Lecture V:
Diagonalization of Symmetric Matrices

In this section, all vector spaces are over $\mathbb{R}$.

Definition: A linear transformation $T: V \to V$ of a (finite dimensional) vector space V with inner product $\langle\ ,\ \rangle$ is called <u>self-adjoint</u> if, for $v, w \in V$

$$\langle Tv, w\rangle = \langle v, Tw\rangle.$$

The name comes from the observation that $\langle v, Tw\rangle = \langle T^{*}v, w\rangle$ so that the condition $\langle Tv, w\rangle = \langle v, Tw\rangle$ is the same as $\langle T^{*}v, w\rangle = \langle Tv, w\rangle$ for all $v, w \in V$ and hence the condition (for all $v, w \in V$) of the definition is equivalent to $T^{*} = T$. Here we are using the fact that in the presence of an inner product, the adjoint T^{*} of $T: V \to V$, which by definition maps V^{*} to V^{*}, can be considered to map V to V since the inner product gives an identification of V and V^{*}: $v \leftrightarrow \langle \cdot, v\rangle$ (as discussed earlier).

Note: The definition as given of self-adjoint makes sense whether or not V is finite dimensional, but we are only interested for the moment in the finite-dimensional case. Also, the remarks on the relationship of $\langle Tv, w\rangle = \langle v, Tw\rangle$ and $T=T^*$ apply directly only in the finite-dimensional case, since in general there might be (if V is infinite dimensional) elements of V^* not arising as $\langle \cdot, v\rangle$, $v \in A$

Example: V = eventually-0 sequences, $\langle (x_1 \ldots, x_n \ldots), (y_1, \ldots, y_n, \ldots)\rangle = \sum x_i y_i$.
The linear function
$$L((x_1, \ldots x_n \ldots)) = \sum x_i$$
does not arise as $\langle \cdot, v\rangle$, v an eventually zero sequence.

Theorem: If $T: V \to V$, V finite dimensional with an inner product $\langle\ ,\ \rangle$, and if T is self-adjoint, then there is an orthonormal basis $v_1, \ldots, v_n$ of V and an n-tuple of numbers $\lambda_1, \ldots \lambda_n$ such that

$$T v_i = \lambda_i v_i \quad \text{for all } i = 1, \ldots n.$$

This theorem is often called "the diagonalization of symmetric matrices" for the following reason: Relative to an orthonormal basis $v_1, \ldots, v_n$ (at both ends, same basis) the adjoint T^* of a linear transformation T with matrix A has matrix A^t.. This is easily checked by direct calculation. Alternatively, it follows from the similar statement for $T^*: V^* \to V^*$ as discussed earlier. Thus T with matrix A (relative to an orthonormal basis) is self-adjoint if and only if $A = A^t$.

A basis satisfying the conclusion of the Theorem of course makes the matrix of the self-adjoint linear transformation T (relative to the basis) into a diagonal matrix: its (i,i) elements are λ_i, $i = 1, \ldots, n$ while its (i,j) elements, $i \neq j$ are 0. This explains the name "diagonalization of symmetric matrices" for the Theorem.

However, our proof of the Theorem makes no use of matrices as such, though the explicit calculation in concrete cases of the basis $v_1 \ldots v_n$ and the numbers $\lambda_1, \ldots, \lambda_n$ will use matrices. This will be discussed later.

Proof of the Theorem:
The proof is by induction on the dimension n. The correctness of the conclusion for $n=1$ is clear, since a linear transformation of a one-dimensional vector space to itself consists of multiplication by a fixed constant, so v_1 can be any vector of unit inner product with itself

(there are exactly two such vectors!).

To treat $n > 1$, assuming $n-1, \ldots, 1$, note first that we can assume $V = \mathbb{R}^n$ with its standard inner product. This entails no loss of generality since any n-dimensional vector space over $\mathbb{R}$ with an inner product is isomorphic to $\mathbb{R}^n$ via an isomorphism that preserves inner products. (Reason: V has an orthonormal basis, by Gram-Schmidt. If $w_1, \ldots, w_n$ is an orthonormal basis, $\sum_{i=1}^{n} \alpha_i w_i \to (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}$ does the job).

