Summary of Lecture I & Exercises

Sets. Unions and intersections as usual (taken for granted). $\bigcup_{\lambda \in \Lambda} A_\lambda$, $\bigcap_{\lambda \in \Lambda} A_\lambda$, Λ "index set".

Basic concept: A_1, A_2 have same cardinality $\overset{\text{def.}}{\Longleftrightarrow}$ $\exists$ 1-1 onto ("bijective") function $f: A_1 \to A_2$

Notation: $A_1 \sim A_2$

(Recall here set theory idea of function $f: A \to B$ = by definition as subset of $A \times B \overset{\text{def.}}{=} \{(a,b): a \in A, b \in B\}$ such that for each $a \in A$, there is exactly one $b \in B$ such that $(a,b) \in$ the subset)

Finite set is one with same cardinality as $\{1, 2, \ldots, n\}$ for some n.
(We take the natural numbers $1, 2, 3 \ldots$ as given)

"Same cardinality" is an equivalence relation on sets. Namely, it has the three properties:

(1) $A \sim A$ ("reflexive" property as it is called though "identity" or "idempotence" might be more sensible)

(2) $A \sim B \Leftrightarrow B \sim A$ ("symmetric" "symmetry")

(3) $A \sim B$ & $B \sim C \Rightarrow A \sim C$ ("transitivity" "transitive property")

[Important concept in general: equivalence relation!]

Idea of one set being bigger than other (larger cardinality):

Cardinality of A is bigger than cardinality of B if $\exists\ f: B \to A$ 1-1 but there is <u>no</u> $g: A \to B$ 1-1. Notation: $\mathrm{Card}(A) > \mathrm{card}(B)$

$\mathrm{Card}(A) \geq \mathrm{Card}(B)$ if $\mathrm{Card}(A) = \mathrm{Card}(B)$ <u>or</u> $\mathrm{card}(A) = \mathrm{card}(B)$.

Basic (nonobvious) facts: (1) $\mathrm{Card}(A) \geq \mathrm{card}(B)$ and $\mathrm{card}(B) \geq \mathrm{card}(A) \Rightarrow \mathrm{card}(A) = \mathrm{card}(B)$

(2) If $\exists\ f: A \to B$ 1-1 and $\exists\ g: B \to A$ 1-1, then $\exists\ F: A \to B$ 1-1 onto.

These follow (as does everything one wants to know here) from:

(*) If $A \supseteq A_1 \supseteq C$ and $\exists\ f: A \to C$ 1-1, onto, then $\exists\ F: A \to A_1$ 1-1 onto.

Proof: Write $A_0 = A$ and define inductively $A_{i+2} = f(A_i)$ (starting from A_0, A_1). So, e.g. $C = A_2$.

Then
$$A_0 = (A_0 - A_1) \cup (A_1 - A_2) \cup (A_2 - A_3) \ldots \cup \left(\bigcap_{j=0}^{+\infty} A_j\right)$$
and
$$A_1 = (A_1 - A_2) \cup (A_2 - A_3) \cup \ldots \cup \left(\bigcap_{j=1}^{+\infty} A_j\right)$$

Then $f: A_0 - A_1 \to A_2 - A_3$ 1-1 onto
$f: A_2 - A_3 \to A_4 - A_5$ 1-1 onto
etc. $f: A_{2n} - A_{2n+1} \to A_{2n+2} - A_{2n+3}$
while identity take $A_1 - A_2 \to A_1 - A_2$
(generally $A_{2n-1} - A_{2n} \to A_{2n-1} - A_{2n}$
$n = 1, 2, 3 \ldots$ by identity).
$\cap A_i$ terms are the same (one starting with A_0, one with A_1 has no effect since $A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$). Put these together to get 1-1 onto map A_0 to A_1 □

So cardinalities have a kind of (partial) order (we have not shown that any two are comparable). They are but we have not proved it (uses Axiom of Choice)
Famous question: If C = cardinality of set of subsets of $N = \{1, 2, 3, 4 \cdots\}$, is there a set A such that $\text{card}(A) > \text{card}(N)$ but $\text{card}(C) > \text{card}\, A$?
"Continuum hypothesis" is that no such A exists.
[We shall see in moment that $\text{card}(C) > \text{card}(N)$ so the exist of such an A is at least possible in terms of our order relation].
Odd answer: Either "yes" or "no" to this question is consistent with everything.
You can choose arbitrarily yes or no without inconsistency.

"Power sets": Given A, 2^A = set of subsets of A.

Lemma: There is no function $f: A \to 2^A$ that is onto.

