Why $T(\eta_r)=0$ for a surface in $\mathbb{R}^3$

Recall $\eta = G_\theta/\|G_\theta\|$. So η is $\perp$ to G_r but also tangent to S and hence $\perp$ to N (the surface normal). Thus $\eta = \pm G_r \times N$ (since $N \perp G_r$ and G_r is a unit vector: the $\pm$ depends on which way we run θ. Wolog $\eta = + G_r \times N$).

So $\quad \eta_r = G_{rr} \times N + N_r \times G_r$

Now G_{rr} is normal to S since $G(r,\theta)$ r varies, θ fixed, curves are geodesics.

So $\quad G_{rr} \times N = 0$. Also since $\langle N, N\rangle \equiv 1$, $\langle N_r, N\rangle \equiv 0$. So N_r is tangent to S.

Thus $N_r \times G_r$ is the cross product of two tangent-to-S vectors and is hence normal. Thus $T(\eta_r) = 0$ as required.