Homework II : Math. 120B Spring 2008

1. Suppose a surface patch $S(u,v)$ has $\langle S_u, S_u \rangle \equiv 1$ and $\langle S_u, S_v \rangle = 0$.
Prove that curves of the form $\gamma(t) = S(t,v)$ [or $t \to S(t,v)$], v fixed, are geodesics.

2. Use problem 1 to show that on a surface of revolution, $S(s,\theta) = (x(s), y(s)\cos\theta, y(s)\sin\theta)$ [where $(x(s), y(s))$ is an arclength parameter curve, $x'(s) > 0$, $y(s) > 0$ as usual], the s-curves $\gamma(s) = S(s,\theta)$, θ fixed, are geodesics.

3. With S as in problem 1, and setting $f(u,v) = \langle S_v, S_v \rangle^{1/2}$, prove that Gauss curvature $= 1 - \frac{\partial^2}{\partial u^2} f / f$.

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4. Recall that (in first homework) we defined area of an (open) set U in a surface patch $S(u,v)$ to be

$$\int_U \|S_u \times S_v\| \, du\, dv$$

and proved that this was independent of parameterization.

(a) Prove that $\|S_u \times S_v\|^2 = EG - F^2$.
so that area $= \int \sqrt{EG - F^2} \, du\, dv$

(b) Let $U_r =$ (open) disc of radius r (around p_0) in $S \stackrel{\text{def}}{=}$ the set of points which are connected to p_0 by a geodesic of length $< r$. Find area of U_r for

(1) $S =$ sphere of radius 1
(2) $S =$ sphere of radius R $(R>0)$
(3) $S = \mathbb{R}^2$
(4) $S =$ "Poincare disc" $\left(= \{(x,y) : x^2+y^2<1\},\ E = G = 4/(1-x^2-y^2)^2,\ F=0 \right)$.

[Suggestion: Reparameterize in geodesic polar coordinates, and use our previous formulas for $\langle G_\theta, G_\theta \rangle$].

5(a) Analogously to the G. curv $= \lim_{r\to 0^+} \frac{3}{\pi r^3}(2\pi r - l(C_r))$ formula, find a formula for the Gauss curvature in terms of the limiting behavior as $r \to 0^+$ of the area of the disc of radius r (in a general surface)

(b) Check your formula in the cases (1) - (4) of the previous problem.

6. Let $\gamma(s)$ be an arclength-parameter-curve in a surface $S(u,v)$. Let $T(s) = \frac{d}{ds}\gamma(s)$ so that $\langle T(s), T(s) \rangle = 1$. Set $\eta(s) =$ T rotated 90° left in the orientation determined by the surface normal $N = (S_u \times S_v)/\|S_u \times S_v\|$, i.e. N, T, η form a "right-handed" orthonormal frame in $\mathbb{R}^3$. Define the <u>geodesic curvature</u> κ_g of γ by $\frac{d^2\gamma(s)}{ds^2} \left(= \frac{dT}{ds} \right) = \kappa_g \eta$.

(a) Show that γ is a geodesic $\iff \kappa_g = 0$.

(b) Show that if $\kappa =$ the curvature of γ as an $\mathbb{R}^3$ space curve, then
$$\kappa^2 = \kappa_g^2 + \|N(\text{acceleration of } \gamma)\|^2$$

7. With γ as in problem 6 but in addition a smoothly closed curve and with $\gamma_t(s)$ a (normal) variation of γ (same notation as earlier), prove (extending our results for plane curves) that

$$\left.\frac{dL(t)}{dt}\right|_{t=0} = -\int_\gamma k_g(s) b(s)\, ds$$

if $\frac{d\gamma_t(s)}{dt} = b(s)\,\eta(s)$, where η is the curve normal.

[Note: normal variation means here just that $\frac{d\gamma_t(s)}{dt}$ has this form!]

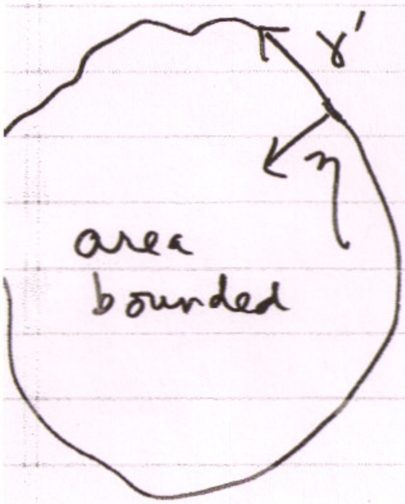
8. With γ as in problem 6 and 7, say $\gamma : [a,b] \to S$, plus γ simple closed, define a function:

$F(s,t)$ = the endpoint of the geodesic (with arclength parameter) that starts at $\gamma(s)$ and has initial tangent $\eta(s)$ and length t, if $t>0$, and length $|t|$ and initial tangent $-\eta$ if $t<0$.

(a) Just the Inverse Function Theorem to show that F maps, for some $\varepsilon>0$, $[a,b]\times(-\varepsilon,\varepsilon)$ one-to-one onto a neighborhood of $\gamma([a,b])$, one-to-one that is except for the obvious identifications $F(a,t) = F(b,t)$ all t.

(b) thinking of F as a reparameterization of the surface near $\gamma([a,b])$, find E, F, G for this parameterization at points of γ. [You do not need to try to find E, F, G elsewhere, just along γ].

(c) Using part (b), compute $\left.\frac{dA(t)}{dt}\right|_{t=0}$ for a normal variation of γ where we assume that γ bounds an area in S which lies to the $+\eta$ side of γ, as shown in the figure.



[Suggestion: This should be compared with the plane curve case as far as answer is concerned. But it is easiest here to use the area $= \int\sqrt{EG-F^2}$ formula for finding $\left.\frac{dA(t)}{dt}\right|_{t=0}$].

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9. Consider the circle C_r of radius r around the north pole in the unit sphere. Let $A(r)$ = the area of the component of the complement of this circle that contains the north pole (i.e., the "interior" of the disc the circle bounds). Investigate whether $A(r)$ is more or less that the area inside the euclidean plane circle of ~~radius~~ length = length of C_r, i.e. of radius $\ell(C_r)/2\pi$.