Summary of Lecture VII

Note from last time: Ideas work (e.g., sequence $\{\vec{a}_n\}$ has associated series $\sum \vec{a}_n$) for elements of $\mathbb{R}^n$ instead of just real numbers. E.g.
$\sum \|a_n\| < +\infty \Rightarrow \sum a_n$ converges.
Think about this (recall $\mathbb{R}^N$ is complete).

Connectedness: Recall definition X metric space is connected if U open in X and closed in X
$\Rightarrow U = \phi$ or $U = X$.
A disconnection of X is a pair of sets $U, V \subseteq X$ neither empty with $U \cap V = \phi$ and $U \cup V = X$.

A metric space X is connected $\Leftrightarrow$ there is no disconnection of X.

Basic fact: $[0,1]$ is connected.
Proof: Suppose $0 \in U$ where U, V is a disconnection.
Let $\hat{U} = \{\alpha \in [0,1] : [0,\alpha] \subset U\}$. Then

$0 \in \hat{U}$. Let $\beta =$ l.u.b. of $\hat{U}$. If $\beta = 1$, then $U \supset [0,1)$, hence $U = [0,1]$ since U is closed in $[0,1]$. Suppose $\beta < 1$. Since U is open, if $\beta \in U$ $\exists\, \varepsilon \ni \beta + \varepsilon < 1$ and $\beta + \varepsilon/2 \in \hat{U}$. This contradicts β being an upper bound for $\hat{U}$.
But if $\beta \in V$, then $\exists\, \varepsilon$ such that $\beta - \varepsilon > 0$ and $(\beta - \varepsilon, \beta] \subset V$. This contradicts $\beta =$ least upper bound of $\hat{U}$ since, e.g., $\beta - \varepsilon/2$ is an upper bound. $\square$

Immediate observations: A_λ $\lambda \in \Lambda$ connected, $\subset X$ and $\bigcap_{\lambda\in\Lambda} A_\lambda \neq \phi \Rightarrow \bigcup_{\lambda\in\Lambda} A_\lambda$ connected.

So, e.g., $\mathbb{R} = \bigcup_{N=1}^{+\infty} [-N, N]$ is connected,

$$(0,1) = \bigcup_{N=2}^{+\infty} \left[\tfrac{1}{N}, 1-\tfrac{1}{N}\right] \qquad \left(\tfrac{1}{2} \in \bigcap_{N=2}^{+\infty}\left[\tfrac{1}{N}, 1-\tfrac{1}{N}\right]\right)$$

disc in $\mathbb{R}^2$, ball in $\mathbb{R}^n$ connected (open or closed)
(union of "line segments" from center)

<u>Definition</u>: A metric space X is <u>arcwise connected</u> if $\forall\, p, q \in X$, $\exists\, \gamma : [0,1] \to X$ γ continuous with $\gamma(0) = p$, $\gamma(1) = q$.

Exercises: (1) arcwise connected $\Rightarrow$ connected.
(2) U open in $\mathbb{R}^n$, U connected $\Rightarrow$ U arcwise connected
(Suggestion: set of pts that are "arc-accessible" from given point $p \in U$ is both open and closed in U: arc accessible means $\exists\, \gamma$ $\ni$ $\gamma(0) = p$, $\gamma(1) = q$, $\gamma : [0,1] \to U$).

Observation: $F : X \to Y$, X, Y metric spaces, F continuous, F onto, X connected $\Rightarrow$ Y connected.
Proof: F^{-1} of a disconnection of Y would be a disconnection of X. $\square$
Application: $f : [0,1] \to \mathbb{R}$, f continuous, then $\alpha \in$ image of f if $\exists\, a, b \in$ image of f with $\alpha \in [a, b]$.

Cor: $f: [0,1] \to \mathbb{R}$ continuous $\Rightarrow$

$f([0,1])$ = a point or a closed interval, namely $f([0,1]) = \{\max f\}$ if $\max f = \min f$ or $f([0,1]) = [\min f, \max f]$ if $\min f < \max f$.

Proof $f([0,1]) \subset [\min f, \max f]$ by definition (and existence of) $\min f$ & $\max f$.

