Lecture XII: Line Integrals, Path Independence and Related Topics, and Classical Vector Calculus

Converse of equality of mixed partials: If $U \overset{\text{open}}{\subset} \mathbb{R}^2$

is an open rectangle (or open disc) and if P and Q are continuous functions on U such that $\partial P/\partial y$ and $\partial Q/\partial x$ exist and are continuous on U and if $\partial P/\partial y \equiv \partial Q/\partial x$ on U, then $\exists\, f: U \to \mathbb{R}$ with $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$ at each point of U.

Proof. Wolog $(0,0) \in U$ (and $(0,0)$ = center of disc if U = disc). Define for $(x,y) \in U$

$$f(x,y) = \int_0^x P(t,0)\,dt + \int_0^y Q(x,s)\,ds$$

By the Fundamental Theorem of Calculus, $\frac{\partial f}{\partial y}$ exists at each (x,y) and $= Q(x,y)$ there.

Also $\left[\frac{d}{dx}\left(\int_0^x P(t,0)\,dt\right)\right]\Big|_{(x,y)} = P(x,0)$

while, by differentiation under the integral sign and using $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$:

$$\frac{d}{dx}\left(\int_0^y Q(x,s)\,ds\right) = \int_0^y \frac{\partial}{\partial x} Q(x,s)\,ds \;\Big|_x$$

$$= \int_0^y \frac{\partial P}{\partial y}\Big|_{(x,s)} ds = P(x,y) - P(x,0)$$

(Fund. Th. of Calculus again)

So $\frac{d}{dx} f(x,y)\Big|_{(x,y)} = P(x,0) + (P(x,y) - P(x,0)) = P(x,y)$

! Exercise: Generalize to $P_1, \ldots P_n$ on open rectangle (or ball) in $\mathbb{R}^n$; $\partial P_j/\partial x_i = \partial P_i/\partial x_j$ all i, j. Use induction.

Non-example: $U = \mathbb{R}^2 - \{\vec{0}\}$ ②

$$P = -y/(x^2+y^2) \qquad Q = x/(x^2+y^2)$$

$$\partial P/\partial y = [(x^2+y^2)(-1) + y(2y)]/(x^2+y^2)^2$$
$$= -x^2+y^2 \ / \ (x^2+y^2)^2$$

$$\partial Q/\partial x = [(x^2+y^2) - x(2x)]/(x^2+y^2)^2$$
$$= (-x^2+y^2)/(x^2+y^2)^2$$

Note: This had to work since $P = \frac{\partial\theta}{\partial x}$ $Q = \frac{\partial\theta}{\partial y}$ where θ is a "local" branch of polar coordinate angle, i.e. $(r,\theta) \to (x,y)$ locally inverted $(r \neq 0)$, $\theta = \theta(x,y)$.

But if $\exists f: \mathbb{R}^2 - \{\vec{0}\} \to \mathbb{R}$ with $\frac{\partial f}{\partial x} = P$, $\frac{\partial f}{\partial y} = Q$ then

$$f((1,1)) - f((1,-1)) = \int_{-1}^{+1} \frac{1}{1+s^2}\, ds$$

$$= \arctan s\Big]_{-1}^{+1} = (\pi/4) - \left(-\frac{\pi}{4}\right) = \pi/2$$

and similarly
$$f((-1,1)) - f((1,1)) = \pi/2$$
$$f((-1,-1)) - f((-1,1)) = \pi/2$$
$$f((1,-1)) - f(-1,-1)) = \pi/2$$

Adding all four $f-f$ equalities gives
$$0 = 2\pi, \text{ a contradiction!}$$

So no such f exists! (smooth or even continuous)

This is a precise version of no "global" polar coordinate angle existing on $\mathbb{R}^2 - \{\vec{0}\}$

Exercise: Prove: If $V^{\text{open}} \subset \mathbb{R}^2 \setminus \{\vec{0}\}$ and if $f: V \to \mathbb{R}$ is continuous and satisfies
$(x,y) = (\sqrt{x^2+y^2}\cos f(x,y), \sqrt{x^2+y^2}\sin f(x,y))$
for all $(x,y) \in V$, then f is actually C^1 (indeed, C^∞).
So our proof that no C^1 polar coordinate choice exists on all of $\mathbb{R}^2 - \{\vec{0}\}$ actually proves no continuous choice exists

Line Integrals

Def: A curve $\gamma: [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^n$) is C^1 on $[a,b]$ if $\exists\ \varepsilon > 0$ and $\hat{\gamma}: (a-\varepsilon, b+\varepsilon) \to \mathbb{R}^2$ (or $\mathbb{R}^n$) which is C^1 differentiable and satisfies $\hat{\gamma}\big|_{[a,b]} = \gamma$.

