = \int_a^b \left\langle \frac{\partial}{\partial s}\left(\frac{\partial \gamma_t}{\partial t}\right), \frac{\partial \gamma}{\partial s} \right\rangle ds$$

$$= \int_a^b \frac{\partial}{\partial s} \left\langle \frac{\partial \gamma_t}{\partial t}, \frac{\partial \gamma_t}{\partial s} \right\rangle - \int_a^b \left\langle \frac{\partial \gamma_t}{\partial t}, \frac{\partial^2 \gamma_t}{\partial s^2} \right\rangle ds$$

with everything evaluate at $t=0$.

But $\left.\int_a^b \frac{\partial}{\partial s} \left\langle \frac{\partial \gamma_t}{\partial t}, \frac{\partial \gamma_t}{\partial s} \right\rangle\right|_{t=0} = \left\langle \frac{\partial \gamma_t}{\partial t}, \frac{\partial \gamma_t}{\partial s} \right\rangle \Big|_{a=s,\ t=0}^{b=s,\ t=0}$

$=0$ since $\gamma_t(a)$ and $\gamma_t(b)$ are constants independent of t.

Thus $\left.\frac{dL(t)}{dt}\right|_{t=0} = -\int_a^b \left\langle \frac{\partial \gamma_t}{\partial t}, \frac{\partial^2 \gamma_t}{\partial s^2} \right\rangle$.

Now we turn to the minimization question. Suppose Γ is a smooth curve from p to q with length Γ minimal among all such curves. Assume $\Gamma : [a,b] \to \mathbb{R}^3$ has arclength as parameter.

Now if γ_t is any variation of the sort we described with $\gamma_0 = \Gamma$, then it must be that

$$\left.\frac{dL(t)}{dt}\right|_{t=0} = 0$$

since length of $\Gamma = L(0) \leq L(t)$, all t.

We want to show that $\dfrac{\partial^2 \gamma_0}{\partial s^2} = 0$ everywhere on $[a,b]$, so that γ_0 must be a straight line.
Suppose, for proof by contradiction, that

$$\left.\frac{\partial^2 \gamma_0}{\partial s^2}\right|_{s=c} \neq 0 \quad \text{for some} \quad c \in (a,b).$$ [This is enough, to do the open interval]

Choose a smooth function on $\mathbb{R}$, $\rho : \mathbb{R} \to \mathbb{R}$ with $\rho \geq 0$, $\rho(c) = 1$, and closure $\{\lambda : \rho(\lambda) \neq 0\} \subset (a,b)$.

Then

$$\gamma_t(s) = \gamma_0(s) + t\, \rho(s) \left(\frac{\partial^2 \gamma_0}{\partial s^2}\right)$$

is a fixed-endpoint variation of the sort already discussed. And, using $\dfrac{\partial \gamma_t}{\partial t} = \rho(s) \dfrac{\partial^2 \gamma_0}{\partial s^2}$, we get

$$\left.\frac{dL(t)}{dt}\right|_{t=0} = -\int_a^b \left\langle \left.\frac{\partial \gamma_t}{\partial t}\right|_{t=0}, \frac{\partial^2 \gamma_0}{\partial s^2} \right\rangle$$

$$= -\int \rho(s) \left\langle \frac{\partial^2 \gamma_0}{\partial s^2}, \frac{\partial^2 \gamma_0}{\partial s^2} \right\rangle < 0.$$

Thus we have the contradiction we sought: it must be that $\frac{\partial^2 \gamma_0}{\partial s^2} \equiv 0$ for all $s \in (a,b)$.

We now want to extend these ideas to help us understand what curves could be minimal connections between two points on a surface, minimal that is, among curves lying entirely on the surface. The answer we shall get is as follows:

Lemma: If $S(u,v)$, $S: U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, is a surface path and if $\Gamma:[a,b] \to U$ is a smooth curve such that $\hat{\Gamma}(s) = S(\Gamma(s))$ has arc length parameter and is minimal length among all curves of this form from $\hat{\Gamma}(a)$ to $\hat{\Gamma}(b)$, then $\hat{\Gamma}$ is a geodesic in the sense that

$$\left.\frac{\partial^2 \hat{\Gamma}}{\partial s^2}\right|_s$$

is normal to the tangent plane of S at $\hat{\Gamma}(s)$ for all $s \in [a,b]$.

(i.e. $\left\langle \left.\frac{d^2 \hat{\Gamma}}{ds^2}\right|_s, \left.S_u\right|_{\Gamma(s)} \right\rangle = 0$ and similarly for S_v).

The idea of the proof is this: Suppose $\frac{\partial^2 \hat{\Gamma}}{\partial s^2}$ has a nonzero inner product with, say, S_u at some $s = c \in (a,b)$. Then if we varied

Γ is the u-direction near $c \in (a, b)$, we could get a fixed-endpoint variation of $\hat{\Gamma}$ that was still in S by taking the S-image of the variation. In fact, this idea just works!

Supposing $\left\langle \frac{\partial^2 \hat{\Gamma}}{\partial s^2}\Big|_c , S_u\Big|_{\Gamma(c)} \right\rangle \neq 0$, we define

$$\Gamma_t(s) = \Gamma(s) + t\, \rho(s)\, (1, 0)$$

[Recall that Γ is $\mathbb{R}^2$ valued, values $\in U$ so $(1,0)$ and the vector addition make sense!]

Where ρ is a function on $\mathbb{R}$, smooth, ≥ 0 everywhere, $\{\lambda : \rho(\lambda) \neq 0\} \subset (a, b)$ and finally that $\left\langle \frac{\partial^2 \hat{\Gamma}}{\partial s^2}\Big|_d , S_u\Big|_{\Gamma(d)} \right\rangle \neq 0$ and

has the same sign ($\pm$) as $\left\langle \frac{\partial^2 \hat{\Gamma}}{\partial s^2}\Big|_c , S_u\Big|_{\Gamma(c)} \right\rangle \neq 0$

for all d with $\rho(d) \neq 0$. This can be done by making $\rho = 0$ except near c. (This is just by continuity). Then, for some $\varepsilon > 0$, for $t \in (-\varepsilon, \varepsilon)$ it must be that

$$\Gamma_t(s) \in U \quad \text{for all } s \in [a, b]$$

(continuity again). Also

if we set $\hat{\Gamma}_t(s) = S(\Gamma_t(s))$, then

$$\frac{\partial \hat{\Gamma}_t}{\partial t}\Big|_{t=0} = \rho(s)\, S_u .$$

Then, using our formula from page (2)

$$\frac{dL(t)}{dt}\Big|_{t=0} = -\int_a^b \left\langle \frac{\partial \hat{\Gamma}_t}{\partial t}\Big|_{t=0}, \frac{\partial^2 \hat{\Gamma}_0}{\partial s^2} \right\rangle$$

we get

$$\frac{dL(t)}{dt}\Big|_{t=0} = -\int_a^b \rho(s) \left\langle S_u, \frac{\partial^2 \Gamma_0}{\partial s^2} \right\rangle ds$$

which is nonzero, since the integrand is either 0 or has the same sign as $\left\langle S_u, \frac{\partial^2 \Gamma_0}{\partial s^2}\Big|_c \right\rangle$ (which is nonzero) and the integrand is nonzero at $s = c$.

Thus geodesics are the only candidates for shortest connections on a surface. Next, we shall prove that "locally" they really are shortest connections.