$f([0,1]) \supset [\min f, \max f]$ by connectedness. If $\alpha \in (\min f, \max f)$ but $\alpha \notin f([0,1])$ then $f^{-1}((-\infty, \alpha))$ and $f^{-1}((\alpha, +\infty))$ would be a disconnection of $[0,1]$. $\square$

Differentiability of one variable functions. Basic ideas and results.

We consider functions on an open interval $f: (a,b) \to \mathbb{R}$.

Definition: The function f is <u>differentiable</u> at $\lambda \in (a,b)$ if $\lim_{h \to 0} \frac{1}{h}(f(\lambda+h) - f(\lambda))$ exists.

The limit is called the derivative of f at λ, notation: $f'(\lambda)$ or $\frac{df}{dx}\big|_\lambda$.

Theorem (Rolle's Theorem): If $f: [a,b] \to \mathbb{R}$ is continuous and $f(a) = f(b) = 0$ and if f is differentiable on (a,b) (i.e., at each point of (a,b)), then $\exists\ \lambda \in (a,b)$ with $f'(\lambda) = 0$.

Proof: If $f \equiv 0$, done. If $f \not\equiv 0$, wolog $\max_{[a,b]} f > 0$. Set λ = a point of (a,b) where $f(\lambda) = \max_{[a,b]} f$. $f'(\lambda) = 0$ easy. $\square$

Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b), then $\exists\, \lambda \in (a, b)$ such that

$$f'(\lambda) = [f(b) - f(a)]/(b-a).$$

Proof: Apply Rolle's Theorem to

$$F(x) = f(x) - f(a) - \left[((f(b) - f(a))/(b-a)\right](x-a)$$

$\square$

Cor.: If $\mathcal{F}$ is a family of functions on $[a, b]$, continuous, differentiable on (a, b) with $|\text{derivative}| \leq M$ (M fixed), all pts $\in (a, b)$ and all functions in $\mathcal{F}$, then $\mathcal{F}$ is equicontinuous on $[a, b]$.

Theorem: If $\{f_n\}$ is a sequence of continuous function $f_n : (a, b) \to \mathbb{R}$ which converge uniformly to the (necessarily continuous) function $f_0 : (a, b) \to \mathbb{R}$, if the f_n are differentiable on (a, b) with $\{f_n'\}$ converging uniformly to a continuous function $g : (a, b) \to \mathbb{R}$, then f_0 is differentiable on (a, b) and $f_0' = g$.

Proof: $\lambda \in (a, b)$ and h ($|h| < \lambda - a$ & $< b - \lambda$) fixed then $\frac{1}{h}(f_n(\lambda+h) - f_n(\lambda))$ converges to $\frac{1}{h}(f_0(\lambda+h) - f_0(\lambda))$.
But $\frac{1}{h}(f_n(\lambda+h) - f_n(\lambda)) = f_n'(\alpha)$ where α is between λ and $\lambda + h$. Given $\varepsilon > 0$, $\exists n$ such that $|f_n'(\alpha) - g(\alpha)| < \varepsilon/2$ for all

$\alpha \in (a,b)$. On the other hand, $\exists\ \delta > 0$ such that $|h| < \delta \Rightarrow \alpha$ (between λ & $\lambda + h$) satisfies $|g(\alpha) - g(\lambda)| < \varepsilon/2$. It follows

that $|h| < \delta \Rightarrow \left| \frac{1}{h}\left(f_0(\lambda+h) - f_0(\lambda)\right) - g_0(\lambda) \right| < \varepsilon$.
Hence $f_0'(\lambda)$ exists and $= g_0(\lambda)$. $\square$

f bounded on $[a,b]$ (not nec. continuous)
Riemann integral: P, partition of $[a,b]$,
$a = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = b$.

$$\text{Upper sum}(P) = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \sup_{[a_i, a_{i+1}]} f$$

$$\text{Lower sum}(P) = \sum_{i=0}^{n-1} (a_{i+1} - a_i) \inf_{[a_i, a_{i+1}]} f$$