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then T is a continuous function on $\mathbb{R}^n$ with its usual topology: this is obvious. So, necessarily, is the function $v \hookrightarrow \langle Tv, v \rangle$, since $\langle\,,\rangle$ is also continuous. Now $\{v \in \mathbb{R}^n : \langle v, v \rangle = 1\}$ is compact. Hence, $\exists\, v_0 \in \mathbb{R}^n$ with $\langle v_0, v_0 \rangle = 1$ and $\langle Tv_0, v_0 \rangle \geq \langle Tv, v \rangle$ for all v with $\langle v, v \rangle = 1$: "a continuous function on a compact set attains its maximum". If $w_0 \in \mathbb{R}^n$ satisfies $\langle w_0, v_0 \rangle = 0$, then $v_0 + t w_0$ satisfies $\langle v_0 + t w_0, v_0 + t w_0 \rangle = \langle v_0, v_0 \rangle + t^2 \langle w_0, w_0 \rangle \geq 1$ so that $v_0 + t w_0 \neq \vec{0}$. Set

$$v(t) = \frac{1}{\sqrt{\langle v_0 + t w_0, v_0 + t w_0 \rangle}} (v_0 + t w_0).$$

Then $\langle v(t), v(t) \rangle = 1$ for all t. Hence

$$\langle T v(t), v(t)\rangle \le \langle T v_0, v_0\rangle \quad \text{for all } t.$$

Note that $v(0) = v_0$ so that
$$\langle T v(0), v(0)\rangle = \langle T v_0, v_0\rangle.$$

Now
$$\langle T v(t), v(t)\rangle$$
is a differentiable (indeed C^∞) function of $t \in \mathbb{R}$, and (since it attains its maximum at $t=0$):

$$\frac{d}{dt}\langle T v(t), v(t)\rangle\Big|_{t=0} = 0.$$

Also $\frac{d}{dt}\sqrt{\langle v_0 + t w_0, v_0 + t w_0\rangle}\Big|_{t=0} = 0.$

from $\langle v_0 + t w_0, v_0 + t w_0\rangle = 1 + t^2\langle w_0, w_0\rangle.$

Thus
$$\frac{d}{dt}\langle T(v(t)), v(t)\rangle\Big|_{t=0}$$
$$= \frac{d}{dt}\langle T(v_0 + t w_0), v_0 + t w_0\rangle\Big|_{t=0}$$
$$= \langle T w_0, v_0\rangle + \langle T v_0, w_0\rangle$$
$$= 2\langle T v_0, w_0\rangle$$
by the self-adjoint property of T. Thus we have shown

$$\langle T v_0, w_0\rangle = 0 \quad \text{for } \forall\, w_0 \ni \langle v_0, w_0\rangle = 0$$

It follows that $\exists\, \lambda \in \mathbb{R}$ such that $T v_0 = \lambda_0 w_0$ (since $T v_0 \in ((\operatorname{span} v_0)^\perp)^\perp = \operatorname{span} v_0$).

Now $(\text{span}(v_0))^\perp$ is mapped to itself by T since $\langle w, v_0 \rangle = 0 \Rightarrow$

$$\langle Tw, v_0 \rangle = \langle w, Tv_0 \rangle = \lambda \langle w, v_0 \rangle = 0.$$

But $(\text{span}(v_0))^\perp$ has dimension $n-1$. And $T|(\text{span}(v_0))^\perp$ is clearly self-adjoint relative to $\langle\, ,\, \rangle$ restricted to $(\text{span}(v_0))^\perp$. Thus there is, by inductive hypoth., a basis $v_1, \ldots, v_{n-1}$ of $(\text{span}(v_0))^\perp$ with $v_1 \ldots v_{n-1}$ orthonormal and

$T v_i = \lambda_i v_i$, some $\lambda_i \in \mathbb{R}$, $\forall\, i = 1, \ldots, n-1$.
Then $v_0, v_1 \ldots v_{n-1}$ satisfy the conditions of the theorem for $T: \mathbb{R}^n \to \mathbb{R}^n$ $\square$

How does one compute things in practice about this situation, in the context that locating a maximum of a continuous function on a compact set is not so easy in practice? Here is a good approach: Consider $T - \lambda I : \mathbb{R}^n \to \mathbb{R}^n$, where $\lambda \in \mathbb{R}$ is some fixed number. Looking a T relative to a diagonalizing basis $v_1, \ldots, v_n$ (with $Tv_i = \lambda_i v_i$), it is clear that the λ_i's that occur are exactly the λ-values such that
$$\ker(T - \lambda I) \neq \{\vec{0}\}.$$
[Note that several v_i's may have the same λ_i attached: this does not affect the argument].

So we can find the λ_i by finding the real solutions of

$$\det(A - \lambda I_n) = 0$$

(we shall see momentarily that all solutions of this equation, a polynomial equation of degree n in λ, are necessarily real when A is symmetric, but this is not needed for the moment).

Notice that if $v \in \ker(T - \lambda_1 I)$, $w \in \ker(T - \lambda_2 I)$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then [T symmetric!]

$$\langle v, w \rangle = \{\vec{0}\}.$$

The reason is that $(\lambda_1 - \lambda_2)\langle v, w\rangle$

$$\lambda_1 \langle v, w\rangle - \lambda_2 \langle v, w\rangle$$

$$= \langle \lambda_1 v, w\rangle - \langle v, \lambda_2 w\rangle = \langle Tv, w\rangle - \langle v, Tw\rangle$$

$= 0$ since T is symmetric.