Proof: Given $f: A \to 2^A$, define A_0 by $a \in A_0 \iff a \notin f(a)$ (for each $a \in A$). Then A_0 is not in the image of f since this would give a contradiction (think about whether $a \in A_0$ if $A_0 = f(a)$ for some particular $a \in A$) $\square$ (cf. website item for details).

An infinite set A is countable if $\text{card}(A) = \text{card}(N) \iff$ the elements of A can be "listed" $a_1, a_2, a_3 \ldots$,

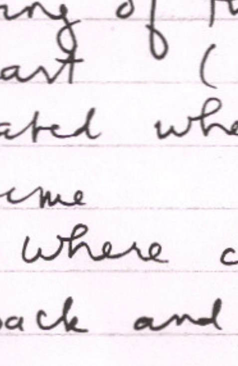
Example $\mathbb{Z}$ is countable: $0, +1, -1, +2, -2, +3, -3 \ldots$

The rationals $\mathbb{Q}$ are countable:

$0, r_1, -r_1, r_2, -r_2, r_3, -r_3$ where $r_1, r_2, r_3 \ldots$ is a listing of the positive fractions from this chart (eliminate fractions that are repeated when you hit them after the first time

	1	2	3	4	5	6	7
1	$1/1$	$2/1$	$3/1$	$4/1$			
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$			
3	$\frac{1}{3}$	$\frac{2}{3}$					
4	$\frac{1}{4}$	$\frac{2}{4}$					
5	$\downarrow$						

where chart is traversed back and forth as shown by arrows.



Similar chart traversal shows countable union $\bigcup_{j=1}^{\infty} A_j$ of countable sets is countable

Write $A_j = \{a_1^j, a_2^j, a_3^j, \dots\}$

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & a_1^1 \rightarrow & a_1^2 & a_1^3 \rightarrow & a_1^4 & \\
2 & a_2^1 & a_2^2 & & & \\
3 & a_3^1 & & & & \\
4 & a_4^1 & & & & \\
5 & & & & &
\end{array}$$

Skip any items that have already occurred.

Axiom of Choice: If A_λ $\lambda \in \Lambda$ is a collection of nonempty sets, then $\exists$ $f : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$ such that $f(\lambda) \in A_\lambda$

for each $\lambda \in \Lambda$.
(The function f "chooses" an element from each A_λ).

"Obvious" in a sense — but it can say something highly nonconstructive.
Example: $\mathbb{R}$ = real numbers, $\sim$ relation defined by $a \sim b$ means $a - b$ rational (reflexive, symmetric, transitive are obvious).
Look at collection of "equivalence classes" (sets of the form $\{a + r : r \in \mathbb{Q}\}$, a a real number)
These are equal or disjoint. There are uncountably many of them (why?) Try to think of a "systematic" way to pick a "representative" from each class!!

Exercises (not an assignment as such!)

1. If $f : A \to B$ is onto, show $\exists\, g : B \to A$ that is 1-1. Think about how this means that the "other way" of defining $\mathrm{card}(A) \geq \mathrm{card}(B)$ is related to the way we did it. (OK to use Axiom of Choice)
2. Is there a "largest" cardinality, i.e., does there exist a set A such that $\mathrm{card}(A) \geq \mathrm{card}(B)$ for all sets B?
3. Is there a "universe", that is, a set of which all sets are subsets?
4. Are you sure this stuff is safe to think about?
5. Try some finite set examples of the proof that $f : A \to 2^A$ cannot be onto to see how it works.
6. A real number α is algebraic if $\exists\, n, a_n, \ldots, a_0$, $a_n \neq 0$ integers $(n > 0)$ such that $a_n \alpha^n + \ldots + a_0 = 0$. Show that the set of algebraic numbers is countable. (Hint: How many α's can there be with, for N $= 1, 2, 3, \ldots$ fixed, $n \leq N$ and $|a_n| \leq N$; is it finite?)

A real number is <u>transcendental</u> if it is not algebraic. Once we show the reals are uncountable, we'll know (from this problem) that transcendental numbers exist. Actually π and e are transcendental, for example: but it is not easy to prove these specific instances.

7. If $\sim$ is a symmetric ($a \sim b \Leftrightarrow b \sim a$) and transitive ($a \sim b$ & $b \sim c \Rightarrow a \sim c$) relation on a set A, we can define a new relation $\hat{\sim}$ by $a \hat{\sim} b$ if $a \sim b$ or $a = b$ or both. Is $\hat{\sim}$ an equivalence relation? How much does it resemble $\sim$?

8. Investigate the difficulties of "filling in" in a similar way a relation that is "two-for-three" on the three properties of equivalence relation ("reflexive", "symmetric", "transitive") in a way that makes sense, i.e., stays reasonably close to the original relationship.