P_1 is a refinement of P_2 if every division point of P_2 is a division point of P_1 (but maybe not vice versa). If P_1 is a refinement of P_2 then

$$\text{lower sum}(P_2) \le \text{lower sum}(P_1) \le \text{upper sum}(P_1) \le \text{upper sum}(P_2).$$

f is Riemann integrable if there is, given $\varepsilon > 0$, a partition P such that upper sum(P) − lower sum$(P) < \varepsilon$.
One thinks of this as lower sum and upper sum converge to a (unique) number as P is "refined".
Specifically, if f is Riemann integrable $\Longleftrightarrow$ $\exists$ a unique number α such that $\alpha \le$ all upper sums and $\alpha \ge$ all lower sums.
(Prove this!). This is the same as "mesh convergence" of upper & lower sums: see exercise set.

Power series $\sum_{n=0}^{+\infty} a_n (x-x_0)^n$ (formal series)

Taylor series of a function that is infinitely differentiable, C^∞ ($f:(a,b) \to \mathbb{R}$ such that f is differentiable on (a,b), f' is diff. on (a,b), f'' is diff on (a,b) etc.).

(Assume $0 \in (a,b)$ to simplify notation)

$f(0) + f'(0)\,x + \frac{1}{2!}f''(0) + \frac{1}{3!}f'''(0) + \ldots + \frac{1}{n!}f^{(n)}(0)\,x^n + \ldots$

(formal series)

Examples: (1) $f(0) = 0$, $f(x) = e^{-1/x^2}$ $x \neq 0$

is C^∞ (Compute this)

(2) $f(x) = 0$, $x \leq 0$, $f(x) = e^{-1/x^2}$, $x > 0$.

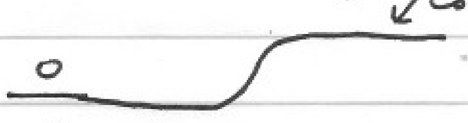
is C^∞

(3) $\exists\ f\ C^\infty \ni$ $f(x) = 0$ $x < 0$ and

$f(x) = 0$ $x > 1$

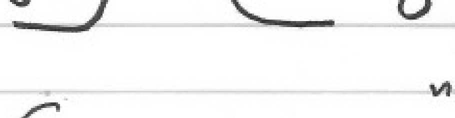
and $f \geq 0$ everywhere, positive somewhere

(take Example 2, multiply $f(x)$ by translate of $f(-x)$). Integrate to get



Flip and translate

Multiply to get



C^∞ "flat-topped bump function"

E. Borel Theorem: Given any sequence a_n, $n = 0, 1, 2 \ldots$, $\exists\ C^\infty$ function $f: \mathbb{R} \to \mathbb{R} \ni$ $f(0) = a_0$, $f'(0) = a_1$, $f''(0) = a_2, \ldots$ $f^{(n)}(0) = a_n, \ldots$ all n.

(not to be proved here).

So Taylor series can have very large coefficents (i.e. large growth increasing $n!$, so fast growing that it converges for no $x \neq 0$, e.g., $a_n = (n!)^2$

Also, Taylor series can converge everywhere but not to f (example 1: $f^{(n)}(0) = 0$, all $n = 0, 1, 2, \ldots$)

General power series picture (proofs next time)

Given $\sum_{n=0}^{+\infty} a_n x^n$, $\exists$ $\underset{\text{"radius of convergence"}}{\overset{\text{(unique)}}{R}} = 0$, pos. no., or $+\infty$

such that series converges if $|x| < R$, diverges (fails to converge) if $|x| > R$.

Fact 1: If $R \neq 0$ and $0 \leq R_1 < R$ then convergence is uniform on $[-R_1, R_1]$.

$R_1 < +\infty$ just means R_1 finite

Fact 2: Radius of convergence of differentiated series $\sum_{n=1}^{+\infty} n a_n x^{n-1}$ is the same as radius of convergence of $\sum_{n=0}^{+\infty} a_n x^n$

Conclusion (from earlier item on differentiation of limits): The function defined on $(-R, R)$ by $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ is C^∞ on $(-R, R)$ and its derivatives are obtained by "term by term [repeated] differentiation".