Thus we can find the "diagonalizing basis" of the Theorem by:

(1) find all real solutions of $\det(A - \lambda I) = 0$

(2) For each such real λ, choose an orthonormal basis of $\ker(A - \lambda I)$

(3) Then these orthonormal bases of the $\ker(A - \lambda I)$ taken together are a diagonalizing orthonormal basis of $\mathbb{R}^n$ fo T (or A).

Illustration: $n=2$, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (a-\lambda)(c-\lambda) - b^2$$
$$= \lambda^2 - (a+c)\lambda + ac - b^2$$

$\det(A-\lambda I) = 0 \iff \lambda = \frac{1}{2}\left((a+c) \pm \sqrt{(a+c)^2 + 4b^2 - 4ac}\right)$

or $\lambda = \frac{1}{2}\left((a+c) \pm \sqrt{(a-c)^2 + 4b^2}\right)$

Two cases (roots are real in either case, since $(a-c)^2+4b^2 \geq 0$)

(1) $(a-c)^2 + 4b^2 = 0$, double root,
happens only if $b=0$ and $a=c$,
Then $\lambda = a = c$, $A = a I_2$.

(2) $(a-c)^2 + 4b^2 > 0$, two real roots λ_1, λ_2,
associated unit perpendicular vectors
v_1, v_2 (unique up to order & $\pm$) with
$$Tv_1 = \lambda_1 v_1 \quad \text{and} \quad Tv_2 = \lambda_2 v_2 .$$
The vectors v_1, v_2 are obtained by solving
$$(a-\lambda_i)x + by = 0 \quad \&$$
$$bx + (c-\lambda_i)y = 0$$
which for $i = 1$ or 2 has a nontrivial solution since it has $\det(\text{coef.}) = 0$.

Note that this is all explicitly doable!

The consideration of $\ker(T-\lambda I)$ or $\det(A-\lambda I_n)$ occurs so frequently that definitions have been made.

Definition: If V is a vector space over a field F and if $T: V \to V$ is a linear transformation of V, then $\lambda \in F$ is an <u>eigenvalue</u> of T if

$\ker(T-\lambda I) \neq \{\vec{0}\}$ and the (nonzero) vectors in $\ker(T-\lambda I)$ are <u>eigenvectors</u> of T for the eigenvalue λ.

If A is an $(n\times n)$ matrix with entries in a field F, then the roots of the degree n polynomial in λ

$$\det(A-\lambda I_n)$$

are called <u>eigenvalues</u> of the matrix A.

This terminology is used when these roots belong to a field extension of F! For example, a real matrix can have complex, nonreal eigenvalues, e.g. $A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But we do not

consider these as eigenvalues of the associated linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. But they are eigenvalues of the associated linear transformation from $\mathbb{C}^2 \to \mathbb{C}^2$ (also given from A)

Def: If λ is an eigenvalue of a linear transformation $T: V \to V$ then the "<u>eigenspace</u> of λ is $\ker(T - \lambda I)$.

Eigenvectors belonging to distinct eigenvalues of T are linearly independent. More exactly:

Theorem: If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of $T: V \to V$ and if $v_1, \ldots, v_k$ are (nonzero) eigenvectors for $\lambda_1, \ldots, \lambda_k$ respectively, then the set $\{v_1, \ldots, v_k\}$ is linearly independent.

Proof: Suppose $\{v_1, \ldots, v_k\}$ is dependent. Consider a nontrivial linear relationship among these v's that involves the minimum number of v's: since no v is $\vec{0}$, this number must be at least two. Relabeling if need be, we can suppose the relationship has the form

$$\sum_{i=1}^{\ell} \alpha_i v_i = \vec{0} \qquad \text{where } \ell \geq 2 \text{ and minimal}$$

(and no $\alpha_i = 0$).

Then, applying T: $\sum_{i=1}^{\ell} \alpha_i \lambda_i v_i = 0$

Hence

$$\vec{0} = \sum_{i=1}^{\ell} \alpha_i \lambda_i v_i - \lambda_\ell \left(\sum \alpha_i v_i\right) = \sum_{i=1}^{\ell-1} \alpha_i (\lambda_i - \lambda_\ell) v_i$$

But since $\lambda_i - \lambda_\ell \neq 0$, $i < \ell$, and $\alpha_i \neq 0$, this is a nontrivial linear relationship among the v's involving only $\ell - 1$ v's $\square$

A related matter is that the orthonormal basis $v_1, \dots, v_n$ of eigenvectors account in effect for all the eigenvalues.

