Summary of Lecture VI (August 14, 2008)

Uniformity of convergence in Arzela-Ascoli Theorem: see separate item.

Bits and pieces related to $\mathbb{R}$ and $\mathbb{R}^n$.

Cauchy Schwarz Inequality and Triangle Inequality for $\mathbb{R}^n$

Def: Inner product of $\vec{a}, \vec{b} \in \mathbb{R}^n$ (notation: $\langle \vec{a}, \vec{b} \rangle$) $= \sum_{i=1}^{n} a_i b_i$. Then

(norm of $\vec{a}$) $\|\vec{a}\| = \langle \vec{a}, \vec{a} \rangle^{\frac{1}{2}}$ & $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| \,(= \|\vec{w} - \vec{v}\|)$.

Cauchy Schwarz Inequality: $|\langle \vec{a}, \vec{b} \rangle| \leq \|\vec{a}\| \|\vec{b}\|$.

Proof: $0 \leq \langle \vec{a} + \lambda \vec{b}, \vec{a} + \lambda \vec{b} \rangle = \lambda^2 (\langle b, b \rangle) + 2\lambda \langle a, b \rangle + \langle a, a \rangle$

Quad. formula $\Rightarrow (2 \langle a, b \rangle)^2 - 4 (\langle b, b \rangle \langle a, a \rangle) \leq 0$: otherwise there are two distinct real λ-roots, hence a sign change.

So $\langle a, b \rangle^2 \leq \langle b, b \rangle \langle a, a \rangle \Rightarrow |\langle a, b \rangle| \leq \|a\| \|b\|$ $\square$.

$\Delta \leq$: $d(a, b) + d(b, c) \geq d(a, c)$?, $\forall a, b, c$?

Since $\|v - w\| = \|(v - a) - (w - a)\|$, all v, w, a, this $\geq$ is equivalent to $d(0, b) + d(b, c) \geq d(0, c)$ $\forall$ b, c. So we need $\|b\| + \|b - c\| \geq \|c\|$. $\forall$ b, c. This in turn is equivalent to $\|v + w\| \leq \|v\| + \|w\|$ (since $\|b - c\| = \|c - b\|$, so $v = b$, $w = c - b$, $v + w = c$ gives equivalence).

But $\|v + w\|^2 = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2$ since $\|v + w\|^2 = \langle v + w, v + w \rangle$. Also

$\|v\|^2 + 2\langle v, w \rangle + \|w\|^2 \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$

by Cauchy Schwarz Ineq. So $\|v + w\|^2 \leq (\|v\| + \|w\|)^2$

$\Rightarrow \|v + w\| \leq \|v\| + \|w\|$ as required.

Interesting and important observation:

Same argument gives "L^2 metric" on $C([0,1])$ (continuous $\mathbb{R}$-valued functions on $[0,1]$), namely $\|f\|_2 \stackrel{\text{def}}{=} \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}}$ from $\langle f,g\rangle \stackrel{\text{def}}{=} \int_0^1 f(x)g(x)\,dx$

$d(f,g) = \|f-g\|$, C.S.≤: $\int_0^1 f(x)g(x)\,dx \le \left(\int f^2\right)^{\frac{1}{2}}\left(\int g^2\right)^{\frac{1}{2}}$

Exercise: Check details

Definition: $L^2([0,1])$ = metric space completion of $C([0,1])$ relative to L^2 norm metric (a central object of study in analysis later on)

Sequences and series in $\mathbb{R}$: $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$

Series (formal) $\sum_1^\infty a_n$, partial sums $S_N = \sum_1^N a_n$. $\sum_1^\infty a_n$ "converges" if $\lim_{N\to+\infty} S_N$ exists. [Note $\Rightarrow \lim a_n = 0$ since $a_n = S_n - S_{n-1}$]

Basic fact: $\sum_1^{+\infty} |a_n|$ converges ($\Leftarrow\Rightarrow$ partial sums bounded above) $\Longrightarrow \sum_1^\infty a_n$ converges.

Proof: Set $A_N = \sum_1^N |a_n|$. Then A_N is a Cauchy sequence. But if $N_1 > N_2$ then [write $S_N = \sum_1^N a_n$]

$$|S_{N_1} - S_{N_2}| = \left|\sum_{N_2+1}^{N_1} a_n\right| \le \sum_{N_2+1}^{N_1} |a_n|$$

$= |A_{N_1} - A_{N_2}|$, so $\{S_N\}$ is a Cauchy sequence, too. Hence $\{S_N\}$ converges and so (by definition) does $\sum_{n=1}^{+\infty} a_n$

Familiar examples

(1) $1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{>\frac{1}{2}}+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{>\frac{1}{2}}+\underbrace{\frac{1}{9}+\cdots}_{>\frac{1}{2}\ (\text{up to } \frac{1}{16})}$

diverges

(2) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\cdots$

$\frac{1}{n^2}\le\frac{2}{n(n+1)}$ and $\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}<+\infty$ (converges)

because $\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}$ so series "telescopes"
(Exercise to do details)

(3) alternating series test $a_1\ge a_2\ge a_3\ldots$
and $\lim a_n=0 \Rightarrow a_1-a_2+a_3-a_4\cdots$ converges
(exercise)

Example: $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots=\pi/4$ (proof later)

Historically important observation:
If $\sum_1^{+\infty} a_n$ converges but $\sum_1^{+\infty}|a_n|$ diverges, then given $\alpha\in\mathbb{R}$ $\exists$ a rearrangement of $\sum a_n$ that converges to α. (also $\exists$ rearrangement "$\to+\infty$" or "$\to-\infty$").

Proof: Choose (in order) pos terms until sum exceeds α (first time)
then (in order) neg terms until sum is less than α,
then more pos. terms until sum exceeds α, etc.
You take at least one pos and one neg term each time, so eventually all used. [Note: $\sum$ pos terms $=+\infty$ and $\sum|$neg terms$|=+\infty$ necessarily]. Terms go to 0 (since $\sum a_n$ converges) so limit of processed sequence $=\alpha$.