The Gauss Map and the Gauss Curvature

Let $S(u,v)$ be a surface patch with $N(u,v) = N = S_u \times S_v / \|S_u \times S_v\|$ = its unit normal. Since $N(u,v)$ is a unit vector, we can think of it as a point of the unit sphere $= \{(x,y,z) : x^2+y^2+z^2 = 1\}$. Now suppose that $S(u,v)$ has the form $S(u,v) = (u, v, f(u,v))$ (a "Monge patch") so that $S_u = (1,0,f_u)$, $S_v = (0,1,f_v)$ and $N(u,v) = (-f_u, -f_v, 1)/\sqrt{1+f_u^2+f_v^2}$.

We also suppose $f_u = f_v = 0$ at $(u,v) = (0,0)$. Then at $(0,0)$, $N = (0,0,1)$ and

$$N_u = (-f_{uu}, -f_{vu}, 0)$$
$$N_v = (-f_{uv}, -f_{vv}, 0)$$

so that

$$N_u \times N_v = (0, 0, f_{uu}f_{vv} - f_{uv}^2)$$
$$= (0,0,K) = K(0,0,1).$$

<u>where</u> K = Gauss curvature.

Thus if we orient the unit sphere by its exterior unit normal, the area multiplication map of the "Gauss map" that takes (u,v) to $N(u,v)$ is K in the sense that $d(\text{area})$ on the surface S ~~is~~ corresponds ~~taken~~ under the Gauss map to $K\, d(\text{area})_{S^2}$ where S^2 = unit sphere. In other words, the area of the Gauss map image of a $\Delta u, \Delta v$ box on S is $\approx |K| \Delta u \Delta v$, and the $\pm$ sign of K indicates orientation preservation or reversal.

Now suppose S is a closed surface such that $\Gamma : S \to S^2$ is one-to-one onto nonsingular, where Γ is the Gauss map. Then by change of variables for double integrals,

$$\int_S K \, d(\text{area}) = \int_{S^2} 1 = 4\pi .$$

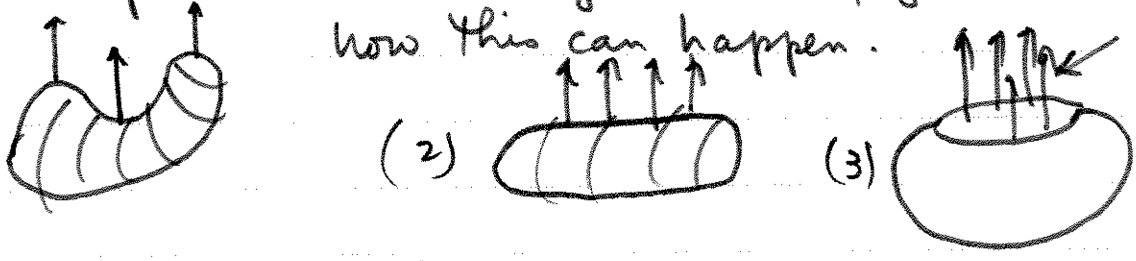
(at least if orientations are set up to make $\det \Gamma > 0$). We shall see that this actually happens if S has $K > 0$ everywhere.

Lemma: If S is a closed (compact, no edges) surface and N is the exterior unit normal, then the Gauss map $\Gamma : S \to S^2$ defined by $F(x) = N(x)$, $x \in S$, is surjective (onto). [We assume here without proof for the moment that S divides $\mathbb{R}^3$ into two connected open sets, the "interior" and "exterior" with S itself as their shared boundary so that the concept of "exterior normal" is well-defined and the exterior normal varies continuously over S].

Proof: It is enough (wolog) to show that $\exists x \in S$ with $\Gamma(x) = (0,0,1)$. Let p = point of S where $\sup_{q \in S} z(q)$ is attained. Then

$\Gamma(p) = (0,0,1)$: $(0,0,1)$ is normal to S at p since the function $z|S$ has a critical point at x and $(0,0,1)$ is the exterior normal since $\{(x,y,z) : z > z(p) = \sup_{q \in S} z(q)\}$ is empty. $\square$

In general, Γ, while it is surjective, may not be injective (on a "closed" surface S, i.e. compact with no "edges"): the figures show how this can happen.



(1) (2) (3)

(third figure has a "flat spot": there is a whole area on which Γ is constant!). But in case $K>0$, these situations cannot occur.

Lemma: If S is a closed surface in $\mathbb{R}^3$ with $K>0$ everywhere, then $\Gamma: S \to S^2$ is bijective and everywhere nonsingular.

Proof: That Γ is surjective we have already shown. Nonsingularity follows from our discussion of the "area multiplication factor" (the Jacobian). It remains to discuss injectivity. It suffices to show that $\exists$ at most one $p \in S$ with $\Gamma(p) = (0,0,1)$. For this, we consider $P_\alpha \underset{\text{def}}{=} \{q \in S : z(q) = \alpha\}$

for each $\alpha \in [\min_{q \in S} z(q), \max_{q \in S} z(q)]$.

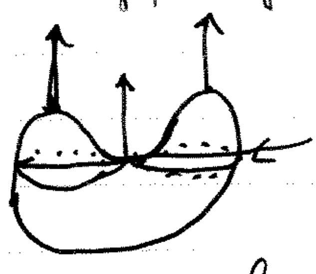
If $q \in P_\alpha$ for some such α and if $\Gamma(q) \neq (0,0,1)$ and $\neq -(0,0,1)$, then $z|S$ is noncritical at q so, near q, $P_\alpha \cap S$ is a smooth curve. If $\Gamma(q) = (0,0,1)$ or $-(0,0,1)$, then q is an isolated point of $P_\alpha \cap S$, since $K > 0$ implies that S lies "strictly" on one side of its tangent plane (that is, near $q \in S$, if T_q is the tangent plane through q, then $T_q \cap S \cap U$ consists, for some open set U in $\mathbb{R}^3$ with $q \in U$, of the point q only).