Theorem: If A is a real symmetric $n \times n$ matrix, then all solutions of $\det(A - \lambda I) = 0$ over $\mathbb{C}$ are in fact in $\mathbb{R}$ and occur among the $\lambda_1, \dots, \lambda_n$ for the diagonalizing basis.

Proof: Let $v_1, \dots, v_n$ be an orthonormal basis of real eigenvectors, $Av_i = \lambda_i v_i$, λ_i real but not necessarily all distinct. Then A can be considering to act on $\mathbb{C}^n$ with $v_1, \dots, v_n$ a $\mathbb{C}$ basis for $\mathbb{C}^n$.* If $\det(A - \lambda_0 I) = 0$, then $\exists$ $\sum \alpha_i v_i$, $\alpha_i \in \mathbb{C}$ with not all $\alpha_i \neq 0$ and

$$A \left(\sum \alpha_i v_i \right) = \sum_1^n (\lambda_0 \alpha_i) v_i .$$

But

$$A \left(\sum \alpha_i v_i \right) = \sum_{i=1}^n \alpha_i \lambda_i v_i$$

$$\text{So} \quad \vec{0} = \sum_1^n \alpha_i (\lambda_0 - \lambda_i) v_i .$$

If $\lambda_0 \notin \{\lambda_1, \dots, \lambda_n\}$, (and since not all $\alpha_i = 0$) this is a nontrivial $\mathbb{C}$-relationship among the v_i's, which is impossible. $\square$

*Exercise: Check that $v_1, \dots, v_n$ form a $\mathbb{C}$-basis: they generate!

It is natural to ask whether, for an arbitrary field F, whether the solutions of

$$\det(A - \lambda I_n) = 0$$

in the algebraic closure of F in fact all lie in F when A is symmetric ($n \times n$). This is false: the eigenvalues of

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ are the solutions of $(1-\lambda)(2-\lambda) - 1 = 0$

namely $\lambda^2 - 3\lambda + 1 = 0$, solutions

$$\lambda = \tfrac{1}{2}\left(3 \pm \sqrt{9-4}\right) = \tfrac{1}{2}\left(3 \pm \sqrt{5}\right)$$

which do not belong to $\mathbb{Q}$, even though the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ has entries in $\mathbb{Q}$. This justifies our use of specific properties of $\mathbb{R}$ in our proof of the real diagonalization over $\mathbb{R}$ of symmetric matrices with $\mathbb{R}$-valued entries.

Another natural question is this: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is self-adjoint, then can we compute not just the eigenvalues of T (= solutions of $\det(A - \lambda I) = 0$) but also the dimensions $\dim(\ker(A - \lambda I))$ from the algebra of the polynomial (the "characteristic polynomial") $\det(A - \lambda I)$. The answer is yes!

This depends on the following fact:

Lemma: If $T : V \to V$ is a linear transformation (not necessarily self-adjoint) and $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are bases for V with matrix of T relative to v-basis (at both ends) $= A$ and matrix of T relative to w-basis is (at both ends) $= B$, then

$$\det(A - \lambda I) = \det(B - \lambda I)$$

where equality means equality as polynomials!

Proof: $B = PAP^{-1}$ for some $n \times n$ invertible matrix P (Exercise). So

$$\det(B - \lambda I) = \det(PAP^{-1} - \lambda I)$$
$$= \det(P(A - \lambda I)P^{-1})$$
$$= (\det P)(\det P^{-1}) \det(A - \lambda I)$$
$$= \det(A - \lambda I)$$

since $(\det P)(\det P^{-1}) = \det(PP^{-1}) = \det I_n = 1$

[Here we have used det product = product of determinants, which we shall prove later].

Corollary: If $T : V \to V$, V ($\mathbb{R}$ vector space) with inner product $\langle , \rangle$, is self-adjoint, then $\forall \lambda_0 \in \mathbb{R}$

$\dim[\ker(T - \lambda_0 I)] =$ the multiplicity of λ_0 as a root of $\det(A - \lambda I) = 0$ where

A is the matrix of A relative to <u>any</u> basis of V (at both ends).

Proof: This is obvious for A_0 = matrix of T relative to a diagonalizing orthonormal basis. But the previous Lemma gives that

$$\det(A - \lambda I) = \det(A_0 - \lambda I)$$

as a polynomial for every A representing T relative to some basis. $\square$

Note that when T is not self adjoint (and A is not symmetric) this can go wrong:

$$\det\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \quad \text{is} \quad (\lambda - 2)^2 = 0.$$

The eigenvalue $\lambda = 2$ has multiplicity 2 as a root of $\det\left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \lambda I\right) = 0$.

But $\ker\left[\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

has dimension 1, namely it is $\text{span}[(1, 0)]$.