More on the Form of the Metric in Geodesic Normal Coordinates

p_0 = center of geodesic coordinates in what follows

Recall that we computed that with $f = \|G_\theta\|$,

$\lim_{r \to 0^+} f(r,\theta) = 0$ and $\lim_{r\to 0^+} \left.\frac{\partial f}{\partial r}\right|_{(r,\theta)} = 1$, each θ fixed

Also $f(r,\theta)$ satisfies $\frac{\partial^2 f}{\partial r^2} = -K(r,\theta)\, f(r,\theta)$.

From these, we see that the Taylor expansion of f, for θ fixed, is

$$f(r,\theta) = 0 + r - \tfrac{1}{6} K|_{p_0}\, r^3 + \text{higher order}$$

because $\frac{\partial^2 f}{\partial r^2} = -K(r,\theta)\, f(r,\theta)$ gives $\lim_{r\to 0^+} \frac{\partial^2 f}{\partial r^2} = 0$ while differentiating one more time gives

$$\lim_{r\to 0^+} \frac{\partial^3 f}{\partial r^3} = -\lim_{r \to 0^+}\left(\frac{\partial}{\partial r} K\right) f - \lim K \frac{\partial f}{\partial r}$$

$$= -K|_{p_0} \cdot 1 = -K|_{p_0}$$

Note that this gives, if $\ell(C(r))$ = length of circle of radius $r = \int_0^{2\pi} \|G_\theta\|\, d\theta = \int_0^{2\pi} f(r)\, d\theta$

that $\ell(C(r)) = 2\pi r - \frac{2\pi}{6} K|_{p_0} r^3 + \text{h.o.t.}$

Thus

$$K|_{p_0} = \lim_{r\to 0^+} \frac{3}{\pi}\left(\frac{2\pi r - \ell(C_r)}{r^3}\right)$$

(result obtained by Gauss).

Example: (1) S^2, $\|G_\theta\| = \sin r$ $(0 < r < \pi)$
$\ell(C_r) = 2\pi \sin r$.

(2) metric on $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ given by
$E = G = 4/(1-x^2-y^2)^2$, $F=0$.
Symmetry suggests that straight lines through the origin with proportional-to-arc-length parameter are geodesics (Think about this: If x-axis, say, is so parameterized then acceleration must be zero since it is $\perp$ but if it were nonzero this would give a preferred "up or down" while the metric is symmetric under $y \to -y$).
Let us compute the parameterization:
Set $\Gamma(t) = (t,0)$. How long is $\Gamma|[0,a]$, $0<a<1$:

$$\ell(\Gamma|[0,a]) = \int_0^a \sqrt{E\left(\frac{dx}{dt}\right)^2 + F\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + G\left(\frac{dy}{dt}\right)^2}$$

$$= \int_0^a \sqrt{E \cdot 1} = \int_0^a \frac{2}{1-x^2-y^2}\bigg|_{(t,0)} dt$$

$$= \int_0^t \frac{2}{1-t^2}\, dt = \int_0^t \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt$$

$$= \log(1+a) - \log(1-a) = \log\left(\frac{1+a}{1-a}\right)$$

What a-value do we need to get a given r

$r = \log((1+a)/(1-a))$: $e^r = \dfrac{1+a}{1-a}$

or $a = \dfrac{e^r - 1}{e^r + 1}$ (note $a \to 1^-$ as $r \to +\infty$!)

So the arclength parameter geodesic along the x-axis is, in (x,y) coordinates

$$\gamma(t) = \left(\frac{e^r-1}{e^r+1}, 0 \right)$$

In general, for θ fixed, the geodesic with angle θ in (euclidean) polar coordinates is

$$\gamma(r) = \left(\underset{\nearrow}{\frac{e^r-1}{e^r+1}}, \theta \right)$$

Euclidean radius.

Now what is G_θ?

The length of the r-fixed, θ goes from 0 to 2π curve C_r is

$$2\pi(\text{Euc. radius}) \cdot \frac{2}{1-r^2}$$

So $$\|G_\theta\| = \frac{e^r-1}{e^r+1} \cdot \frac{2}{1-\left(\frac{e^r-1}{e^r+1}\right)^2}$$

$$= \frac{2(e^r-1)(e^r+1)}{(e^r+1)^2-(e^r-1)^2}$$

$$= \frac{2\,e^{2r}-1}{4e^r} = \frac{e^r-e^{-r}}{2} = \sinh r.$$

Note that $\|G_\theta\|\big|_{(r,\theta)} \to 0$ as $r \to 0^+$ and $\frac{\partial}{\partial r}\|G_\theta\|\big|_{(r,\theta)} \to 1$ $\left(= \cosh r\big|_{r=0}\right)$ as $r \to 0^+$.

This fits in with the fact that the metric we are talking about has Gauss curvature $= -1$ (as you check earlier):

$$f(r,\theta) = \sinh r \quad \text{so}$$

$$\frac{\partial^2}{\partial r^2} f(r,\theta) = \frac{\partial}{\partial r}(\cosh r) = \sinh r$$

$$= -(-1)\, f(r,\theta)$$

so Gauss curvature $= -1$.

Returning to the general situation, we can find the first few terms (up to and including quadratic terms) of E, F, and G for (x,y) geodesic (nonpolar) rectangular coordinates from the $\|G_\theta\| = f$ expansion as follows:

$$\hat{S}_x = \frac{\partial}{\partial x} \hat{S}(r\cos\theta, r\sin\theta) = \frac{\partial}{\partial x} G(r,\theta)$$

and

$$\hat{S}_y = \frac{\partial}{\partial y} \hat{S}(r\cos\theta, r\sin\theta) = \frac{\partial}{\partial y} G(r,\theta)$$

So $$\hat{S}_x = \frac{\partial r}{\partial x} G_r + \frac{\partial \theta}{\partial x} G_\theta = \frac{x}{r} G_r - \frac{y}{r^2} G_\theta$$

$$\hat{S}_y = \frac{\partial r}{\partial y} G_r + \frac{\partial \theta}{\partial y} G_\theta = \frac{y}{r} G_r + \frac{x}{r^2} G_\theta$$

So $$E = \left\langle \frac{x}{r} G_r - \frac{y}{r^2} G_\theta, \frac{x}{r} G_* - \frac{y}{r^2} G_\theta \right\rangle$$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^4} f^2 = \frac{x^2}{r^2} + \frac{y^2}{r^4}\left(r - \frac{r^3}{6} K|_{p_0} + \cdot\right)^2 =$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^4}\left(r^2 - \frac{r^4}{3}K|_{p_0} + \cdots\right)$$

$$= \frac{x^2}{r^2} + \frac{y^2}{r^2} - \frac{y^2}{3}K|_{p_0} + \text{h.o. in } (x, y)$$

$$= 1 - \frac{y^2}{3}K|_{p_0} + \text{h.o.}$$

Similarly $\|\hat{S}_y\|^2 = 1 - \frac{x^2}{3}K|_{p_0} + \text{h.o.}$

and

$$\langle \hat{S}_x, \hat{S}_y \rangle = \frac{xy}{r^2} - \frac{xy}{r^4}\left(r^2 - \frac{r^4}{3}K|_{p_0} + \text{h.o.}\right)$$

$$= -\frac{xy}{3}K|_{p_0}.$$

Thus the quadratic part of the metric (and constant and linear part) in geodesic normal (rectangular) coordinates is determined at po by $K|_{p_0}$.