# Summary of Lecture II

$\varepsilon, \delta$ definition of continuity: $f: X \to Y$, need only idea of distance on $X, Y$ to define.

Definition of metric space $(X, d)$, $d: X \times X \to \mathbb{R}^+$ with $d(x,y) \geq 0$, $= 0 \iff x = y$, $d(x,y) = d(y,x)$, $\underbrace{d(x,y) + d(y,z) \geq d(x,z)}_{\text{triangle inequality}}$

[cf. Chap 1, Sec 1 of Gamelin & Greene text].

Examples: (0) $X$ a set, define $d(x,y) = 1$ if $x \neq y$, $d(x,x) = 0$.
Every set occurs as the underlying set of a metric space.
[not too interesting: any $f$: such a (0,1) space $\to$ metric space $Y$ is continuous: take $\delta = 1/2$ for any $\varepsilon > 0$ ]

Formal def. of continuity (to be explicit)
$f: (X, d_1) \to (Y, d_2)$ is continuous at $x_0 \in X$ if, $\forall \varepsilon > 0$, $\exists \delta \ni \quad d_1(x_0, x) < \delta \Rightarrow d_2(f(x_0), f(x)) < \varepsilon$.

Examples (1): $\mathbb{R}^n$, possible metrics

$$d_1(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \quad \left(\begin{array}{c}\text{triangle inequality} \\ \text{requires proof!}\end{array}\right)$$

$$d_2(\vec{x}, \vec{y}) = \max_i |x_i - y_i|$$

$$d_3(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

Here $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$.

These metrics all give same idea for continuity.

Example (2): $C([0,1])$ = continuous $\mathbb{R}$-valued functions on $[0,1]$

Possible metrics:

"sup norm" $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$ ← like $d_2$ in Ex 1 $\left(\begin{array}{c}\text{will prove} \\ \text{later max exists}\end{array}\right)$

or

"$L^1$ norm" $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ ← like $d_3$ in Ex 1

or

"$L^2$ norm" $d(f,g) = \left(\int_0^1 |f(x) - g(x)|^2\right)^{1/2}$ $\left(\begin{array}{c}\text{triangle inequality} \\ \text{needs proof here}\end{array}\right)$

like $d_1$ in Ex. 1

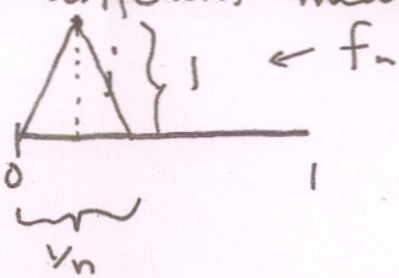
$(X,d)$ metric space, sequence $\{x_j\}$, $x_j \in X$, converges to $x_0 \in X$ if $\forall\, \varepsilon > 0$, $\exists\, N_\varepsilon$ such that $j \geq N_\varepsilon \Rightarrow d(x_0, x_j) < \varepsilon$ (Notation: $x_j \to x_0$)

Examples: (0) In 0,1 metric space (Example 0 of previous page), convergence is the same as eventually constant.

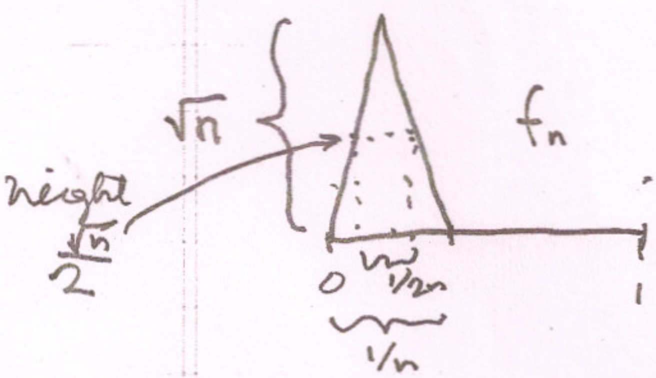
(1) $\mathbb{R}^n$ convergence in any of $d_1, d_2, d_3$ is same as convergence in any other (exercise)

(2) The three metrics on $C([0,1])$ have different meanings for convergence.