By compactness, there are only finitely many $q \in S$ such that $\Gamma(q) = \pm(0,0,1)$. If we remove this finite set from S, then what remains has, for each $\alpha \in [\min z, \max z]$, the property that $P_\alpha \cap S$ is a union of smooth closed curves. But since S is connected, there can be for each such α only one such curve. Since each point with $\Gamma = \pm(0,0,1)$ gives rise to such a family

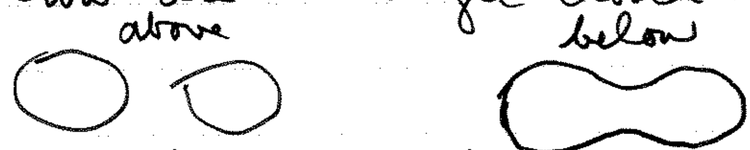
with each family coming from at most

two points, one for $+(0,0,1)$, one for $-(0,0,1)$) it follows that only two points of S have $\Gamma = \pm(0,0,1)$. Since one of these has $\Gamma = -(0,0,1)$, there is exactly one with $\Gamma = (0,0,1)$. $\square$.

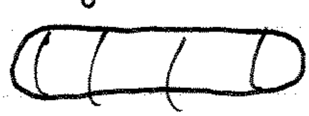
It is worthwhile to think about how this argument breaks down when the hypothesis that $K>0$ everywhere is dropped. In figure 1, two pages back (repeated for convenience)



there is a z-level where the intersection with the surface is not a simple closed curves but two closed curves meeting. The levels just above are two disjoint closed curves, the levels just below are a single closed curve:



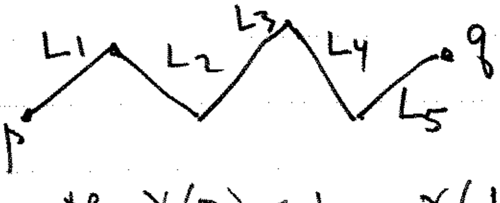
At the transition, the level is not a union of two smooth simple disjoint curve. In figure 2, the max (and min) levels are line segments. And



in figure 3, the max level is a closed disc — it has interior points, in contrast to the single point (local) maxima of the $K>0$ situation.

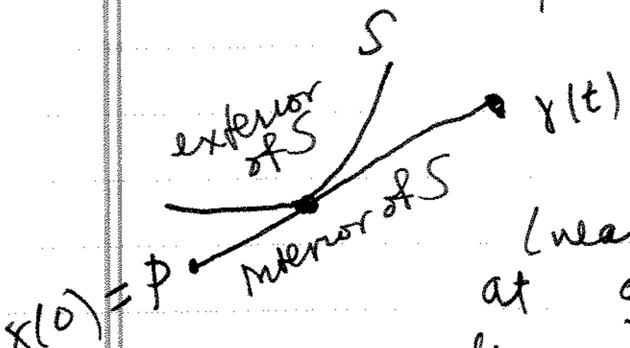
Closed surfaces of Gauss curvature > 0 have another important property: their interior is a convex open set. (Recall that a set $C \subset \mathbb{R}^3$ is <u>convex</u> by definition if for all $p, q \in C$ and $\lambda \in [0,1]$, the point $(1-\lambda)p + \lambda q$ is in C).

To prove this convexity property, we argue as follows: Given p, q in the interior of the surface S (i.e., in the bounded component of $\mathbb{R}^3 - S$), there is certainly a polygonal curve from p to q. (Proof: The interior is a connected open set. The set of points in this set reachable from p via a polygonal curve is clearly nonempty— p is in it!, it is open and it is closed, both the latter by easy arguments. So it is the whole interior). Choose such a polygonal curve and parametrize it by arclength, say, for convenience:



$\gamma : [0, L] \to$ interior of S with $\gamma(0) = p$, $\gamma(L) = q$. Let L_t = the line segment from p to $\gamma(t)$. Let t_0 = the minimum t_0 such that $L_t \cap S$ is nonempty (if such a t_0 exists in $[0, L]$). Since the condition $L_t \cap S \neq \phi$ is closed, such a t_0 exists unless $L_t \subset$ interior for all $t \in [0, L]$ — and in that case we are done.

Moreover, $L_t \subset$ interior if $t < t_0$ so $L_{t_0} \cap \text{exterior} = \phi$. Now the endpoints $\gamma(0) = p$ and $\gamma(t_0)$ are in the interior of S, by construction. Thus the point(s) of $L_{t_0} \cap S$ are not endpoints of L_{t_0}. Let q = the point of $L_t \cap S$ closest to p. Since $L_{t_0} \cap \text{exterior} = \phi$, but $q \in S$, L_{t_0} is tangent to S at q.



Moreover, since S touches (near q) its tangent plane at q only at q, the line L_{t_0} lies on one side of S near q, as shown. But this is a contradiction since S's interior lies (locally) at each point on the side opposite the exterior normal: this is so at the max level for z and is hence so everywhere by continuity! This completes the proof by contradiction. $\square$

[It is actually the case that a closed surface with $K \geq 0$ is the boundary of a convex open set, but the proof is more difficult.]