Definition: If $U^{\text{open}} \subset \mathbb{R}^n$ and $P_1, \ldots, P_n$ are continuous function (into $\mathbb{R}$) on U and if $\gamma: [a,b] \to U$ is C^1 then

$$\oint_\gamma P_1\,dx_1 + \ldots + P_n\,dx_n$$
$$= \int_a^b \sum_{j=1}^n P_j(\gamma(t)) \cdot \frac{dx_j(t)}{dt}\,dt$$

where the x_j are determined by $\gamma(t) = (x_1(t), \ldots, x_n(t))$

Example $\mathbb{R}^2$ $\displaystyle\oint_\gamma P\,dx + Q\,dy = \int_a^b P(\gamma(t))\frac{dx}{dt} + Q(\gamma(t))\frac{dy}{dt}$

[Here $P_j(\gamma(t)) = P_j\big|_{\gamma(t)}$, the function P evaluated at $\gamma(t)$ etc.]

Basic fact If f is C^1 on U and $\gamma : [a,b] \to U$ is C^1 then

$$\oint_\gamma \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n = f(\gamma(b)) - f(\gamma(a)).$$

Proof: $\frac{d}{dt} f(\gamma(t)) = \left.\frac{\partial f}{\partial x_1}\right|_{\gamma(t)} \frac{dx_1}{dt} + \dots + \left.\frac{\partial f}{\partial x_n}\right|_{\gamma(t)} \frac{dx_n}{dt}$

by the Chain Rule. Now apply the Fund. Th. of Calculus. □

Notation: grad f (read: "gradient of f") is, at $\vec{x} \in U$, the vector

$$\left(\left.\frac{\partial f}{\partial x_1}\right|_{\vec{x}}, \left.\frac{\partial f}{\partial x_2}\right|_{\vec{x}}, \dots \left.\frac{\partial f}{\partial x_n}\right|_{\vec{x}} \right).$$

If f is C^1 on U, then grad f is a continuous function from U to $\mathbb{R}^n$. We think of this as a "vector field", i.e., an assignment of a vector to each point of U. [A vector field is no different formally in this context than an $\mathbb{R}^n$-valued function, but the psychology is distinctive!]

Exercise $\frac{d}{dt} f(\vec{x}_0 + t\vec{v}) = \langle \text{grad} f|_{x_0}, \vec{v} \rangle$.

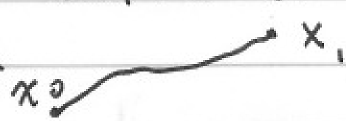
Note: The Basic Fact applies also to γ which are "piecewise C^1", i.e. to γ such that γ is continuous and $\exists$ a partition $a = a_0 < a_1 \dots < a_n = b \ni$ each $\gamma|[a_i, a_{i+1}]$ $i = 0, \dots n-1$ is C^1. (Proof: Exercise)

"Path Independence": If $P_1, \dots, P_n$ (continuous functions on U) are $\ni$

$$\oint_\gamma P_1 dx_1 + \dots + P_n dx_n$$

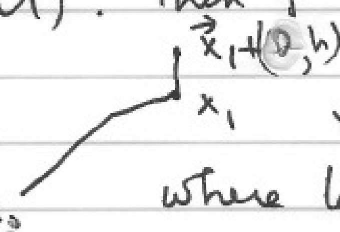
depends only on the endpoints $\gamma(a), \gamma(b)$ of γ for all piecewise C^1 $\gamma : [a,b] \to U$, then $\exists$ f such that $f : U \to \mathbb{R}$ is C^1 and $\partial f/\partial x_i = P_i$, $i = 1, \dots n$, everywhere on U.
(Converse is obvious)

Proof Define (assuming wolog U is connected) with x_0 fixed



$$f(x_1) = \oint_\gamma P_1 dx_1 + \dots + P_n dx_n$$

where γ is a piecewise C^1 curve from x_0 to x_1. (exists since piecewise C^1 accessible (from x_0) pts are an open & closed set in U). Then f satisfies the derivative condition by using pictures like that shown, (to show $\frac{\partial f}{\partial y} = P_2(\vec{x}_1)$)



where last segment of γ is along a coordinate direction: details are an exercise)

$\square$

Path independence $\iff \oint_\gamma = 0$ for all piecewise C^1 γ that are closed ($\gamma(b) = \gamma(a)$). Proof: easy exercise.