E.g.




$\{f_n\}$ converges to 0 function in $L^1$ and $L^2$ but $d(f_n, 0) = 1$, all $n$, in "sup norm" metric




$\{f_n\}$ converges to 0 in $L^1$ but $d(f_n, 0) > 1/8$ all $n$ in $L^2$ norm metric (since $f_n \geq \sqrt{n}/2$ on interval of length $1/2n$)

Convergence in "sup norm" metric does imply convergence in $L^1$ and $L^2$ metrics

Exercise for later: $f_n \to 0$ in $L^2 \Rightarrow f_n \to 0$ in $L^1$

Proof via $\left(\int_0^1 |f_n|\right)^2 \leq \int_0^1 |f_n|^2$ (Cauchy Schwarz for Integrals: will be shown soon)

If $\{x_j\}$ converges to $x_0$ in $(X, d)$ then for all $\varepsilon > 0$, $\exists N_\varepsilon \ni j, k \geq N_\varepsilon \Rightarrow d(x_j, x_k) < \varepsilon$

Proof: $x_j, x_k$ are $\varepsilon/2$ close to $x_0$, $j, k$ large. Use triangle inequality.

Def: A sequence $\{x_j\}$ in $(X, d)$ is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N_\varepsilon \ni j, k \geq N_\varepsilon \Rightarrow d(x_j, x_k) < \varepsilon$.

Cauchy sequences need not converge, e.g. $(\mathbb{Q}, d)$ $\mathbb{Q}$ = rationals $d(x, y) = |x - y|$, sequence $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ (decimal expansion of $\sqrt{2}$)

Fact (Pythagoras) $\sqrt{2}$ is not rational.

Proof If $p/q = \sqrt{2}$, i.e. $p^2 = 2q^2$, may assume (by cancelling powers of 2) that one of $p, q$ is odd. Then $p^2 = 2q^2$ implies that $p$ is even so $q$ is odd. But if $p = 2a$, then $q^2 = \frac{1}{2}p^2 = 2a^2$, so $q$ is even. X $\square$

So "decimal expansion of $\sqrt{2}$" is a Cauchy sequence with no limit

(Important philosophical point: the decimal expansion here could be defined entirely in the $\mathbb{Q}$ context, no reference to $\mathbb{R}$, e.g. $n$th term = largest number of form integer$/10^{n-1}$ such that its square $\leq 2$).

The real numbers are what you get if you "complete" $\mathbb{R}$, that is, fill in the missing Cauchy sequence limits in some way. Two explicit constructions

(1) "cuts": look at sets $C$ of rationals such that $\alpha \in C$ and $\beta < \alpha \Rightarrow \beta \in C$ and $\exists N \ni \alpha \in C \Rightarrow \alpha < N$. We think of $C$ as "being" a real number, intuitively $C =$ set of rationals $\leq \alpha_0$ where $\alpha_0$ is the intuitive real number $C$ "is". E.g. we identify $\sqrt{2}$ with the set of all rational numbers $\leq \sqrt{2}$. Note that $C$ "is" rational if $C = \{\alpha \leq \alpha_0 ; \alpha \in \mathbb{Q}\}$ for some rational number $\alpha_0$. E.g. $1$ is identified with $\{\alpha \leq 1 ; \alpha \in \mathbb{Q}\}$. Arithmetic inequalities etc. are all straightforward

(2) $\mathbb{R} =$ metric space completion of $(\mathbb{Q}, d)$ where $d(x, y) = |x - y|$, $x, y \in \mathbb{Q}$ (cf. p 13 of text G & G)

Either way, we get:

Least Upper Bound "Axiom":

First a definition: A (real) number $M$ is an upper bound for a set $A$ if $M \geq a$ for all $a \in A$. A set $A$ is bounded above if it has some upper bound, i.e., an upper bound $M$ exists.

Least Upper Bound Property of $\mathbb{R}$: If $A \subset \mathbb{R}$, $A \neq \phi$, is a set that is bounded above, then there exists an upper bound $M_0$ such that; if $M$ is an upper bound for $A$, then $M_0 \leq M$.
($M_0$ is called the "least upper bound" of $A$)