Summary of Lecture I

1st half: Quick trip through determinants:

Permutation $i \to \pi(i)$.

Form list $\pi(1)\ \pi(2) \dots$

Count: no. of numbers in list after $\pi(1)$ that are less than $\pi(1)$

no of numbers in list after $\pi(2)$ that are less than $\pi(2)$

etc.

Add these together.

Def: $\operatorname{sign}\pi$ [$= (-1)^\pi$, alternate notation] [also $\operatorname{sgn}\pi$]

$= +1$ if total is even

$= -1$ if total is odd

So adjacent interchange alters $\operatorname{sign}\pi$ by multiplication by $-1 \Rightarrow$ every interchange does this.

Thus if π is product of k interchanges,

$\operatorname{sign}\pi = (-1)^k$

[Exercise: Every permutation is product of interchanges or "transpositions" as they are sometimes called.].

If A = square $(n \times n)$ matrix (a_{ij})

$i = 1, \dots n$ row index

then $j = 1, \dots n$ column index

$$\det A \stackrel{\text{def}}{=} \sum_{\pi} (-1)^\pi\, a_{1\pi(1)}\, a_{2\pi(2)} \cdots a_{n\pi(n)}$$

where sum is over all permutations of $1, \dots n$.

(sum over "symmetric group on n elements")

Consequences of definition:

(1) $\det A = \det A^t \qquad (A^t)_{ij} \overset{\text{def}}{=} A_{ji}$

Proof: Since $\operatorname{sign}\pi = (-1)^{\text{no of interchanges to generate } \pi}$,

$\operatorname{sign}\pi \cdot \operatorname{sign}\sigma = \operatorname{sign}(\pi \circ \sigma)$. Since then

$1 = \operatorname{sign}(\text{identity}) = \operatorname{sign}(\pi \circ \pi^{-1}) = \operatorname{sign}\pi \operatorname{sign}\pi^{-1}$
it follows that $\operatorname{sign}\pi = \operatorname{sign}\pi^{-1}$, from which (1) follows.

(2) two rows equal or two columns equal
$\Rightarrow \det = 0$

Proof: Interchange of two rows or two columns reverses sign. □

(3) det linear in individual rows or columns

(4) Replacing row by same row + constant multiple of another row (same for columns) leaves determinant unaltered.

Proof: Apply (3) and (2) combined.

(5) $\det(\mathrm{Id}) = 1 \qquad$ Def of Id: $\big((\mathrm{Id})_{ij} = 1 \text{ if } i=j,\ 0 \text{ if } i \neq j\big)$

2nd half: Vector spaces & linear transformations

Vector space definition, as usual
V over field F, V has "additive"
structure (abelian group relative to +).
Also $\alpha(v+w) = \alpha v + \alpha w \quad \alpha \in F, v, w \in V$

$$(\alpha+\beta)v = \alpha v + \beta v$$
$$1v = v$$

etc. (see text for detail).
Basic examples: Q over Q, F over F (F any field)
$\mathbb{R}^n$ over $\mathbb{R}$ $\{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$ component-wise operations

$\mathbb{C}^n$ over $\mathbb{C}$ $\{(z_1, \dots, z_n) : z_j \in \mathbb{C}\}$
$\mathbb{C}^n$ over $\mathbb{R}$: same set, only $\mathbb{R}$-multiplication allowed.

Function spaces $C([0,1])$; C^∞ functions on $\mathbb{R}$,
$\{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{+\infty} |f|^2 < +\infty\}$ etc.

"Free vector space" over a set X: set of functions $f: X \to \mathbb{R} \ni f(x) = 0$ for all but a finite number of x.
(also for arbitrary field $f: X \to F$)

E over F if F is a subfield of a field E

Quotient objects:
W is a subspace of V if ($0 \in W$ and) W is closed under vector space operations [$0 \in W$ is automatic: listed for emphasis only, to distinguish from "affine subspace"]

Given W subspace of V, there is a quotient object (vector space) denoted V/W (read "V mod W") which is itself a vector space

set: equivalence classes of V under

$$v_1 \sim v_2 \quad \text{means} \quad v_1 - v_2 \in W$$

operations $[v_1] + [v_2] = [v_1 + v_2]$

$$\alpha [v_1] = [\alpha v_1].$$

(Exercise: These are "well defined" and result is a vector space).

Definition: A vector space V is <u>finite</u> <u>dimensional</u> if $\exists$ $v_1, \ldots, v_n$ such that for each $v \in V$ $\exists$ $\alpha_1, \ldots \alpha_n$ with $v = \sum_{i=1}^{n} \alpha_i v_i$.

Slogan: $v_1, \ldots v_n$ generate V.

Exercise: $\mathbb{R}^n$ is finite dimensional

Basic fact: If $v_1, \ldots, v_n$ generate V, then every set $w_1, \ldots, w_N$, $N > n$, in V is linearly dependent

(Recall definitions: $w_1, \ldots, w_N$ is linearly independent if $\sum \beta_j w_j = 0 \Rightarrow \beta_j = 0$ all j. The set $w_1, \ldots, w_N$ is linearly dependent if it is not linearly independent, e.g., $\exists$ a "nontrivial" linear combination of the vectors which $= \vec{0}$).

Note: An infinite set is linearly independent if every finite subset is (by definition).

Linear transformations: A function $T: V \to W$ is a linear transformation if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ all $\alpha, \beta \in F$, $v_1, v_2 \in V$. [Note: V, W are both over same field].

Subspaces $\ker T = \{ v \in V : T(v) = 0 \}$
$\operatorname{im} T = \{ w \in W : \exists v \in V \ni T(v) = w \}$.
(sometimes range T)

Exercise: These are subspaces.

Examples (not usual ones!)

$V = C^\infty$ functions on $\mathbb{R} = W$ $\quad T = \frac{d}{dx}$

$Tf = df/dx$

$V =$ space of eventually zero sequences
$(x_1, \dots, x_n, \dots) \ni \exists N$ with $x_n = 0$, $n \geq N$.
$(x_1, \dots x_n \dots) \to (0, x_1, x_2, \dots$
("shift operator")

More familiar: $V = \mathbb{R}^n = W$ $\quad Tv = \alpha v$, $\alpha \in F$ fixed
"dilation"

Rotations of $\mathbb{R}^2$, etc.