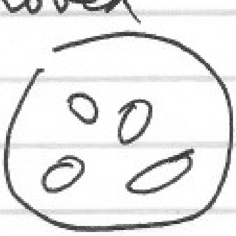
Definition: A "differential" $P\,dx + Q\,dy$ is <u>closed</u> if $\partial P/\partial y = \partial Q/\partial x$ (P, Q C^1).
The differential $P\,dx + Q\,dy$ is <u>exact</u> if $\exists\, f \ni df = P\,dx + Q\,dy$ where $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ by definition.

["Differentials" can be thought of as formal, or as, at each point, having value in $(\mathbb{R}^2)^*$ via $dx((\alpha,\beta)) = \alpha$, $dy((\alpha,\beta)) = \beta$]

(Similar ideas for $\mathbb{R}^n$).
Exact $\Rightarrow$ Closed (equality of mixed partials: we assume everything C^∞ from here on)

Closed differentials and exact differentials are vector spaces (over $\mathbb{R}$)
The quotient space closed / exact is called the (first) deRham cohomology of U. For a disc with k "holes" removed, it has dimension k. It is a "topological invariant": if U is homeomorphic to V, their deRham cohomologies are isomorphic (= same dimension)

4 holes



Topological invariance is hard in general.
Diffeomorphism invariance is easier (see exercises).
<u>See</u> exercises for some calculations, also.

Classical vector calculus operators on vector fields (on open sets in $\mathbb{R}^3$)

V a vector field on $U \subset \mathbb{R}^3$, i.e. $V: U \to \mathbb{R}^3$.
We assume as much differentiability of V as convenient in the context!

Write $V = (v_1, v_2, v_3)$ $\quad v_j$'s $\mathbb{R}$ valued

$$\operatorname{curl} V \overset{\text{def.}}{=} \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial v_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)$$

$$\operatorname{div} V = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$ (this is a number! valued item!)

Facts: (a) $\operatorname{div}(\operatorname{curl} V) = 0$ everywhere

Check

$$\frac{\partial}{\partial x_1}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) = \frac{\partial^2 v_3}{\partial x_1 \partial x_2} - \frac{\partial^2 v_2}{\partial x_1 \partial x_3}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_4}\right) = \boxed{\frac{\partial^2 v_1}{\partial x_2 \partial x_3}} - \frac{\partial^2 v_3}{\partial x_2 \partial x_1}$$

$$\frac{\partial}{\partial x_3}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) = \frac{\partial^2 v_2}{\partial x_3 \partial x_1} - \boxed{\frac{\partial^2 v_1}{\partial x_3 \partial x_2}}$$

Terms cancel as indicated (by = of mixed partials).

Of course, we rigged up curl to make this work!

(b) curl (grad f) = 0   Proof: Use = of mixed partials

Facts (a) and (b) have "local" converses (e.g. on open "rectangles")

(α) div $V \equiv 0 \Rightarrow \exists W \ni$ curl $W = V$

(β) curl $V \equiv 0 \Rightarrow \exists f \ni$ grad $f = V$

Item (β) is equivalent to something we have already shown (top of p.2, analogue in $\mathbb{R}^3$ of page 1).
For item (α), see Exercise .

Important operator on functions div (grad f)

$$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \quad \left(\text{or } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \text{ on } \mathbb{R}^2\right)$$

$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ (or $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or obvious generalization to $\mathbb{R}^n$)

is the Laplacian (definition)

$\Delta f \equiv 0 \overset{\text{def}}{\Longleftrightarrow} f$ is harmonic.

Important examples:

(1) $\log\sqrt{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$

(2) $1/\sqrt{x^2+y^2+z^2}$ on $\mathbb{R}^3 - \{\vec{0}\}$, $\frac{1}{r^{n-2}}$ on $\mathbb{R}^n - \{\vec{0}\}$

(3) $\mathrm{Re}\,(x+iy)^n$ and $\mathrm{Im}\,(x+iy)^n$ on $\mathbb{R}^2 = \mathbb{C}$

E.g. $x^2 - y^2 = \mathrm{Re}(x+iy)^2$

$x^3 - 3xy^2 = \mathrm{Re}\,(x+iy)^3$ (by inspection)

General reason: $\mathrm{Re}(x+iy)^n = x^n - \frac{n(n-1)}{2}x^{n-2}y^2 + \binom{n}{4}x^{n-4}y^4 \ldots$

2nd x deriv of each term is cancelled by 2nd y deriv of next one (Ex).