Gauss Curvature and the Form of the Metric in Geodesic Polar Coordinates.

We are interested in computing our usual quantities, $E, F, G, L_{11}, L_{12}, L_{22}$ in (r,θ) geodesic normal coordinates. We have already shown that $E \overset{\text{def.}}{=} \langle G_r, G_r \rangle = 1$ and $F \overset{\text{def.}}{=} \langle G_r, G_\theta \rangle = 0$ while $\langle G_\theta, G_\theta \rangle > 0$ but is otherwise not determined.

With θ_0 fixed and r varying over (small) positive numbers, we define

$$\eta(r) = G_\theta / \|G_\theta\| = G_\theta / \langle G_\theta, G_\theta \rangle^{1/2}.$$

(We use η for "normal" to the r-curve, reserving N for the surface normal).

Lemma: $T(\eta_r) = 0$ for all (small) $r > 0$.

Proof: Since $\langle \eta, \eta \rangle \equiv 1$, $\langle \eta_r, \eta \rangle = 0$. So we need only check that $\langle \eta_r, G_r \rangle = 0$ to see that $T(\eta_r) = 0$. For this, differentiate $\langle \eta, G_r \rangle \equiv 0$ to get $\langle \eta_r, G_r \rangle + \langle \eta, G_{rr} \rangle = 0$. But $T(G_{rr}) = 0$ since the r-curves are geodesics, while η is in the tangent space. So $\langle \eta_r, G_r \rangle = 0$ $\square$

For convenience, write $f(r,\theta) = \langle G_\theta, G_\theta \rangle^{1/2}\big|_{(r,\theta)}$

so $G_\theta = f\,\eta$.

In this notation, $G = \langle G_\theta, G_\theta \rangle = f^2(r,\theta)$
So we are really trying to derive information about $f(r,\theta)$. For this, we experiment with looking at third derivatives of G, more or less along the same lines as our earlier proof that Gauss curvature is intrinsic. It turns out that a good thing to look at is $\langle G_{rr\theta}, G_\theta \rangle$. First, note that

$$\langle G_{rr\theta}, G_\theta \rangle = \frac{\partial}{\partial\theta} \langle G_{rr}, G_\theta \rangle - \langle G_{rr}, G_{\theta\theta} \rangle$$

Since G_{rr} is normal to the surface (r-curves are geodesics!), $\langle G_{rr}, G_\theta \rangle \equiv 0$. Also

$$\langle G_{rr}, G_{\theta\theta} \rangle = \langle N(G_{rr}), N(G_{\theta\theta}) \rangle = L_{11}L_{22}.$$

So $\quad \langle G_{rr\theta}, G_\theta \rangle = -L_{11}L_{22}. \qquad (1)$

Now we use that $G_{rr\theta} = G_{\theta rr}$ and compute $\langle (G_{\theta r})_r, G_\theta \rangle$ as follows:

$$G_{\theta r} = (f\eta)_r = f'\eta + f\eta_r, \text{ where } f' = \frac{\partial f}{\partial r}$$

Note that $f'\eta$ is tangent while η_r is normal to S by the Lemma proved earlier.
So $\quad G_{\theta r} = f'\eta + N(G_{\theta r})$
Differentiating and taking inner product with G_θ gives

$$\begin{aligned}
&\langle (G_{\theta r})_r, G_\theta \rangle \\
&= \langle f''\eta, G_\theta \rangle + \langle f'\eta_r, G_\theta \rangle + \left\langle \frac{\partial}{\partial r}\big(N(G_{\theta r})\big), G_\theta \right\rangle \\
&= f''f \quad + \underset{(\text{since } \eta_r \perp S)}{0} \quad - \langle N(G_{\theta r}), N(G_{\theta r}) \rangle \qquad (2)
\end{aligned}$$

where the last term comes from

$$0 = \frac{\partial}{\partial r} \langle N(G_{\theta r}), G_\theta \rangle = \left\langle \frac{\partial}{\partial r}\big(N(G_{\theta r})\big), G_\theta \right\rangle + \langle N(G_{\theta r}), G_{\theta r} \rangle$$

So $\quad \left\langle \frac{\partial}{\partial r}\big(N(G_{\theta r})\big), G_\theta \right\rangle = -\langle N(G_{\theta r}), N(G_{\theta r}) \rangle$

Combining items (1) and (2) we get

$$-L_{11}L_{22} = f''f - \langle N(G_{\theta r}), N(G_{\theta r})\rangle$$

or

$$-L_{11}L_{22} = f f'' - L_{12}^2$$

Recall that $G = f^2$, $E = 1$, $F = 0$

So

$$f'' = -\left(L_{11}L_{22} - L_{12}^2\right)/f$$

$$= -f\left(\frac{L_{11}L_{22} - L_{12}^2}{EG - F^2}\right)$$

So finally $\quad f''(r,\theta) = -K_{r,\theta} f(r,\theta)$

where $\quad K_{r,\theta}$ = Gauss curvature at $G(r,\theta)$

and $\quad f''(r,\theta)$ is shorthand for $\left.\frac{\partial^2 f}{\partial r^2}\right|_{(r,\theta)}$.

So f (which was $\langle G_\theta, G_\theta\rangle^{1/2}$) satisfies an ordinary differential equation along each r-curve, θ fixed, with the (variable) coefficient being Gauss curvature!

This means in effect that the metric is determined (locally) by the Gauss curvature: all we need to know to make this explicit is to find the "initial conditions" $\lim_{r\to 0^+} f(r,\theta)$ and $\lim_{r\to 0^+} f'(r,\theta)$, for θ fixed. These, it turns out, have a simple form:

Lemma: $\lim_{r \to 0^+} f(r,\theta) = 0$, $\lim_{r \to} f'(r,\theta) = 1$.

Proof: Write $\hat{S}(x,y)$ in place of our previous $\hat{S}(a,b)$, just so things look more usual, with x, y as variables. So $x = r\cos\theta$, $y = r\sin\theta$

Thus $\hat{S}_x = \frac{\partial}{\partial x} G(r,\theta)$

$$= \frac{\partial r}{\partial x} G_r + \frac{\partial \theta}{\partial x} G_\theta$$

$$= \frac{x}{\sqrt{x^2+y^2}} G_r - \frac{y}{x^2+y^2} G_\theta$$

So

$$\langle \hat{S}_x , \hat{S}_x \rangle = \frac{x^2}{x^2+y^2} + \frac{y^2}{(x^2+y^2)^2} f^2(r,\theta)\Big|_{\theta=0}$$

For this to be finite-valued and smooth ~~on~~ the y-axis ($\theta = \pi/2$), i.e., $y \to 0^+$, $x = 0$, it must be that $f(r, \pi/2) \to 0$ as $r = \sqrt{x^2+y^2} \to 0$. And in fact

$\frac{y^2 f^2}{y^4}$ must go to 1 (since at $(0,0)$, S_x has length 1). So $f/y \to 1$. Since $y = r$ on this axis, $\lim_{r \to 0^+} f'(r,\theta) = 1$ in this $\theta = \pi/2$ case.

But clearly, the choice of $\theta = \pi/2$ is arbitrary (the argument could be made along any axis). $\square$