Geodesic Normal Coordinates on a Surface.

To help us in understanding the properties of geodesics on surface, we shall be making use of a special kind of coordinate system. To define this, we start with a surface S, a point p_0 in S (and hence in $\mathbb{R}^3$) and a parameterization $S(u,v)$ of S near p_0 with $S(0,0)=p_0$ and $S_u|_{(0,0)}$ and $S_v|_{(0,0)}$ an orthonormal basis for the tangent space of S at p_0 (or, equivalently, $E|_{(0,0)}=1$, $G|_{(0,0)}=1$ and $F|_{(0,0)}=0$). We can do this without loss of generality, since we can always arrange these conditions, starting with an arbitrary parameterization, by making a translation and linear tranformation of the parameters. [Alternatively, one can think of "graphing the surface over its tangent plane" as we discussed earlier, that is writing it as $S(u,v)=(u,v,f(u,v))$ after a translation and rotation). With S as described, we define a new parameterization $\hat{S}$ by $\hat{S}(a,b) = \gamma_{a,b}(1)$ where $\gamma_{a,b}$ = the geodesic in S with $\gamma_{a,b}(0)=p_0$ and $\gamma'_{a,b}(0) = aS_u+bS_v$. The basic Existence and Uniqueness Theorem from differential equations applied to the equations of geodesics gives that $\hat{S}$ is defined on some neighborhood W of $(0,0)$ in $\mathbb{R}^2$ and is C^∞ on W. We can also show that

$\hat{S}$ is nonsingular at $(0,0)$ so that $\hat{S}$ is a local parameterization of S near p_0. For this, we need to compute $\hat{S}_a$ and $\hat{S}_b$. Now note that $\gamma_{\alpha a, \alpha b}(1) = \gamma_{a,b}(\alpha)$ for each α (so small that the maps are defined): this comes from Uniqueness together with the homogenity in t of the geodesic equations: $\gamma(t)$ is a geodesic if and only if $\gamma(\alpha t)$ is a geodesic $(\alpha \neq 0)$.

So $\left.\frac{d}{dt}\hat{S}(ta, tb)\right|_{t=0} \overset{\text{def.}}{=} \left.\frac{d}{dt}\gamma_{ta,tb}(1)\right|_{t=0}$

$= \left.\frac{d}{dt}\gamma_{a,b}(t)\right|_{t=0} = aS_u + bS_v.$

In particular $\hat{S}_a = S_u$ and $\hat{S}_b = S_v$

since $\hat{S}_a =$ (by definition) $\left.\frac{d}{dt}\hat{S}_{ta}\right|_{t=0}$ and similarly for $\hat{S}_b$. Thus $\hat{S}_a|_{(0,0)} \times \hat{S}_b|_{(0,0)} =$

$S_u|_{(0,0)} \times S_v|_{(0,0)} \neq 0.$

The parameterization of S we have just defined is called a geodesic normal coordinate system at $p_0 \in S$. It is in fact unique up to linear orthogonal changes of (a, b).

What is particularly interesting is the associated idea of geodesic polar coordinates obtained from polar coordinates on the (a,b) plane!
We take the a-axis as $\theta = 0$ and $r = \sqrt{a^2+b^2}$ as usual. So we get geodesic polar coordinates (by definition) on S) by sending

$$(r,\theta) \to \hat{S}(r\cos\theta, r\sin\theta) \overset{\text{def}}{=} G(r,\theta)$$

These are coordinates on S (near p_0) in the same sense that polar coordinates are coordinates on $\mathbb{R}^2$: i.e., $(r,\theta) \to \hat{S}(r\cos\theta, r\sin\theta)$ is C^∞ but it is singular at $r=0$ and many-to-one as far as θ is concerned.

We are going to want to study $\langle G_r, G_r\rangle$, $\langle G_r. G_\theta\rangle$ and $\langle G_\theta, G_\theta\rangle$

(G for geodesic!). Note first that by definition a curve $r \to G(r,\theta_0)$, θ_0 fixed, $r>0$, is a geodesic. (Look at the definition of $\hat{S}$). In fact it is a geodesic with arclength parameter.
So $\langle G_r, G_r\rangle \equiv 1$ $(r>0)$.
Our next main goal is to show that $\langle G_r, G_\theta\rangle = 0$. But before we do that, let us look at some examples.

