Summary of Lecture VIII: Power Series & Taylor Expansion

Power series: $\sum_{n=0}^{+\infty} a_n x^n$, $\sum_{n=0}^{+\infty} a_n (x-a)^n$

Sufficient to treat first case.

Basic observation: If $\sum_{n=0}^{+\infty} a_n \alpha^n$ converges, then $\sum_{n=0}^{+\infty} a_n x^n$ converges for each x with $|x| < |\alpha|$.

Moreover, if $0 < r < |\alpha|$, then $\sum_{n=0}^{+\infty} |a_n||x|^n$ converges uniformly on $[-r, +r]$ and hence $\sum_{n=0}^{+\infty} a_n x^n$ converges uniformly on $[-r, +r]$.

Proof: $\sum a_n \alpha^n$ converges $\Rightarrow \lim_{n\to+\infty} a_n \alpha^n = 0 \Rightarrow$

$\exists M > 0 \ni |a_n \alpha^n| \le M$, $\forall n = 1, 2, 3 \ldots$

So $|a_n||x|^n \le M\left(\frac{|x|}{|\alpha|}\right)^n \quad \forall n = 1, 2, 3 \ldots$

$\Rightarrow \sum |a_n||x|^n$ converges if $|x| < |\alpha|$.

Also $\Rightarrow$ uniform convergence on $[-r, r]$ (Use Weierstrass M-test with $\sum M|\alpha|^{-n} r^n$ as comparison series of constants).

So what does the "convergence set $= \{x \in \mathbb{R} : \sum a_n x^n$ converges$\}$" look like? Possibilities

(1) convergence set $= \{0\}$ (only) "$R = 0$"

(2) convergence set $= (-\infty, +\infty) = \mathbb{R}$ "$R = +\infty$"

(3) convergence set $= (-R, R) \cup$ possibly R or $-R$ or both $R =$ rad of convergence

We describe the cases as (1) radius of convergence $= 0$.
(2) radius of convergence $= +\infty$
(3) radius of convergence $= R$

Formula $\quad R = 1 / \limsup \sqrt[n]{|a_n|}$ (interpreted to mean $= +\infty$ if denom. $= 0$ etc.)

(Recall lim sup of a sequence = largest $\beta \in \mathbb{R}$ such that some subsequence converges to β. By convention, this is $+\infty$ is there is an unbounded-above (subsequence.))

Proof: Exercise using methods already indicated.

Corollary: The radius of convergence of the differentiated series $\sum_{n=0}^{\infty} (1+n) a_{n+1} x^{n+1}$

and of the original series $\sum a_n x^n$ are equal

Proof: $\quad x$ (differentiated series) $= \sum^{\infty} n a_n x^n$
has same radius of convergence as the differentiated series itself (obvious). But
$$\limsup \sqrt[n]{n|a_n|} = \limsup \sqrt{|a_n|}$$

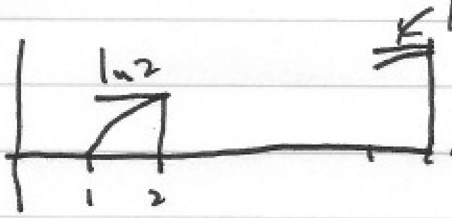
because $\lim \sqrt[n]{n} = 1$ (Reason $\ln(\sqrt[n]{n}) = \frac{1}{n} \ln n \to 0$ as $n \to +\infty$).

Cor: If $\sum a_n x^n$ has radius of convergence R then $\sum a_n x^n$ is C^∞ on $(-R, R)$ and
$(\sum a_n x^n)' = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.

Proof: Apply Theorem that limit of convergent sequence is differentiable with derivative of limit = limit of derivatives, if convergence is uniform and derivatives converge uniformly to a continuous function (done earlier) ∎

Examples: $\sqrt[n]{n!} \to +\infty$ as $n \to +\infty$

($\ln n! = \ln 2 + \dots + \ln n \geq \int_1^n \ln t \, dt$



$= 1 + n(\ln n) - n = n(\ln n - 1) + 1$

& $\frac{1}{n} \ln n! \to +\infty$ [OR

$\ln n! \geq \left(\frac{n}{2}\right)^{n/2}$ if n is even

so $\frac{1}{n} \ln n! \geq \frac{1}{n} \left(\frac{n}{2} \cdot (\ln n - \log 2) \right) \to +\infty$

similarly for odd (just go one down)]

So radius of convergence of $1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$ is $+\infty$.

