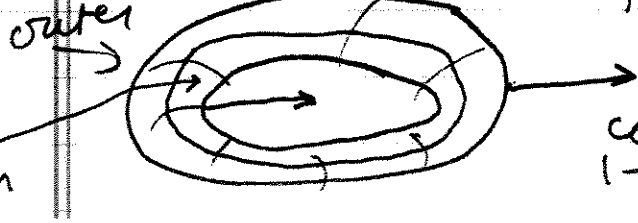
# More on the Gauss Map for Closed Surfaces with $K > 0$

If $S$ is SUCH A closed surface in $\mathbb{R}^3$, then, as we have already seen, the Gauss map $\Gamma: S \to$ the unit sphere $S^2$ is everywhere nonsingular. (Recall that $\Gamma$ is defined by $\Gamma(p)$ = exterior unit normal of $S$ at $p$ and that the Jacobian of $\Gamma$ at $p$ = the Gauss curvature of $S$ at $p$). If one assumes known some basic algebraic topology, then one can deduce rapidly that $\Gamma$ is a diffeomorphism (1-1, onto, with differentiable inverse as well as $\Gamma$ itself being differentiable, as we already know). Namely, since $S$ is compact, its image under $\Gamma$ is also compact, but its image is also open (in $S^2$) because of $\Gamma$'s being nonsingular everywhere. So $\Gamma(S) = S^2$: $\Gamma$ is onto (surjective). Since $\Gamma$ is locally one-to-one by nonsingularity again, each point $u \in S^2$ has only finitely many preimages $p_1, \ldots, p_k$ ($k$ possibly depending on $u$) [Just consider the situation that would obtain if some sequence of distinct images of $u$ accumulated at $p_0 \in S$; this would contradict 1-1 locally at $p_0$]. From this, one concludes easily that there is a neighborhood of $u$, say $V$, in $S^2$ such that $\Gamma^{-1}(V)$ is a disjoint union of open sets $U_1, \ldots, U_k$ in $S$, $p_j \in U_j$, $\forall j$, and with $\Gamma | U_j$ a diffeomorphism onto $V$. In other words $\Gamma : S \to S^2$ is a (differentiable) covering space in the sense of algebraic topology. [V is an "evenly covered" neighborhood of u, and u was arbitrarily chosen]. Since $S^2$ is simply-connected, $\Gamma$ must be one-to-one (injective).

We can consider this situation from the viewpoint of the Gauss-Bonnet Theorem and Euler characteristic, without reference to covering space theory as such. Namely, since $\Gamma$ is nonsingular, we can define a new (abstract) metric on $S$ by declaring $\Gamma$ to be a local isometry. (This new metric on $S$ has the same Gauss curvature as $S^2$, namely $+1$). Since $\Gamma$ is everywhere orientation preserving, one sees readily that if $\Gamma$ covers some parts of $S^2$ more than once then $\int_S K \, d(\text{area})$ for this new abstract metric on $S$ is $> 4\pi$. So the Euler characteristic $\chi(S) > 2$. But there are no (compact) surfaces with Euler characteristic $> 2$, by standard classification of surfaces. So $\Gamma$ again must be 1-1 (injective).

One should contrast this situation with Gauss map of the usual torus of revolution in $\mathbb{R}^3$. There, "generic" points of $S^2$ are covered twice by the Gauss map: the "outer" part of the torus covers $S^2 - \{(0,0,1), (0,0,-1)\}$ 1-1 and the "inner" part covers $S^2 - \{(0,0,1), (0,0,-1)\}$ 1-1 but with reverse



orientation. Gauss curvature on the outer part is $> 0$ and on the inner part is $< 0$. $\int K \, d(\text{area})$ on the outer part is thus $+4\pi$ while $\int K \, d(\text{area})$ on the inner part is $-4\pi$ (whole sphere but with reverse orientation). This gives

$$\int_{\text{whole torus}} K \, d(\text{area}) = 0 \qquad \text{as expected.}$$

In this case, the "pulled back" new abstract metric that arises by declaring $\Gamma$ to be a local isometry is not defined over the whole (torus) surface: where $\Gamma$ is singular (i.e. on the circles that are $\Gamma'^{-1}((0,0,1))$ and $\Gamma'^{-1}((0,0,-1))$) the "metric" is not positive definite. So the Gauss-Bonnet Theorem cannot be applied and no contradiction arises. [The torus cannot admit an everywhere defined and positive definite abstract metric with Gauss curvature $\equiv +1$, by the Gauss Bonnet Theorem again].