(1) $\mathbb{R}^2 \subset \mathbb{R}^3$ $\quad S(u,v) = (u, v, 0)$, $p_0 = (0,0,0)$

Geodesic polar coordinates are just polar coordinates on $\mathbb{R}^2$, that is

$$G(r,\theta) = (r\cos\theta, r\sin\theta, 0)$$

(2) S^2 (unit 2-sphere $= \{(x,y,z) : x^2+y^2+z^2=1\}$) in $\mathbb{R}^3$. $p_0 = (0,0,1)$.

$$S(u,v) = (u, v, \sqrt{1-u^2-v^2}).$$

$G(r,\theta) =$ endpoint of geodesic starting at $(0,0,1)$, arclength parameter, traversed up to distance r. Since geodesics are great circles

$$G(r,\theta) = (\sin r \cos\theta, \sin r \sin\theta, \cos r)$$

$$G_r = (\cos r \cos\theta, \cos r \sin\theta, -\sin r)$$

and

$$\langle G_r, G_r \rangle = \cos^2 r\,(\cos^2\theta + \sin^2\theta) + (-\sin r)^2 =$$
$$= \cos^2 r + \sin^2 r = 1$$ as expected.

$$G_\theta = (-\sin r \sin\theta, \sin r \cos\theta, 0)$$

and

$\langle G_r, G_\theta \rangle = 0$ as we shall prove in general.

Finally

$$\langle G_\theta, G_\theta \rangle = (-\sin r \sin\theta)^2 + (\sin^2 r \cos^2\theta)$$
$$= \sin^2 r.$$

Think about how this gives the expected answer for the $r =$ positive constant, θ goes from 0 to 2π (a curve which is a circle, radius $|\sin r|$)

We now turn to our goal:
Gauss's Lemma: $\langle G_r, G_\theta \rangle \equiv 0$.

Proof: We compute (for $r > 0$)

$$\frac{d}{dr}\langle G_r, G_\theta\rangle = \langle G_{rr}, G_\theta\rangle + \langle G_r, G_{\theta r}\rangle$$

$$= \langle \underbrace{T(G_{rr})}_{0 \text{ since } r\text{-curve is godesic!}}, G_\theta\rangle + \langle G_r, G_{r\theta}\rangle$$

$$= 0 + \frac{1}{2}\frac{\partial}{\partial\theta}\langle G_r, G_r\rangle = 0$$

since $\langle G_r, G_r\rangle \equiv 1$. So $\langle G_r, G_\theta\rangle\big|_{\substack{\theta \text{ fixed}\\ r>0}}$ is constant. But $\lim_{r\to 0^+}\langle G_r, G_\theta\rangle\big|_{\theta \text{ fixed}} = 0$ (since the Δ-change in S associated to change $\Delta\theta$ in θ goes to 0 as r goes to 0: write everything in (a, b) terms to see this). So $\langle G_r G_\theta\rangle \equiv 0$ (θ fixed) any r, as required. $\square$

This Lemma actually implies that geodesics are locally minimal connections. Here is the idea: A curve $\sigma(t) = (r(t), \theta(t))$ has $\sigma'(t) = \left(\frac{dr}{dt}, \frac{d\theta}{dt}\right)$ in r, θ terms, that is its $\mathbb{R}^3$ tangent vector is

$$\frac{dr}{dt} G_r + \frac{d\theta}{dt} G_\theta .$$

Using what we know ($\langle G_r, G_r \rangle = 1$, $\langle G_r, G_\theta \rangle = 0$) we get

$$\| \sigma'(t) \|^2 = \left(\frac{dr}{dt}\right)^2 + \langle G_\theta, G_\theta \rangle \left(\frac{d\theta}{dt}\right)^2$$

Now suppose we want to go from p_0 ($r=0$) to some $q = (r_0, \theta_0)$.

Then

$$\text{length}(\sigma) = \int \sqrt{\left(\frac{dr}{dt}\right)^2 + \langle G_\theta, G_\theta \rangle \left(\frac{d\theta}{dt}\right)^2}$$

$$\geq \int \left| \frac{dr}{dt} \right| \geq \text{change in } r = r_0 .$$

So $\text{length}(\sigma) \geq r_0$. On the other hand, the geodesic $t \to (t, \theta_0)$ in (r, θ) coordinates has length r_0. So no other curve (with same endpoints) is shorter. Also if $\text{length}(\sigma) = r_0$, then (since $\langle G_\theta, G_\theta \rangle > 0$) it must be that $\frac{d\theta}{dt} = 0$ and

since $\int \left| \frac{dr}{dt} \right| = \text{change in } r$, it must also be that $r(t)$ is monotone. So σ is the geodesic indicated with some (monotone) parameterization.

This of course all only works if q is "close" to p_0, namely $q \in \hat{S}$ (some disc around $(0,0)$ in $\mathbb{R}^2$ on which $\hat{S}$ is 1-1, onto its image nonsingularly.