(Can also do with ratio test: $\left| x^{n+1}/(n+1)! \Big/ (x^n/n!) \right| = |x/n+1| < 1$ if n is large)

Similarly for $\sin x$ series: $x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$

$\cos x$ series $1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

Note by differentiation term-by-term

$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$ so $\sin^2 x + \cos^2 x$ has derivative 0 so is cons. $\equiv 1$ (since $\sin^2 0 + \cos^2 0 = 1$).

So if used series as definition would recover properties! (some of them: others take some work).

Similarly e^x series differentiated = same series etc.

Side observation: Whole radius of convergence theory works for $\sum a_n z^n$, $a_n \in \mathbb{C}$, $z \in \mathbb{C}$. This explains some things that otherwise seem mysterious. For instance $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \ldots$ has radius of convergence 1 but $1/(1+x^2)$ is C^∞ on $\mathbb{R}$. But $1/(1+z^2)$ has a zero denominator (hence "singularity") at $z = \pm i$. Distance 1 from 0! so radius of convergence of series could not be larger than 1.

Interesting example from integration term by term (which also works, by uniform convergence, inside radius of convergence): $0 < x < 1 \Rightarrow$

$$\int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^{2n} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots$$

As $x \to 1^-$, $\int_0^x \frac{1}{1+t^2}\,dt = \arctan x \to \frac{\pi}{4}$.

RHS, as $x \to 1^-$, goes to $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$ (reason: |error| of stopping at a term $\leq$ |next nonzero term| because of alternating series with | | terms nonincreasing. Apply this: exercise)

Similarly $\int_1^x \frac{1}{1-x}$

$$= -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \text{ if } |x| < 1.$$

Let $x \to -1$ (alternating series again to prove legal) to get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$

But we can also put $x = \frac{1}{2}$ to get

$$-\ln \tfrac{1}{2} = \ln 2 = \tfrac{1}{2} + \tfrac{1}{2}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^4 + \dots$$

— which is not obviously $= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$!

A given constant can have more than one "natural" series representation!

Taylor Expansion:

formal series (assuming f C^∞ in a neighborhood of $x=a$)

$$f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Enough to consider case $a = 0$:

Theorem: If f is $(n+1)$ times differentiable on $(-\varepsilon, b+\varepsilon)$, $b>0$, for some $\varepsilon > 0$ then

$$f(b) = f(0) + bf'(0) + \dots + f^{(n)}(0)\frac{b^n}{n!} + f^{(n+1)}(\lambda)\frac{b^{n+1}}{(n+1)!}$$

for some $\lambda \in (0,b)$.

(Similar estimate for negative b case).

Proof (cf. extract from Thomas Calculus, proof by James Wolfe).

Set $F(x) = f(x) - f(0) - xf'(0) \dots - f^{(n)}(0)\frac{x^n}{n!} - Kx^{n+1}$

where K is such that $F(b) = 0$.

Since $F(0) = 0$ and $F(b) = 0$, $\exists\, c_1 \in (0,b)$ with $F'(c_1) = 0$

Since $F'(0) = 0$ also, $\exists\, c_2 \ni F''(c_2) = 0$, $c_2 \in (0, c_1)$.

Continuing, get c_j, $0 < c_j < c_{j-1}$ with $F^{(j)}(c_j) = 0$.

until $j = n+1$. At c_{n+1}, $F^{(n+1)}(c_{n+1}) = 0$

But $F^{(n+1)}(c_{n+1}) = f^{(n+1)}(c_{n+1}) - K(n+1)!$

So $K = f^{(n+1)}(c_{n+1}) / (n+1)!$. So $c_{n+1} = \lambda$ works. $\square$

Example: Since e^x has all derivative $= e^x$, the derivatives are uniformly bounded by e^R on $[-R,R]$ all R. So
error term $\leq \left(f^{(n+1)}(\lambda) / (n+1)!\right) R^{n+1} \to 0$ as $n \to +\infty$.
So (infinite) Taylor series converges to function e^x (here we are supposing e^x was not defined by the series but rather was defined otherwise, e.g. as the inverse function of $\int_1^x \frac{1}{t}\,dt = \ln x$, $x > 0$: then $(e^x)' = e^x$, Taylor series is usual series and convergence is obtained as described. If we defined e^x by the series, then the remainder estimate would not ... since series would converge to function e^x automatically! — but remainder estimate would still show about rate of convergence!)
Exercise: Think similar thoughts about $\sin x$ and $\cos x$ (also defined as inverse functions to their own (branch of) inverses defined by integrals, e.g. $\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$ $|x| < 1$

etc.)